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Variational Methods of Analysis of Signals Based on the Frequency of Ideas

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Abstract : The article considers the approach to the signal analysis on the base of variational principles reflecting the views of the efficiency of the results obtained. These principles are formed by means of the natural language of frequency domain representations for the theory and practice of the signal processing, which allows to reach the adequacy from the point of view of the reflection of the physical entity of analysis and synthesis problems. The results of the problems solution of accurate values calculation of the energy shares of signals parts in an arbitrary frequency domain (accurate time-frequency analysis) were obtained, the criterion was formulated and the corresponding solution of the frequency filtration and optimal subband transformation (optimal subband frequency analysis) was obtained.

Keywords : Variational principles of the signal analysis on the base of frequency presentations, accurate values of energy shares of signals parts in an arbitrary frequency domain, optimal subband frequency analysis.

1. INTRODUCTION AND PROBLEM ASSIGNMENT

A signal is understood as a time function, characteristics of which contain data intended for the participants of the information exchange in an encoded form. The analysis of signals provides the distinguishing of these data on the base of the data processing.

Frequency representations are most often used as a base of the signal analysis algorithm. The analysis procedure of signals often leads to the distinguishing their components possessing these or those frequency properties, for example, the formation of spectrogram, filtration and subband transformations.

Solving of the specified problems shall satisfy some criteria of optimality. Such conditions are formed in this work on the basis of some variational principles allowing to build up the appropriate processing methods. It is the mathematical apparatus that determines the particular algorithms of data processing and reachable efficiency of the problem solving.

Frequency representations as a base of the signal analysis

Apparently, the frequency representations of the following type are most widely used in the theory of signals and at the development of the methods of its synthesis and analysis [1-5]:

$$f(t) = \int_{-\infty}^{\infty} \mathrm{F}(\omega)\exp(j\omega t)d\omega/2\pi, \tag{1}$$

where $f(t)$ is a function with a finite or unbounded definition domain $t \in [a, b]$. In the context of the question considered of the analysis and synthesis of signals it is sufficient to consider Fourier transform (main frequency characteristics) as the function $\mathrm{F}(\omega)$

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$$\mathrm{F}(\omega) = \int_a^b f(t)\exp(-j\omega t)dt \quad (2)$$

And the variable $\omega = 2\pi v$

is commonly called a circular frequency and that reflects the periodicity (cyclicity) of the changes of the initial function components $f(t)$ with the change of the argument t, which means that v is a value opposite to the period. According to this, the Fourier transform is often called a frequency characteristic of the studied function.

Concerning the content, the role of frequency representations in the problems of the analysis and synthesis of the signals is determined not only by the mathematical dualism of the representations (1) and (2), but also by the fact that the representation (1) is the expression of the superposition principle in a general case of the infinite number of the periodic components to which this initial function is expanded.

Parseval equation is true

$$\int_a^b f^2(t)dt = \int_{-\infty}^{\infty} |\mathrm{F}(\omega)|^2 \, d\omega / 2\pi, \quad (3)$$

and can be easily transformed into

$$\int_a^b f^2(t)dt = \sum_{r=0}^{\infty} \int_{\omega \in D_r} |F(\omega)|^2 \, d\omega / 2\pi, \quad (4)$$

where domains D_k determine the partition of the axes of the frequencies

$$\mathrm{D}_r = [-\Omega_{r+1}, -\Omega_r] \cup [\Omega_r, \Omega_{r+1}] \; \Omega_0 = 0. \quad (5)$$

Thus, it is possible to perform the frequency analysis of the energy characteristics of the studied function because the integrals determine the shares of energy belonging to the selected frequency domains.

$$\mathrm{P}_r = \int_{\omega \in \mathrm{D}_r} |\mathrm{F}(\omega)|^2 \, d\omega / 2\pi \quad (6)$$

In particular, the frequency domains can be distinguished where the biggest energy share is concentrated, or almost periodic components of the initial function which energies are concentrated in the different domains, if the last ones are rather narrow in comparison with the value

$$\mathrm{D} = 2\pi/(b-a).$$

The subintegral function in the right part of the ratio (3) is often called the spectral power density that emphasizes the "physical" sense of the characteristic. At the same time, it seems more reasonable to use the integrals of a type (6) as the physical characteristic of the signals.

Computer processing of the signals leads to the necessity of the domains discretization of the studied functions. Further the use of the equidistant discretization with a constant pitch is presupposed when

$$t_{k+1} = t_k + \Delta.$$

For the simplicity of the designation, we suppose

$$\begin{aligned} a &= 0\,;\, b = \mathrm{T}\,; \\ t_k &= (k-1)^* \Delta; \\ \Delta &= \mathrm{T}/(\mathrm{N}-1)\,; \\ k &= 1, \ldots, \mathrm{N}\,; \\ f_k &= f(t_k). \end{aligned} \quad (7)$$

For the discretized (discrete) functions, the analogue of the representation (1) is

$$f_k = \int_{-\Omega}^{\Omega} \mathrm{F}_d(\omega)\exp(j\omega\Delta(k-1))d\omega / 2\pi, \Omega = \pi/\Delta \quad (8)$$

and the Fourier transform is determined by the ratio

$$F_d(\omega) = \sum_{k=1}^{N} f_k \exp(-j\omega\Delta(k-1)) \tag{9}$$

and the Parseval equation is the following

$$\sum_{k=1}^{N} f_k^2 = \int_{-\Omega}^{\Omega} |F_d(\omega)|^2 \, d\omega / 2\pi, \Omega = \pi / \Delta \tag{10}$$

The natural analogue of the determination (6) of the energy share of the discretized function that gets into the selected frequency domain is

$$P_r = \int_{\omega \in D_r} |F_d(\omega)|^2 \, d\omega / 2\pi \tag{11}$$

Here it is the same as above

$$D_r = [-\Omega_{r+1}, -\Omega_r) \cup [\Omega_r, \Omega_{r+1}), \Omega_0 = 0$$

but the equation shall be satisfied

$$\Omega_{r+1} \le \Omega = \pi/\Delta \tag{12}$$

It is evident that the main problem is to provide the adequacy of the conclusions of the frequency properties of the studied functions on the base of the processing of its discretized values. The Nyquist theory gives the answer to some extent; one of its results is the ratio connecting the Fourier transforms of the continuous function and its discretized variant.

$$F_d(\omega) = \sum_{k=-\infty}^{\infty} F(\omega + 2\pi k / \Delta) / \Delta, |\omega| \le \pi / \Delta \tag{13}$$

It follows herefrom that at the finiteness of the domain of the Fourier transform of the continuous function and the selection of the discretion interval value, on the assumption of

$$F(\pi / \Delta + \varepsilon) = 0, \varepsilon > 0, \tag{14}$$

for the corresponding Fourier transforms the following equality will be performed

$$F_d(\omega) = F(\omega) / \Delta, |\omega| \le \pi / \Delta. \tag{15}$$

It should be mentioned that the condition (14) for the functions with a finite domain can be performed approximately only. In particular, the Fourier transform of the function with the finite domain of the definition is the entire frequency function, and, consequently, it can be equal to zero in any frequency domain of the finite length.

It is reasonable to use the following energy share as an accuracy measure of the condition fulfillment

$$P_\infty = \int_{\pi/\Delta}^{\infty} |F(\omega)|^2 \, d\omega / 2\pi / \| f \|^2$$

which shall be compared with the unit; here $\| f \|^2$ is the Euclidean norm (energy) of the signal.

Thus, one of the most important tasks of the analysis of signals is the definition of accurate values of shares of their energies in the set frequency domains on the base of the variational principle of the inaccuracy minimization.

Solution of the calculation problem of the accurate values of the energy shares

It can be shown that the substitution into the definition (11) of the right part of the ratio (9) allows to obtain the representation for the part of energy P_r concentrated in the frequency domain V_r

$$V_r = [-v_{r+1}, -v_r) \cup [v_r, v_{r+1}), v_0 = 0$$

of the studied vector f

$$P_r = \int_{\omega \in V_r} |F(\omega)|^2 \, d\omega / 2\pi = \vec{f}\,' A_r \vec{f} \tag{16}$$

where A_r is a subband matrix, $Ar = \{a^r_{ik}\}$,

$$a^r_k = (\sin(v_{r+1}(i-r)) - \sin(v_r(i-k)))/(\pi(i-k)); i, k = 1,.., N. \tag{17}$$

Let us consider some properties of the subband matrices A_r.

It is obvious that a subband matrix is symmetrical and nonnegative definite. Therefore [6-7], it possesses the complete system of the orthogonormal eigenvectors corresponding to the nonnegative eigenvalues and satisfying to their ratios

$$\lambda_{kr}\vec{q}_{kr} = A_r \vec{q}_{kr} \;;$$

$$(\vec{q}_{kr}, \vec{q}_{ir}) = \sum_{m=1}^{N} q^r_{mk} * q^r_{mi} = 1, i = k;$$

$$(\vec{q}_{kr}, \vec{q}_{ir}) = 0, i \neq k;$$

$$\lambda_{1r} > \lambda_{2r} > \dots \lambda_{Nr} \geq 0;$$

$$A_r = \sum_{\pi =} \lambda_{kr} \vec{q}_{kr} \vec{q}\,'_{kr} = GLG'$$

$$G = \{\vec{q}_{kr}\}, k = 1,.., N$$

$$L = diag\,(\lambda_{1r}, \lambda_{2r}, \dots, \lambda_{Nr}).$$

Besides, the Fourier transforms of the eigenvectors possess the double orthogonormality

$$\int_{\omega \ni V_r} Q_{kr}(\omega) Q_{ir}(\omega) d\omega = 0, \; i \neq k,$$

$$\int_{-\pi}^{\pi} Q_{kr}(\omega) Q_{ir}(\omega) d\omega = 0, \; i \neq k,$$

$$\int_{-\pi}^{\pi} |Q_{kr}(\omega)|^2 \, d\omega / 2\pi = 1, \; k = 1, 2, \dots, N,$$

$$\lambda_{kr} = \int_{\omega \ni V_r} |Q_{kr}(\omega)|^2 \, d\omega / 2\pi \tag{18}$$

Thus, in the discrete case the eigenvalues are equal qualitatively to the energy shares concentrated in the selected frequency domains and corresponding to their eigenvectors, which is important for the signal synthesis.

As a consequence, we obtain the inequality that determines the range of changes of the eigenvalues,

$$0 < \lambda_{kr} \leq 1, \; k = 1, \dots, N.$$

It is easy to obtain the equality from the ratios (17)

$$\text{trec}\, A_r = \sum_{k=1}^{N} \lambda_{kr} = N / R_r \,,$$

where $Rr = \pi / (v_{r+1} - v_r).$

The inequality is performed due to the properties of the subband matrix

$$\det A_r = \prod_{k=1}^{N} \lambda_{kr} \leq 1 / R_r^N$$

from which and from the inequality (18) it follows that some of the eigenvalues will be very small.

The calculations show that the eigenvalues, the indices of which exceed the value

$$J_r = 2[N/2R_r] + 4 = M + 4,$$

are negligible in comparison with the unit (square brackets means the integral part of the value). Therefore, the representation of the subband matrix with a sufficient degree of accuracy can be replaced by the following approximation

$$A_r \approx \sum_{\pi=1}^{J_r} \lambda_{kr} \vec{q}_{kr} \vec{q}_{kr}^{'} = G_{1r} L_{1r} G_{1r}^{'}$$

Therefore, $$P_r \approx P_{Jr} = \sum_{k=1}^{J_r} \lambda_{kr} \alpha_{kr}^2$$

where $$\alpha_{kr} = (\vec{q}_{kr}, \vec{f}).$$

Relative computational error of the energy shares is defined by the equation

$$\delta_r = 1 - P_{Jr} / P_r = \sum_{k=J_r+1}^{N} \lambda_{kr} \alpha_{kr}^2 / \sum_{i=1}^{N} \lambda_{ir} \alpha_{ir}^2 \le N / R \sum_{k=J_r+1}^{N} \lambda_{kr}$$

Optimal filtration

The other example of the more frequently used procedure is the division (filtration) of the signals for additive components that are defined by the use of the frequency representations.

Let $f(t)$, $t \in [0, T]$ be a continuous part of empiric data. We designate its Fourier transform $F_d(\rceil)$. These data shall be divided into additive components

$$\vec{f} = \vec{f}_1 + \vec{f}_2;$$

so that the first of them in its turn possesses the Fourier transform that satisfies the conditions

$$F_{d1}(\omega) = F_d(\omega),\ \omega \in D_k;$$
$$F_{d1}(\omega) = 0,\ \omega \notin D_k;$$
$$\vec{f}_1 = (f_{11}, \dots, f_{N1})';$$
$$D_k = [-\Omega_{k+1}, -\Omega_k] \cup [\Omega_k, \Omega_{k+1}),\ \Omega_0 = 0.$$

It is clear that these conditions express the ideal requirements to the obtained components as a result of the filtration that can be satisfied approximately only.

Therefore, the implementation of the variational principle of error minimization of the approximization to the ideal case is a natural approach

$$S^2(\vec{f}, \vec{f}_1) = \int_{\omega \in V_r} |F_d(\omega) - F_{d1}(\omega)|^2\, d\omega + \int_{\omega \notin V_r} |F_{d1}(\omega)|^2\, d\omega = \min,$$

where $$F_{d1}(\omega) = \sum_{k=1}^{N} f_{k1} \exp(-j(k-1)\omega).$$

$$\vec{f}_1^{'} = A_r \vec{f}. \quad (19)$$

Solution (19) has an important property

$$f_{i1} = \int_{\omega \in V_r} F(\omega) \exp(-j(i-1)\omega) d\omega / 2\pi,\ i = 1, .., N \quad (20)$$

Thus, the components of the vectors obtained as a result of the optimal filtration are determined only by the signal energy in the corresponding frequency domain. This property distinguishes profitably the offered method of filtration from the use of finite impulse response filters [8-10].

2. CONCLUSION

It was recommended to use the characteristic of the energy share in the set frequency domain as a base for the analysis of signals. This allowed to obtain the subband matrices that are adequate mathematical apparatus of the problem solution of signal analysis on the base of the frequency representations.

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