Communications on Stochastic Analysis
Vol. 12, No. 2 (2018) 137-156

Serials Publications
www.serialspublications.com

DIRECTIONAL MALLIAVIN DERIVATIVES: A CHARACTERISATION OF INDEPENDENCE AND A GENERALISED CHAIN RULE

STEFAN KOCH*

Abstract. We define a directional Malliavin derivative connected to a continuous linear operator. We show that this directional Malliavin derivative being zero is equivalent to some measurability or independence condition on the random variable. Using this, we obtain that two random variables, whose classical Malliavin derivatives live in orthogonal subspaces, are independent. We also extend the chain rule to directional Malliavin derivatives and a broader class of functions with weaker regularity assumptions.

1. Introduction

This work is separated into two main parts. The first part covers the definition and study of the directional Malliavin derivative together with a characterisation of independence, and in the second part we extend the chain rule of Malliavin calculus to a directional Malliavin derivative and a broader class of functions. We consider an isonormal Gaussian process $W = \{W(h), h \in H\}$ associated with a separable Hilbert space H and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the σ-algebra $\mathcal{F}$ is generated by W.

First we introduce the notation and state some preliminary results in Section 2 before defining our directional Malliavin derivative in Section 3. Two types of directional Malliavin derivatives are widely used in the literature and both are covered by the definition we use. The first one is given by

$$D^h F = \langle DF, h \rangle_H, \qquad h \in H,$$

where $\langle \cdot, \cdot \rangle_H$ denotes the inner product on H, and which appears, among others, in [8], [3], [6]. Further, letting $W = (W_t)_{t \geq 0}$ be a d-dimensional Brownian motion, $H = L^2([0,T], \mathbb{R}^d)$ and $W(h) = \int_0^T h(t)\, \mathrm{d}W_t$, $h \in H$, we have that $D^{(j)}$, the Malliavin derivative with respect to the j-th Brownian motion, is a directional Malliavin derivative used e.g. in [9]. It is well-known that $DF = 0$ is equivalent to F being almost surely constant. This raises the question whether the directional Malliavin derivative being zero also corresponds to a different property of the random variable F. To give an intuition, we take a look at the result in the context of the example $H = L^2([0,T], \mathbb{R}^d)$, using $d = 2$. It is clear that if F is measurable

Received 2018-7-9; Communicated by S. Tindel.
2010 *Mathematics Subject Classification.* Primary 60H07; Secondary none.
Key words and phrases. Malliavin derivative, independence, chain rule.
* Corresponding author.

with respect to $\sigma^{(1)} = \sigma(W_t^{(1)} : t \in [0,T])$, then $D^{(2)}F = 0$. It turns out that the converse also holds. This is done, in this example, by first proving that $D^{(2)}F = 0$ implies that F is independent of $\sigma^{(2)} = \sigma(W_t^{(2)} : t \in [0,T])$. In a second step we show that independence of $\sigma^{(2)}$ is close enough to measurability with respect to $\sigma^{(1)}$ to allow for the reverse statement. This result can be used to shed some new light on the characterisation of independence of random variables. In [11] the authors have shown that $\langle DF, DG\rangle = 0$ a.s. is not sufficient to ensure independence of $F, G \in \mathbb{D}^{1,2}$ and conjectured that the conditions that imply independence have to be more complicated. We will see that only slightly stricter conditions suffice, namely, if there exists a closed subspace $\mathcal{H}$ of H such that almost surely $DF \in \mathcal{H}$ and $DG \in \mathcal{H}^\perp$, it follows that $F, G \in \mathbb{D}^{1,1}$ are independent. These results are presented in Section 4.

In Section 5 we derive a chain rule for our directional Malliavin derivative that also extends the existing chain rule in standard Malliavin calculus. Letting $p, d \in \mathbb{N}$ and $F = (F^1, \ldots, F^d)$ be a d-dimensional random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ where $F^i \in \mathbb{D}^{1,p},\ i \in \{1, \ldots, d\}$, the chain rule for Malliavin calculus states that for a continuously differentiable Lipschitz function $\varphi : \mathbb{R}^d \to \mathbb{R}$, we have $\varphi(F) \in \mathbb{D}^{1,p}$ and

$$D\varphi(F) = \sum_{i=1}^{d} \partial_i \varphi(F) DF^i. \tag{1.1}$$

Let $L : H \to \mathcal{H}$ be a bounded linear operator. The directional Malliavin derivative D^L, which we will define later on, extends the standard Malliavin derivative in the sense that $D^L F = LDF,\ F \in \mathbb{D}^{1,2}$. We obtain a chain rule for this directional derivative and a less restrictive class of functions stating that, under certain conditions on φ and for $F^i \in \mathbb{D}^{1,p,L},\ i \in \{1, \ldots, d\}$, we have

$$D^L \varphi(F) = \sum_{i \in J} \partial_i \varphi(F) D^L F^i,$$

where

$$J = \{1, \ldots, d\} \setminus \{ i \mid F^i \text{ independent of } \sigma(W(h) : h \in \ker(L)^\perp)\ \}.$$

This helps e.g. to check Malliavin differentiability in the Heston model (see [2]) as the square root is not globally Lipschitz but an admissible function in our theorem. Some more elementary lemmata that we used can be found in the appendix.

2. Preliminaries

Let $W = \{W(h), h \in H\}$ be an isonormal Gaussian process associated with a separable Hilbert space H and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the σ-algebra $\mathcal{F}$ is generated by W. The following definitions and conventions are in line with [8]. Denote by $C_p^\infty(\mathbb{R}^d)$ all functions $f : \mathbb{R}^d \to \mathbb{R}$ that are infinitely often differentiable, and f and all its partial derivatives have polynomial growth. We define $\mathcal{S}$ to be the set of all random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)),$$

where $n \in \mathbb{N}$, $f \in C_p^\infty(\mathbb{R}^n)$ and $h_1, \dots, h_n \in H$. This set is called the set of smooth random variables. Similar we define $\mathcal{S}_b$ to be the set of all smooth random variables such that

$$f \in C_b^\infty(\mathbb{R}^n) := \{g \in C^\infty(\mathbb{R}^n) : g \text{ and all its partial derivatives are bounded}\}.$$

It holds that $\mathcal{S}_b \subseteq \mathcal{S}$ and both are dense in $L^p(\Omega)$. On $\mathcal{S}$ the Malliavin derivative is defined as

$$DF = \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n))h_i$$

and $\mathbb{D}^{1,p}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = \left(\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_H^p]\right)^{\frac{1}{p}}.$$

The same definition can be extended to Hilbert space-valued random variables. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{S}_\mathcal{H}$ a family of $\mathcal{H}$-valued random variables of the form

$$F = \sum_{i=1}^n F_j h_j,$$

where $F_j \in \mathcal{S}$, $h_j \in \mathcal{H}$ for all $j \in \{1, \dots, n\}$. Define $DF = \sum_{j=1}^n DF_j \otimes h_j$. We denote by $\mathbb{D}^{1,p}(\mathcal{H})$ the closure of $\mathcal{S}_\mathcal{H}$ with respect to the norm

$$\|F\|_{1,p,\mathcal{H}} = \left(\mathbb{E}[\|F\|_\mathcal{H}^p] + \mathbb{E}[\|DF\|_{H\otimes\mathcal{H}}^p]\right)^{\frac{1}{p}}.$$

Note that $\mathcal{S}_\mathcal{H}$ is dense in $L^2(\Omega; \mathcal{H})$. This way we can also define higher order Malliavin derivatives D^k and their respective domains $\mathbb{D}^{k,p}$.

Our first auxiliary result is the following small lemma.

Lemma 2.1. *Let $\mathfrak{B} = \{e_j, j \in I\}$ be an orthonormal basis of H, where $I = \{1, \dots, N\}$ or $I = \mathbb{N}$, depending on the dimension of H. Define*

$$\mathfrak{S} := \{F \in \mathcal{S}_b : F = f(W(e_1), \dots, W(e_n)),\, n \in I,\, f \in C_b^\infty(\mathbb{R}^n)\}.$$

Then $\mathfrak{S}$ is dense in $\mathcal{S}_b$ and therefore in $L^p(\Omega)$.

Proof. We prove the result for infinite dimensional H. The proof for finite dimensional H follows trivially.

Let $F = f(W(h_1), \dots, W(h_m)) \in \mathcal{S}_b$, i.e. $f \in C_b^\infty(\mathbb{R}^m)$ and $h_1, \dots, h_m \in H$. We have that

$$h_i := \sum_{j=1}^\infty \underbrace{\langle h_i, e_j\rangle_H}_{=:\lambda_{ij}} e_j.$$

Because of the linearity of W, there exists some $g \in C_b^\infty(\mathbb{R}^n)$ such that

$$F_n := f\left(W\Big(\sum_{j=1}^n \lambda_{1j}e_j\Big), \dots, W\Big(\sum_{j=1}^n \lambda_{mj}e_j\Big)\right) = g(W(e_1), \dots, W(e_n)).$$

So, $F_n \in \mathfrak{S}$ for all $n \in \mathbb{N}$. Since all $W(h)$, $h \in H$ are normally distributed with mean zero and variance $\|h\|_H^2$, there exists a constant $c_p > 0$ such that

$$\left\| W(h_i) - W\Big(\sum_{j=1}^{n} \lambda_{ij} e_j\Big) \right\|_{L^p(\Omega)}^p = \mathbb{E}\left[\Big| W\Big(\sum_{j=n}^{\infty} \lambda_{ij} e_j\Big)\Big|^p \right] \le c_p \Big(\sum_{j=n}^{\infty} \lambda_{ij}^2\Big)^{p/2}.$$

Because the right-hand side converges to zero as $n \to \infty$ and f is Lipschitz continuous, we obtain $F_n \overset{L^p(\Omega)}{\longrightarrow} F$. □

3. Directional Malliavin derivative

In this section we generalise the idea of Malliavin derivatives to the concept of directional Malliavin derivatives in the style of [1].

Let $\mathcal{H}$ be a Hilbert space and $L : H \to \mathcal{H}$ a bounded linear operator. On the set $\mathcal{S}$ of smooth random variables, we define the directional Malliavin derivative D^L as $L \circ D$, i.e.

$$D^L F = \sum_{i=1}^{m} \partial_i f(W(h_1), \dots, W(h_m)) L h_i,$$

where $F = f(W(h_1), \dots, W(h_m))$, $f \in C_p^\infty(\mathbb{R}^d)$, $h_1, \dots, h_m \in H$. This implies that $D^L F = LDF$, $F \in \mathcal{S}$.

Lemma 1.2.1 and 1.2.2 in [8] state the following: Let $F, G \in \mathcal{S}$ and $h \in H$. Then

$$\mathbb{E}[\langle DF, h\rangle_H] = \mathbb{E}[FW(h)] \tag{3.1}$$

and

$$\mathbb{E}[G\langle DF, h\rangle_H] = \mathbb{E}[-F\langle DG, h\rangle_H + FGW(h)].$$

With the help of these result we can prove the corresponding statements for D^L.

Lemma 3.1. *Let $F, G \in \mathcal{S}$ and $h \in \mathcal{H}$. We denote the adjoint of L by $L^\star$. We have*

$$\mathbb{E}[\langle D^L F, h\rangle_{\mathcal{H}}] = \mathbb{E}[FW(L^\star h)] \tag{3.2}$$

and

$$\mathbb{E}[G\langle D^L F, h\rangle_{\mathcal{H}}] = \mathbb{E}[-F\langle D^L G, h\rangle_{\mathcal{H}} + FGW(L^\star h)] \tag{3.3}$$

Proof. Using (3.1) yields

$$\mathbb{E}[\langle D^L F, h\rangle_{\mathcal{H}}] = \mathbb{E}[\langle DF, L^\star h\rangle_H] = \mathbb{E}[FW(L^\star h)].$$

To prove (3.3) first note that by linearity of L we have

$$D^L(FG) = L(D(FG)) = L(FDG + GDF) = FD^L G + GD^L F.$$

Using this result and (3.2) we obtain

$$\mathbb{E}[FGW(L^\star h)] = \mathbb{E}[\langle D^L(FG), h\rangle_{\mathcal{H}}] = \mathbb{E}[\langle FD^L G, h\rangle_{\mathcal{H}} + \langle GD^L F, h\rangle_{\mathcal{H}}].$$

□

Proposition 3.2. *The operator D^L is closable from $L^p(\Omega)$ to $L^p(\Omega;\mathcal{H})$.*

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}$ converging to zero in $L^p(\Omega)$ such that $D^L F_n$ converges to η in $L^p(\Omega;\mathcal{H})$. By equation (3.3) we have for any $h\in\mathcal{H}$ and any

$$G\in\{F\in\mathcal{S}_b : FW(L^\star h) \text{ is bounded}\}=:\beta(h)$$

that

$$\lim_{n\to\infty}\mathbb{E}[\langle D^L F_n,h\rangle_{\mathcal{H}}G]=\lim_{n\to\infty}\mathbb{E}[-F_n\langle D^L G,h\rangle_{\mathcal{H}}+F_nGW(L^\star h)]=0,$$

since $\langle D^L G,h\rangle_{\mathcal{H}}$ and $GW(L^\star h)$ are bounded. It remains to be shown that $\beta(h)$ is dense in $\mathcal{S}_b$ which is itself dense in $L^p(\Omega)$. Then, $\eta=0$ in $L^p(\Omega)$ and the assertion follows. So, let $G\in\mathcal{S}_b$ and set $G_n:=G\exp\left(-\frac{W(L^\star h)^2}{n}\right)$ for $n\in\mathbb{N}$. Then we have that $(G_n)_{n\in\mathbb{N}}$ is a sequence in $\beta(h)$ with $G_n\overset{L^p(\Omega)}{\longrightarrow}G$. □

This proposition allows us to define $\mathbb{D}^{1,p,L}$ as the domain of D^L in $L^p(\Omega)$, i.e. $\mathbb{D}^{1,p,L}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p,L}=\left(\mathbb{E}[|F|^p]+\mathbb{E}[\|D^L F\|_{\mathcal{H}}^p]\right)^{\frac{1}{p}}.$$

For $p=2$, the space $\mathbb{D}^{1,2,L}$ is a Hilbert space with the inner product

$$\langle F,G\rangle_{1,2,L}=\mathbb{E}[FG]+\mathbb{E}[\langle D^L F,D^L G\rangle_{\mathcal{H}}].$$

We remark that a different approach would be to define

$$\widetilde{D}^L:\mathbb{D}^{1,p}\to L^p(\Omega;\mathcal{H});\ F\mapsto L(DF).$$

In fact we have $\mathbb{D}^{1,p}\subseteq\mathbb{D}^{1,p,L}$ and $\widetilde{D}^L F=D^L F$ for $F\in\mathbb{D}^{1,p}$ but in general $\mathbb{D}^{1,p}\neq\mathbb{D}^{1,p,L}$.

Similar to the divergence operator δ in standard Malliavin calculus it is possible to define δ^L as the adjoint of D^L and many properties of δ carry over to δ^L.

The following theorem shows that in some cases, which include the ones usually considered, the directional Malliavin differentiability implies Malliavin differentiability. In some set-ups this might make it easier to check for Malliavin differentiability.

Proposition 3.3. *Let $d\in\mathbb{N}$ and $H_j,\ j\in\{1,\dots,d\}$ orthogonal subspaces of H, such that $H=\bigoplus_{j=1}^d H_j$. We denote by $L_j:H\to H_j$ the projections of H onto H_j. If $F\in\bigcap_{j=1}^d\mathbb{D}^{1,p,L_j}$, then $F\in\mathbb{D}^{1,p}$ and*

$$DF=\sum_{j=1}^d D^{L_j}F.$$

Proof. It is evident that there exists a sequence $(F_n)_{n\in\mathbb{N}}\subseteq\mathcal{S}_b$ such that $F_n\overset{L^p(\Omega)}{\longrightarrow}F$. We have, for some $m=m(n)\in\mathbb{N}$, that

$$D^{L_j}F_n=\sum_{i=1}^m\partial_i f_n(W(h_1),\dots,W(h_m))L_jh_i,$$

where $f_n \in C_b^\infty(\mathbb{R}^m)$. Since $\sum_{j=1}^d L_j$ is the identity on H, it follows that

$$\sum_{j=1}^d D^{L_j} F_n = \sum_{i=1}^m \left[\partial_i f_n(W(h_1), \ldots, W(h_m))\Big(\sum_{j=1}^d L_j\Big) h_i\right] = DF_n.$$

Since the left hand side of the equation converges in $L^p(\Omega; H)$ to $\sum_{j=1}^d D^{L_j} F$ and the operator D is closed, we obtain $F \in \mathbb{D}^{1,p}$ and

$$DF = \sum_{j=1}^d D^{L_j} F.$$

□

The following is a common example of a directional Malliavin derivative. Let $I \subseteq \mathbb{R}$ be an interval. Consider $H = L^2(I; \mathbb{R}^d)$ and the isonormal Gaussian process $W = \{W(h) : h \in H\}$ that is defined by a Wiener integral over a d-dimensional Brownian motion $(B_t)_{t\in[0,T]} = ((B_t^{(1)}, \ldots, B_t^{(d)})^\top)_{t\in[0,T]}$. Putting $\mathcal{H} = L^2(I; \mathbb{R})$ and defining

$$L_j : H \to \mathcal{H};\ L_j h = h_j, \quad \text{where } h = (h_1, \ldots, h_d)^\top \in H$$

for $j \in \{1, \ldots, d\}$, we can understand $D^{L_j} := D^{(j)}$ as the (directional) Malliavin derivative with respect to the j^{th} Brownian motion. If $F \in \mathbb{D}^{1,1}$, then

$$DF = \begin{pmatrix} (DF)_1 \\ \vdots \\ (DF)_d \end{pmatrix} = \begin{pmatrix} D^{(1)}F \\ \vdots \\ D^{(d)}F \end{pmatrix}.$$

4. Characterisation of independence

In this section we will see what can be inferred about $F \in \mathbb{D}^{1,p,L}$ if $D^L F = 0$. This allows us to formulate a condition on the Malliavin derivatives that implies independence of the random variables.

The following lemma follows from Lemma 1.2.4 in [8].

Lemma 4.1. *Let $\sigma^{\ker^\perp}$ denote the σ-algebra generated by $\{W(h) : h \in \ker(L)^\perp\}$. Then we have that $H_L^\perp := \ker(L)^\perp$ with the inner product of H is a Hilbert space and the set*

$$T^\perp = \big\{1, W(h)G - \langle DG, h\rangle_H : G \in \mathcal{S}_b^\perp,\ h \in H_L^\perp\big\},$$

where

$$\mathcal{S}_b^\perp := \{F = f(W(h_1), \ldots, W(h_m)) : f \in C_b^\infty(\mathbb{R}^m),\ h_1, \ldots, h_m \in H_L^\perp\}$$

is a total set in $L^2(\Omega, \sigma^{\ker^\perp}, \mathbb{P})$.

Proposition 4.2. *Let $F \in \mathbb{D}^{1,1,L}$. If F is measurable with respect to the σ-algebra $\sigma^{\ker} := \sigma\big(W(h) : h \in \ker(L)\big)$, then $D^L F = 0$. On the other hand, $D^L F = 0$ implies that F is independent of $\sigma^{\ker^\perp}$. Note that equality is meant in the $L^1(\Omega; \mathcal{H})$ sense.*

Proof. First we assume that F is $\sigma^{\ker}$-measurable. Then, there exists a sequence $(F_n)_{n\in\mathbb{N}}$, where

$$F_n = f(W(h_1),\dots,W(h_m)), \quad f \in C_b^\infty(\mathbb{R}^m),\ h_1,\dots,h_m \in \ker(L)$$

for all $n \in \mathbb{N}$ and $F_n \overset{n\to\infty}{\longrightarrow} F$ in $L^1(\Omega)$. We have $D^L F_n = 0$ for all $n \in \mathbb{N}$ and thus $D^L F = 0$.

Now we suppose that $D^L F = 0$. It holds that $L : H_L^\perp \to \operatorname{im}(L)$ is an isomorphism and consequently so is $L^\star : \operatorname{im}(L) \to H_L^\perp$. Let $G \in \mathcal{S}_b^\perp$ be arbitrary and bounded by $c > 0$ and fix an $h \in H_L^\perp$. There exists a $g \in \operatorname{im}(L) \subseteq \mathcal{H}$ such that $h = L^\star g$ and we have

$$\mathbb{E}[\langle DG, h\rangle_H] = \mathbb{E}[\langle DG, L^\star g\rangle_H] = \mathbb{E}[\langle D^L G, g\rangle_{\mathcal{H}}].$$

Let $(F_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}_b$ such that $F_n \overset{L^1(\Omega)}{\longrightarrow} F$ and $\mathbb{E}[\|D^L F_n\|_{\mathcal{H}}] \to 0$ as $n \to \infty$.

In addition, let $\psi : \mathbb{R} \to \mathbb{R}$ be a bounded, measurable function. As the law of F, denoted by $\mathbb{P}^F$, is a Radon measure on the Borel sets of $\mathbb{R}$, Lusin's Theorem (see e.g. [5], Theorem 7.10) states that ψ can be approximated in $L^2(\mathbb{R}, \mathbb{P}^F)$ by continuous, compactly supported functions. The approximations can be chosen to be uniformly bounded by $\|\psi\|_\infty$. A mollifying argument yields that there exists a sequence $(\psi_N)_{N\in\mathbb{N}} \subseteq C_b^\infty(\mathbb{R})$ such that $\psi_N \to \psi$ in $L^2(\mathbb{R}, \mathbb{P}^F)$, or, in other words,

$$\mathbb{E}[(\psi_N(F) - \psi(F))^2] \overset{N\to\infty}{\longrightarrow} 0.$$

For the moment let $N \in \mathbb{N}$ be fixed. So, we have $\psi_N \in C_b^\infty(\mathbb{R})$ and, for all $n \in \mathbb{N}$, $F_n, G \in \mathcal{S}$, which implies $\psi_N(F_n)G \in \mathcal{S}$. It follows by equation (3.2) that

$$\mathbb{E}[\langle D^L(\psi_N(F_n)G), g\rangle_{\mathcal{H}}] = \mathbb{E}[\psi_N(F_n) W(L^\star g) G]. \tag{4.1}$$

Note that, for $X = x(W(h_1),\dots,W(h_n)), Y = y(W(h_1),\dots,W(h_n)) \in \mathcal{S}$, we have

$$\begin{aligned} D^L(XY) &= \sum_{i=1}^n \Big[x(W(h_1),\dots,W(h_n)) \partial_i y(W(h_1),\dots,W(h_n)) \\ &\qquad + \partial_i x(W(h_1),\dots,W(h_n)) y(W(h_1),\dots,W(h_n)) \Big] Lh_i \\ &= X D^L Y + Y D^L X, \end{aligned} \tag{4.2}$$

and

$$D^L(\psi_N(X)) = \psi_N'(X) \sum_{j=1}^n \partial_j x(W(h_1),\dots,W(h_n)) Lh_j = \psi_N'(X) D^L X. \tag{4.3}$$

Using (4.1)-(4.3) and $h = L^\star g$, we obtain

$$\begin{aligned} &\mathbb{E}[\psi_N(F_n)(W(h)G - \langle DG, h\rangle_H)] \\ &\qquad = \mathbb{E}[\psi_N(F_n) W(L^\star g) G - \langle D^L(\psi_N(F_n)G), g\rangle_{\mathcal{H}}] \\ &\qquad\quad + \mathbb{E}[G \langle D^L \psi_N(F_n), g\rangle_{\mathcal{H}}] \\ &\qquad = \mathbb{E}[G \psi_N'(F_n) \langle D^L F_n, g\rangle_{\mathcal{H}}] \\ &\qquad \le c\gamma_N \mathbb{E}[\|D^L F_n\|] \|g\|_{\mathcal{H}} \overset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

where $\gamma_N = \sup_{x\in\mathbb{R}} |\psi'_N(x)|$ is the Lipschitz constant of ψ_N. In particular, using dominated convergence and the continuity of ψ_N, we obtain $\mathbb{E}[\psi_N(F)(W(h)G - \langle DG, h\rangle_H)] = 0$ for all $N \in \mathbb{N}$, and thus

$$\mathbb{E}[\psi(F)(W(h)G - \langle DG, h\rangle_H)] = 0. \tag{4.4}$$

Let X be a bounded $\sigma^{\ker^\perp}$-measurable random variable. Then $X \in L^2(\Omega)$ and by Lemma 4.1 there exist $Y_i \in T^\perp$ and $a_i \in \mathbb{R}$, $i \in \mathbb{N}$ such that

$$X_n := \sum_{i=1}^{n} a_i Y_i \overset{L^2(\Omega)}{\longrightarrow} X, \text{ as } n \to \infty.$$

The linear functional $\phi : L^2(\Omega) \to \mathbb{R}$, $X \mapsto \mathbb{E}[\psi(F)(X - \mathbb{E}X)]$ is continuous and, by (4.4), we have $\mathbb{E}[\psi(F)(X_n - \mathbb{E}X_n)] = 0$ for all $n \in \mathbb{N}$. Thus, $\mathbb{E}[\psi(F)X] = \mathbb{E}[\psi(F)]\mathbb{E}[X]$. The choices of the bounded, measurable function ψ and the bounded $\sigma^{\ker^\perp}$-measurable random variable X were arbitrary. Consequently, F is independent of $\sigma^{\ker^\perp}$. □

The following proposition provides a useful characterisation of independence of random variables. This result, being of rather basic nature, was surely shown before but unfortunately we were unable to find it or references to it in the literature.

Proposition 4.3. *Let $(\Omega, \mathfrak{A}, \mathcal{P})$ be a probability space and $\mathfrak{A} = \sigma(\sigma_1 \cup \sigma_2)$, where σ_1, σ_2 are two independent σ-algebras. A random variable $X \in L^1(\Omega, \mathfrak{A}, \mathcal{P})$ is independent of σ_2 if and only if there exists a σ_1-measurable random variable $\widetilde{X} \in L^1(\Omega, \sigma_1, \mathcal{P}) \subseteq L^1(\Omega, \mathfrak{A}, \mathcal{P})$ such that $X = \widetilde{X}$ almost surely.*

Proof. First, let $\widetilde{X} \in L^1(\Omega, \mathfrak{A}, \mathbb{P})$ be a σ_1-measurable random variable and $X = \widetilde{X}$ almost surely. For any bounded σ_2-measurable random variable G and any bounded measurable function $h : \mathbb{R} \to \mathbb{R}$ we have

$$\mathbb{E}[h(X)G] = \mathbb{E}[h(\widetilde{X})G] = \mathbb{E}[h(\widetilde{X})\mathbb{E}[G|\sigma_1]] = \mathbb{E}[h(\widetilde{X})]\mathbb{E}[G] = \mathbb{E}[h(X)]\mathbb{E}[G].$$

This implies that X is independent of σ_2.

It remains to show the reverse implication. Assume X is independent of σ_2 and define $\widetilde{X} := \mathbb{E}[X|\sigma_1]$. The properties of the conditional expectation give us $\widetilde{X} \in L^1(\Omega, \mathfrak{A}, \mathcal{P})$ and $\widetilde{X}$ is σ_1-measurable. We have that $\Pi := \{A \cap B : A \in \sigma_1, B \in \sigma_2\}$ is a π-system with $\sigma(\Pi) = \mathfrak{A}$. To see this, we note that any $A \in \sigma_1$ or $B \in \sigma_2$ is clearly also an element of Π and therefore $\sigma_1 \cup \sigma_2 \subseteq \Pi$, which implies $\mathfrak{A} = \sigma(\sigma_1 \cup \sigma_2) \subseteq \sigma(\Pi)$. As finite intersection of elements in $\mathfrak{A}$ are also in $\mathfrak{A}$, we have $\Pi \subseteq \mathfrak{A}$, which implies $\sigma(\Pi) \subseteq \mathfrak{A}$. We put $C := A \cap B \in \Pi$, where $A \in \sigma_1$ and $B \in \sigma_2$. Because X and $\widetilde{X}$ are both independent of σ_2, we obtain

$$\mathbb{E}[\mathbf{1}_C(X - \widetilde{X})] = \mathbb{E}[\mathbf{1}_A\mathbf{1}_B(X - \widetilde{X})] = \mathbb{E}[\mathbf{1}_B]\mathbb{E}\big[\mathbf{1}_A\big(X - \mathbb{E}[X|\sigma_1]\big)\big] = 0$$

because $\mathbb{E}[\mathbf{1}_A(X - \mathbb{E}[X|\sigma_1])] = 0$ by the definition of conditional expectation. Applying Lemma A.2 yields $X = \mathbb{E}[X|\sigma_1] = \widetilde{X}$ almost surely. □

Proposition 4.3 allows us to reformulate and improve Proposition 4.2 into Theorem 4.4 below. The Theorems 4.4 and 4.5 constitute one of the main results of this paper.

Theorem 4.4. *Let $F \in \mathbb{D}^{1,1,L}$. The following statements are equivalent.*

(1) $D^L F = 0$ *in* $L^1(\Omega; \mathcal{H})$.
(2) F *is independent of* $\sigma^{\ker^\perp}$.
(3) *There exists a random variable* $G \in L^1(\Omega)$ *such that* $F = G$ *a.s. and* G *is* $\sigma^{\ker}$*-measurable.*

Proof.

(1) $\Rightarrow$ (2)**:** Let $D^L F = 0$. By Proposition 4.2 we have that F is independent of $\sigma^{\ker^\perp}$.
(2) $\Rightarrow$ (3)**:** Let F be independent of $\sigma^{\ker^\perp}$. It follows from Proposition 4.3 that there exists a $\sigma^{\ker}$-measurable random variable G such that $F = G$ almost surely.
(3) $\Rightarrow$ (1)**:** Let $X, Y \in L^1(\Omega)$ with $X = Y$ *a.s.*, then $X = Y$ in $L^1(\Omega)$. By the definition of the operator D^L we have $X \in \mathbb{D}^{1,1,L}$ if and only if $Y \in \mathbb{D}^{1,1,L}$ and, if in addition $X \in \mathbb{D}^{1,1,L}$, then $D^L X = D^L Y$ in $L^1(\Omega; \mathcal{H})$. Therefore, $G \in \mathbb{D}^{1,1,L}$ and $D^L F = D^L G = 0$ by Proposition 4.2.

□

From this theorem we can derive a condition on the standard Malliavin derivatives of two random variables that implies independence of said random variables.

Theorem 4.5. *Let $F, G \in \mathbb{D}^{1,1}$. If there exists a closed subspace $\mathcal{H}$ of H such that*

$$DF \in \mathcal{H} \text{ a.s.} \qquad \text{and} \qquad DG \in \mathcal{H}^\perp \text{ a.s.},$$

then F and G are independent.

Proof. Let L be the projection of H onto $\mathcal{H}$. Then $D^L G = 0$. Theorem 4.4 yields that G is independent of $\sigma^{\ker^\perp}$ and there exits a random variable $\widetilde{G} \in L^1(\Omega)$ such that $\widetilde{G} = G$ *a.s.* and $\widetilde{G}$ is $\sigma^{\ker}$-measurable. In the same way we obtain F is independent of $\sigma^{\ker}$ and it follows that F and G are independent. □

Using a result in [11], the reverse implication follows quickly in the case of $H = L^2([0, T])$ and under slightly stricter conditions.

Corollary 4.6. *Let $H = L^2([0, T])$ and $W(h) = \int_0^T h(t)\, \mathrm{d}W_t$. Suppose $F, G \in \mathbb{D}^{1,2}$. Then the following are equivalent:*

(1) *There exists a closed subspace $\mathcal{H}$ of H such that*

$$DF \in \mathcal{H} \text{ a.s.} \qquad \text{and} \qquad DG \in \mathcal{H}^\perp \text{ a.s.}$$

(2) *The random variables F and G are independent.*

Proof. Theorem 4.5 proves (1) $\Rightarrow$ (2). Now let F, G be independent. The random variables can be expanded into a series of multiple stochastic Wiener integrals

$$F = \sum_{n=0}^{\infty} I_n(f_n), \qquad G = \sum_{n=0}^{\infty} I_n(g_n),$$

where $f_n, g_n \in L^2([0,T]^p)$ are symmetric functions. For $n \in \mathbb{N}_0$, denote by J_n the projection onto the n-th Wiener chaos. For $n, m \in \mathbb{N}_0$, we have

$$\begin{aligned}\mathbb{P}(J_nF \in A, J_mG \in B) &= \mathbb{P}(F \in J_n^{-1}(A), G \in J_m^{-1}(B)) \\ &= \mathbb{P}(F \in J_n^{-1}(A))\mathbb{P}(G \in J_m^{-1}(B)) \\ &= \mathbb{P}(J_nF \in A)\mathbb{P}(J_mG \in B)\end{aligned}$$

for all $A, B \in \mathcal{B}(\mathbb{R})$. Thus, $J_nF = I_n(f_n)$ and $J_mG = I_m(g_m)$ are independent for all $n, m \in \mathbb{N}$. Define

$$\mathcal{H} := \left\{\varphi \in L^2([0,T]) \;\middle|\; \forall m \in \mathbb{N} : \Big\| \int_0^T g_m(t,\cdot)\varphi(t)\,\mathrm{d}t \Big\|_{L^2([0,T]^{m-1})} = 0\right\},$$

which is a closed subspace of H.

In what follows let $\cdot$ and $\bullet$ be placeholders for different variables. In iterated integrals we always integrate over the variables represented by $\cdot$ and never over those represented by $\bullet$. The justification of the stochastic Fubini results used in this proof is given in Lemma A.3.

Let $m \in \mathbb{N}$ and $\varphi \in \mathcal{H}$. Applying stochastic Fubini, we have almost surely

$$\begin{aligned}\langle DI_m(g_m), \varphi\rangle_{L^2([0,T])} &= m \int_0^T I_{m-1}(g_m(t,\cdot))\varphi(t)\,\mathrm{d}t \\ &= mI_{m-1}\Big(\int_0^T g_m(t,\cdot)\varphi(t)\,\mathrm{d}t\Big) = 0,\end{aligned}$$

and it follows

$$DI_m(g_m) = mI_{m-1}(g_m(t,\cdot)) \in \mathcal{H}^\perp \ a.s.$$

for all $m \in \mathbb{N}$. Theorem 6 in [11] yields

$$\|f_n \otimes_1 g_m\|_{L^2([0,T]^{m+n-2})} = 0$$

for any choice of $n, m \in \mathbb{N}$, where

$$f_n \otimes_1 g_m = \int_0^T f_n(t,\cdot)g_m(t,\bullet)\,\mathrm{d}t$$

Again applying stochastic Fubini, we obtain for any $n, m \in \mathbb{N}$ that

$$\begin{aligned}\int_0^T D_tI_n(f_n)g_m(t,\bullet)\,\mathrm{d}t &= n\int_0^T I_{n-1}(f_n(t,\cdot))g_m(t,\bullet)\,\mathrm{d}t \\ &= nI_{n-1}\Big(\int_0^T f_n(t,\cdot)g_m(t,\bullet)\,\mathrm{d}t\Big) = 0 \ a.s,\end{aligned}$$

where the last zero denotes the zero function in $L^2([0,T]^{m-1})$. Thus,

$$DI_n(f_n) = nI_{n-1}(f_n(t,\cdot)) \in \mathcal{H} \ a.s.$$

for all $n \in \mathbb{N}$. Since $\mathcal{H}$ and $\mathcal{H}^\perp$ are closed subspaces it follows that

$$t \mapsto D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(t,\cdot)) \in \mathcal{H} \ a.s.,$$
$$t \mapsto D_t G = \sum_{m=1}^{\infty} m I_{m-1}(g_m(t,\cdot)) \in \mathcal{H}^\perp \ a.s.$$

□

It might be conjectured that the statement above holds for general $F, G \in \mathbb{D}^{1,1}$. The following example shows that, for $F, G \in \mathbb{D}^{1,2}$, the condition $\langle DF, DG\rangle = 0$ a.s. is not sufficient to imply independence of F and G.

Example 4.7. Let $W(h) = \int_0^1 h(t)\,\mathrm{d}W_t,\ h \in H = L^2([0,1],\mathbb{R})$ and $W = (W_t)_{t\geq 0}$ a standard Brownian motion. Put

$$F = \mathbf{1}_{[0,\infty)}(W_1)W_1$$
$$G = -\mathbf{1}_{(-\infty,0]}(W_1)W_1.$$

Then F, G are not independent as

$$\mathbb{E}[FG] = \mathbb{E}|W_1| = \sqrt{\frac{2}{\pi}} \neq \frac{1}{2\pi} = \mathbb{E}[F]\mathbb{E}[G].$$

But, using the generalised chain rule, which is proven in the next section, we obtain

$$D_t F = \mathbf{1}_{(0,\infty)}(W_1)\mathbf{1}_{[0,1]}(t)$$
$$D_t G = \mathbf{1}_{(-\infty,0)}(W_1)\mathbf{1}_{[0,1]}(t),$$

and therefore $\langle DF, DG\rangle = 0$.

5. Chain rule in Malliavin calculus

In this section let $p, d \in \mathbb{N}$, $F = (F^1,\dots,F^d)$ be a d-dimensional random variable on $(\Omega,\mathcal{F},\mathbb{P})$, and let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$. We want to quickly restate the standard chain rule in Malliavin calculus that can, e.g., be found in [8], Proposition 1.2.3.

Proposition 5.1. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable function with bounded derivative and $F^i \in \mathbb{D}^{1,p}$, $i \in \{1,\dots,d\}$, then $\varphi(F) \in \mathbb{D}^{1,p}$ and* (1.1) *holds, i.e.*

$$D\varphi(F) = \sum_{i=1}^{d} \partial_i \varphi(F) DF^i.$$

Our aim is to transfer this result to the directional Malliavin derivative and find a larger class of function such that (1.1) still holds.

Let $f : \mathbb{R}^d \to \mathbb{R}$, $I \subseteq \{1,\dots,d\}$ and $e_i \in \mathbb{R}^d$ the vector that has a one in the i-th position and zeros otherwise. We make the following definitions

(1) We say that f is Lipschitz continuous in direction I if there exists a constant $\gamma > 0$ such that for all $x \in \mathbb{R}^d$ and $h \in \mathbb{R}$ we have

$$|f(x + he_i) - f(x)| \leq \gamma\|h\|, \qquad i \in I.$$

(2) We say that f is locally Lipschitz in direction I if for every $x \in \mathbb{R}^d$ there exist positive constants $\varepsilon(x)$ and $\gamma(x)$ such that for all $\|h\| \le \varepsilon(x)$ we have

$$|f(x+he_i) - f(x)| \le \gamma(x)\|h\|, \qquad i \in I.$$

(3) For $p \in \mathbb{N}$, we say $f \in C_I^p(\mathbb{R}^d)$ if, for all $k \le p$ and $i_1, \dots, i_k \in I$, we have that the partial derivative $\partial_{i_1,\dots i_k} f$ exists and is continuous on $\mathbb{R}^d$. Further, define

$$C_I^\infty(\mathbb{R}^d) = \bigcap_{p \in \mathbb{N}} C_I^p(\mathbb{R}^d).$$

Let $\alpha \in C^\infty(\mathbb{R}^d)$ be a nonnegative function with support on the unit ball and $\int_{\mathbb{R}^d} \alpha(x)\,\mathrm{d}x = 1$. Then, for $n \in \mathbb{N}$, we define

$$\alpha_n : \mathbb{R}^d \to \mathbb{R},\, x \mapsto n^d \alpha(nx).$$

This so-called mollifier function will be needed in the proofs that follow. To simplify notation for the rest of Section 5, we make the following definition. If $g : \mathbb{R}^d \to \mathbb{R}$ is not partially differentiable at $x \in \mathbb{R}^d$ in the i-th component, we set $\partial_i g(x) := 0$.

The proof of the following lemma can be found in the Appendix.

Lemma 5.2. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function and set $f_n = f * \alpha_n,\ n \in \mathbb{N}$ with α_n as defined above. The following properties hold:*

(1) *For all $n \in \mathbb{N}$ we have $\displaystyle\int_{\mathbb{R}^d} \|x\|\alpha_n(x)\,\mathrm{d}x \le \frac{1}{n}$.*
(2) *Let f be continuous at $x_0 \in \mathbb{R}^d$. Then $f_n(x_0) \to f(x_0)$ for $n \to \infty$.*
(3) *Let f be continuous on $\mathbb{R}^d$. Then $f_n \in C^\infty(\mathbb{R}^d)$.*
(4) *In addition to the continuity assumption in (3), let f be Lipschitz continuous in direction $I \subseteq \{1, \dots, d\}$ with Lipschitz constant γ. Then, $\|\partial_i f_n\|_\infty \le \gamma$ for all $i \in I$. Moreover, for higher partial derivatives of f_n we have that for every $k \in \mathbb{N}$ there exists $c_k > 0$ such that*

$$\sup_{x \in \mathbb{R}^d} \left|\partial_{i_1,\dots i_k} f_n(x)\right| \le c_k$$

for all $i_1, \dots, i_k \in I$.
(5) *Assume that f is locally Lipschitz continuous in direction I. Then*

$$\partial_i(f * \alpha_n) = \partial_i f * \alpha_n$$

almost everywhere for all $i \in I$.

As the following assumption will be needed in all the chain rule results that follow, we state it here once and only refer to it henceforth.

Assumption 5.3. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ and $F = (F^1, \dots, F^d)$ be a d-dimensional random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with $F^i \in \mathbb{D}^{1,p,L},\ i \in \{1, \dots, d\}$, and

$$J := \{1, \dots, d\} \setminus \{i \mid F^i \text{ independent of } \sigma^{\ker^\perp}\},$$

where $\sigma^{\ker^\perp} = \sigma(W(h), h \in \ker(L)^\perp)$ is the same as in Lemma 4.1 above.

Note that it follows from Assumption 5.3 and Theorem 4.4 that $DF^i = 0$ for all $i \notin J$. We now have the necessary notation to extend Proposition 5.1 to the directional derivative. The result is generalised step-by-step by making the conditions on φ less restrictive, e.g. while the first proposition assumes φ to be bounded, the final result (Theorem 5.7) does not require boundedness.

Proposition 5.4. *Under Assumption 5.3, let φ be bounded, continuous and $\varphi \in C^1_J(\mathbb{R}^d)$ with bounded partial derivatives $\partial_i \varphi,\ i \in J$. Then $\varphi(F) \in \mathbb{D}^{1,p,L}$ and*

$$D^L \varphi(F) = \sum_{i\in J} \partial_i \varphi(F) D^L F^i. \tag{5.1}$$

Proof. Because $F^i \in \mathbb{D}^{1,p,L}$, there exists a sequence $(F_k)_{k\in\mathbb{N}} = ((F_k^1, \ldots, F_k^d)^\top)_{k\in\mathbb{N}}$ with $(F_k^i)_{k\in\mathbb{N}} \subseteq \mathcal{S}_b,\ i \in \{1, \ldots, d\}$ and F_k converging component-wise in $\mathbb{D}^{1,p,L}$ to F. We can write

$$F_k = f_k(W(h_1), \ldots, W(h_m)) = \begin{pmatrix} f_k^1(W(h_1), \ldots, W(h_m)) \\ \vdots \\ f_k^d(W(h_1), \ldots, W(h_m)) \end{pmatrix} = \begin{pmatrix} F_k^1 \\ \vdots \\ F_k^d \end{pmatrix},$$

where $h_1, \ldots, h_m \in H$ and $f_k = (f_k^1, \ldots, f_k^d)^\top \in C_p^\infty(\mathbb{R}^m)$. We define $\varphi_n := \varphi * \alpha_n$, where α_n is the mollifier function from above. We have $\varphi_n \circ f_k \in C_p^\infty(\mathbb{R}^m)$ and obtain by definition that

$$\begin{aligned} D^L \varphi_n(F_k) &= \sum_{j=1}^m \partial_j (\varphi_n \circ f_k)(W(h_1), \ldots, W(h_m)) L h_j \\ &= \sum_{i=1}^d \sum_{j=1}^m \partial_i \varphi_n \big(f_k(W(h_1), \ldots, W(h_m))\big) \partial_j f_k^i (W(h_1), \ldots, W(h_m)) L h_j \\ &= \sum_{i=1}^d \partial_i \varphi_n(F_k) D^L F_k^i. \end{aligned}$$

By Theorem 4.4 the sequence $(F_k)_{k\in\mathbb{N}}$ can be chosen such that $F_k^i \subseteq \mathcal{S}_b^\perp,\ i \notin J,\ k \in \mathbb{N}$, where $\mathcal{S}_b^\perp$ is defined in Lemma 4.1. This yields $D^L F_k^i = 0,\ i \notin J,\ k \in \mathbb{N}$ and thus

$$D^L \varphi_n(F_k) = \sum_{i\in J} \partial_i \varphi_n(F_k) D^L F_k^i.$$

Since $F_k^i \overset{L^p(\Omega)}{\longrightarrow} F^i$ as $k \to \infty$, there exists a subsequence $(F_{k_l})_{l\in\mathbb{N}}$ such that this subsequence converges almost surely to F. We choose such a subsequence as our initial sequence $(F_k)_{k\in\mathbb{N}}$, i.e. we can assume w.l.o.g. that $F_k \overset{k\to\infty}{\longrightarrow} F$ almost surely. It remains to show that

$$\lim_{n\to\infty} \lim_{k\to\infty} \|\varphi_n(F_k) - \varphi(F)\|_{1,p,L} = 0.$$

So, the limits in this proof, if not state otherwise, are obtained by first letting $k \to \infty$ and then $n \to \infty$. Using the triangle inequality we obtain

$$\|\varphi_n(F_k) - \varphi(F)\|_{L^p(\Omega)} \le \|\varphi_n(F_k) - \varphi_n(F)\|_{L^p(\Omega)} + \|\varphi_n(F) - \varphi(F)\|_{L^p(\Omega)}.$$

Because φ_n is continuous and bounded by $\|\varphi\|_\infty$, we have that $|\varphi_n(F_k) - \varphi_n(F)|$ converges almost surely to zero as $k \to \infty$ and applying dominated convergence yields that the first summand converges to zero. By Lemma 5.2(2), we have that $\varphi_n(F)$ converges pointwise to $\varphi(F)$ as $n \to \infty$. Using again dominated convergence, we see that the second summand converges to zero. Moreover, for $i \in J$, the triangle inequality yields

$$\begin{aligned}\|\partial_i\varphi_n(F_k)D^LF_k^i - \partial_i\varphi(F)D^LF^i\|_{L^p(\Omega;\mathcal{H})} &\leq \|\partial_i\varphi_n(F_k)(D^LF_k^i - D^LF^i)\|_{L^p(\Omega;\mathcal{H})} \\ &\quad + \|(\partial_i\varphi_n(F_k) - \partial_i\varphi_n(F))D^LF^i\|_{L^p(\Omega;\mathcal{H})} \\ &\quad + \|(\partial_i\varphi_n(F) - \partial_i\varphi(F))D^LF^i\|_{L^p(\Omega;\mathcal{H})}.\end{aligned}$$

Note that $|\partial_i\varphi_n|$ and $|\partial_i\varphi|$ are bounded by some constant C. So the first summand is bounded by $C\|D^LF_k^i - D^LF^i\|_{L^p(\Omega;\mathcal{H})}$, which converges to zero as $k \to \infty$. The absolute value of the term inside the last norm is bounded by $2C|D^LF^i| \in L^p(\Omega;\mathcal{H})$ and by Lemma 5.2(2) $\partial_i\varphi_n(F(\omega)) \overset{n\to\infty}{\longrightarrow} \partial_i\varphi(F(\omega))$ for all $\omega \in \Omega$. So the third summand converges to zero as $n \to \infty$ by the dominated convergence theorem. The absolute value of the term inside the norm of the second summand is also bounded by $2C|D^LF^i| \in L^p(\Omega;\mathcal{H})$ and since $F_k \to F$ *a.s.*, we have by the continuous mapping theorem and dominated convergence that the second summand converges to zero as $k \to \infty$. Thus, we have shown that $\lim_{n\to\infty}\lim_{k\to\infty}\|\varphi_n(F_k) - \varphi(F)\|_{1,p,L} = 0$ and the proof is complete. □

Lemma 5.5. *Under Assumption 5.3, let φ be Lipschitz continuous in direction J with Lipschitz constant γ. Further, suppose that there exists a set $N \in \mathcal{B}(\mathbb{R}^d)$ with $\mathbb{P}(F \in N) = 0$ such that φ is bounded, continuous, and continuously differentiable in direction J on $\mathbb{R}^d \setminus N$. Then, $\varphi(F) \in \mathbb{D}^{1,p,L}$ and (5.1) holds.*

Proof. We set $\varphi_n := \varphi * \alpha_n$. By property (2) in Lemma 5.2 we have $\varphi_n(F) \to \varphi(F)$ *a.s.* and it follows by dominated convergence that $\varphi_n(F) \overset{L^p(\Omega)}{\longrightarrow} \varphi(F)$. By property (4) of Lemma 5.2 we have that φ_n is differentiable on $\mathbb{R}^d \setminus N$ and its first order partial derivatives are bounded by γ. Now let $\omega \in \Omega_0 := \{\omega \in \Omega : F(\omega) \notin N\}$ be fixed and $i \in J$. Property (5) in Lemma 5.2 implies $\partial_i\varphi_n(F(\omega)) = (\partial_i\varphi * \alpha_n)(F(\omega))$. Since $\partial_i\varphi$ is continuous at $F(\omega)$, property (2) in Lemma 5.2 yields $\partial_i\varphi_n(F(\omega)) \overset{n\to\infty}{\longrightarrow} \partial_i\varphi(F(\omega))$. Thus, we have $\partial_i\varphi_n(F)D^LF^i \to \partial_i\varphi(F)D^LF^i$ almost surely. Because $|\partial_i\varphi_n(F)| \leq \gamma$ and $D^LF^i \in L^p(\Omega,\mathcal{H})$, the dominated convergence theorem yields $\partial_i\varphi_n(F)D^LF^i \overset{L^p(\Omega;\mathcal{H})}{\longrightarrow} \partial_i\varphi(F)D^LF^i$. □

Corollary 5.6. *Under Assumption 5.3, let $B \in \mathcal{B}(\mathbb{R}^d)$ with $\mathbb{P}(F \in B) = 1$. We assume that on B the function φ is bounded and continuous as well as continuously partially differentiable in direction J. Further, suppose $\varphi_{|_B}$ is Lipschitz in direction J. Then $\varphi(F) \in \mathbb{D}^{1,p,L}$ and relation (5.1) holds.*

Proof. By Kirszbraun's Theorem, see e.g. Theorem 2.10.43 in [4], there exists an extension $\widetilde{\varphi}$ of $\varphi_{|_B}$ on $\mathbb{R}^d$ such that $\widetilde{\varphi}$ is globally Lipschitz continuous in direction J with the same Lipschitz constant as $\varphi_{|_B}$. Since $\Omega \setminus B$ is a $\mathbb{P}^F$-null set Proposition 5.5 yields that (5.1) holds for $\widetilde{\varphi}$. The result now follows from the fact that $\varphi(F) = \widetilde{\varphi}(F)$ in $L^p(\Omega)$. □

Theorem 5.7. *Under Assumption 5.3, let φ be locally Lipschitz in direction J on a closed set $B \in \mathcal{B}(\mathbb{R}^d)$, where $\mathbb{P}(F \in B) = 1$. Further, suppose that φ is continuous as well as continuously differentiable in direction J on $B \backslash N$, where $\mathbb{P}(F \in N) = 0$. In addition, we assume $\varphi(F) \in L^p(\Omega)$ and $\partial_i \varphi(F) D^L F^i \in L^p(\Omega; \mathcal{H})$ for all $i \in J$. Then the chain rule (5.1) holds.*

Proof. The proof is divided into two steps. We first suppose that φ is also bounded and show that (5.1) holds and then extend this result to the more general setting stated in the theorem.
Step 1: So, let φ be bounded and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ such that $\mathbb{P}(F^i \neq a_n, \forall i \in \{1,\dots,d\}) = 1$ for all $n \in \mathbb{N}$ and $a_n \to \infty$ as $n \to \infty$. Set $\varphi_n(x) = \varphi(-a_n \vee x \wedge a_n)$, where the minimum and maximum are understood component-wise, i.e.

$$-a_n \vee x \wedge a_n := \begin{pmatrix} h_n(x_1) \\ \vdots \\ h_n(x_d) \end{pmatrix}, \qquad h_n : \mathbb{R} \to \mathbb{R};\ y \mapsto \begin{cases} -a_n, & y < -a_n \\ y, & -a_n \leq y \leq a_n \\ a_n, & y > a_n \end{cases}.$$

Define $A_n := \{y \in B \setminus N \,|\, \forall i \in \{1,\dots,d\} : y_i \neq a_n\}$. We have $\mathbb{P}(F \in A_n) = 1$, φ_n is, on A_n, continuous and continuously differentiable in direction J, and $\varphi_n|_{A_n}$ is globally Lipschitz in direction J. Thus, $\varphi_n(F) \in \mathbb{D}^{1,p,L}$ and (5.1) holds for all φ_n by Corollary 5.6. We have $\varphi_n \to \varphi$ pointwise and $\|\varphi_n\|_\infty \leq \|\varphi\|_\infty$. Therefore, by dominated convergence, $\varphi_n(F) \overset{L^p(\Omega)}{\longrightarrow} \varphi(F)$. Moreover, we have $|\partial_i \varphi_n(x)| \leq |\partial_i \varphi(x)|$, $x \in \mathbb{R}^d$ and $|D^L F^i| \in L^p(\Omega, \mathcal{H})$ for $i \in J$, and it follows $\partial_i \varphi_n(F) D^L F^i \overset{L^p(\Omega,\mathcal{H})}{\longrightarrow} \partial_i \varphi(F) D^L F^i$ for all $i \in J$.
Step 2: We now drop the assumption of φ being bounded and let $(b_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ such that $\mathbb{P}(|\varphi(F)| = b_n) = 0$ for all $n \in \mathbb{N}$ and $b_n \to \infty$ as $n \to \infty$. With a similar notation to above we set $\varphi_n(x) := -b_n \vee \varphi(x) \wedge b_n$. It follows that φ_n is bounded, locally Lipschitz in direction J on B, and partially continuously differentiable in direction J for all $x \in B \backslash \big(N \cup \{x : |\varphi(x)| = b_n\}\big)$. By *step 1*, the chain rule holds for all φ_n. Using the dominated convergence theorem we obtain $\varphi_n(F) \overset{L^p(\Omega)}{\longrightarrow} \varphi(F)$ and $\partial_i \varphi_n(F) D^L F^i \overset{L^p(\Omega;\mathcal{H})}{\longrightarrow} \partial_i \varphi(F) D^L F^i$. □

Note that choosing L as the identity operator, Theorem 5.7 also gives a more general chain rule result for the standard Malliavin derivative.

In the context of an absolute continuous random variable F on $\mathbb{R}$, the function φ, in general, cannot be discontinuous for a chain rule to hold. Consider, e.g., $\varphi : \mathbb{R} \to \mathbb{R},\ x \mapsto \mathbf{1}_{(-\infty,0]}(x)$ and $F = W_1 = W(\mathbf{1}_{[0,1]})$ in the setup of Example 4.7. As for $A \in \mathcal{F}$, $\mathbf{1}_A$ is Malliavin differentiable if and only if $\mathbb{P}(A) \in \{0,1\}$ (cf. e.g. Proposition 1.2.6 in [8]), we have that $\varphi(F) = \mathbf{1}_{(-\infty,0]}(W_1) = \mathbf{1}_{\{W_1 \leq 0\}}$ is not Malliavin differentiable.

A. Appendix

In this appendix we denote by $B_r(z)$ the ball around $z \in \mathbb{R}^d$ with radius $r > 0$ and by $\overline{B_r(z)}$ its closure, i.e.

$$B_r(z) := \{y \in \mathbb{R}^d : \|z - y\| < r\},$$
$$\overline{B_r(z)} := \{y \in \mathbb{R}^d : \|z - y\| \leq r\}.$$

The proof of the next lemma can be found in standard text books on analysis. Nevertheless its proof is given for completeness.

Lemma A.1. *Let $\beta \in C_0^\infty(\mathbb{R}^d)$, i.e. an infinitely differentiable and compactly supported function, and $f : \mathbb{R}^d \to \mathbb{R}$ be continuous. Then $f * \beta$ is continuous.*

Proof. Fix $q \in \mathbb{R}$ such that $\operatorname{supp} \beta \subseteq B_q(0)$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ converging to $x \in \mathbb{R}^d$. W.l.o.g. $\|x - x_n\| \leq q$. For $y \in \mathbb{R} \setminus \overline{B_{2q}(x)}$ we have

$$\|y - x_n\| = \|y - x - x_n + x\| \geq \big| \|y - x\| - \|x_n - x\| \big| \geq q.$$

Thus, $\beta(x_n - y) = 0$ for $y \notin \overline{B_{2q}(x)}$ and

$$f(y)\beta(x_n - y) \leq \|\beta\|_\infty f(y) \mathbf{1}_{\overline{B_{2q}(x)}}(y),$$

where the right hand side is integrable. By dominated convergence, we have

$$\begin{aligned}\lim_{n\to\infty} (f * \beta)(x_n) &= \lim_{n\to\infty} \int_{\mathbb{R}^d} f(y)\beta(x_n - y)\,\mathrm{d}y = \int_{\mathbb{R}^d} f(y) \lim_{n\to\infty} \beta(x_n - y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(y)\beta(x - y)\,\mathrm{d}y = (f * \beta)(x).\end{aligned}$$

□

Let $\alpha \in C^\infty(\mathbb{R}^d)$ be a nonnegative function with support on the unit ball and $\int_{\mathbb{R}^d} \alpha(x)\,\mathrm{d}x = 1$. Then, for $n \in \mathbb{N}$, we define

$$\alpha_n : \mathbb{R}^d \to \mathbb{R},\ x \mapsto n^d \alpha(nx).$$

The following lemma was given in the text.

Lemma 5.2. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function and set $f_n = f * \alpha_n,\ n \in \mathbb{N}$ with α_n as defined above. The following properties hold:

1. For all $n \in \mathbb{N}$ we have $\displaystyle\int_{\mathbb{R}^d} \|x\|\alpha_n(x)\,\mathrm{d}x \leq \frac{1}{n}$.
2. Let f be continuous at $x_0 \in \mathbb{R}^d$. Then $f_n(x_0) \to f(x_0)$ for $n \to \infty$.
3. Let f be continuous on $\mathbb{R}^d$. Then $f_n \in C^\infty(\mathbb{R}^d)$.
4. In addition to the continuity assumption in (3), let f be Lipschitz continuous in direction $I \subseteq \{1, \ldots, d\}$ with Lipschitz constant γ. Then, $\|\partial_i f_n\|_\infty \leq \gamma$ for all $i \in I$. Moreover, for higher partial derivatives of f_n we have that for every $k \in \mathbb{N}$ there exists $c_k > 0$ such that
$$\sup_{x\in\mathbb{R}^d} \big|\partial_{i_1,\ldots i_k} f_n(x)\big| \leq c_k$$
for all $i_1, \ldots, i_k \in I$.

(5) Assume that f is locally Lipschitz continuous in direction I. Then

$$\partial_i(f * \alpha_n) = \partial_i f * \alpha_n$$

almost everywhere for all $i \in I$.

Proof.

(1) We have

$$\int_{\mathbb{R}^d} \|x\|\alpha_n(x)\,\mathrm{d}x = \int_{\{x:\|x\|\leq 1/n\}} \|x\|\alpha_n(x)\,\mathrm{d}x \leq \frac{1}{n}.$$

(2) Let $\varepsilon \geq 0$. Since f is continuous at x_0 there exists an $N \in \mathbb{N}$ such that

$$|f(y) - f(x_0)| \leq \varepsilon, \quad y \in \left\{ y \in \mathbb{R}^d : \|x_0 - y\| \leq \frac{1}{N} \right\}.$$

Thus, we have for $n \geq N$ that

$$\begin{aligned} |f_n(x_0) - f(x_0)| &= |(f * \alpha_n)(x_0) - f(x_0)| \\ &= \left| \int_{\mathbb{R}^d} f(x_0 - y)\alpha_n(y)\,\mathrm{d}y - f(x_0) \int_{\mathbb{R}^d} \alpha_n(y)\,\mathrm{d}y \right| \\ &\leq \int_{\mathbb{R}^d} |f(x_0 - y) - f(x_0)|\alpha_n(y)\,\mathrm{d}y \leq \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, the assertion follows.

(3) Let $i \in \{1, \dots, d\}$. By the mean value theorem there exists ξ_h between 0 and h such that

$$\begin{aligned} \partial_i(f * \alpha_n)(x) &= \lim_{h\to 0} \int_{\mathbb{R}^d} f(y)\frac{1}{h}\big(\alpha_n(x - y + he_i) - \alpha_n(x - y)\big)\,\mathrm{d}y \\ &= \lim_{h\to 0} \int_{\mathbb{R}^d} f(y)\partial_i\alpha_n(x + \xi_h e_i - y)\,\mathrm{d}y \\ &= \lim_{h\to 0} (f * \partial_i\alpha_n)(x + \xi_h e_i) = (f * \partial_i\alpha_n)(x), \end{aligned}$$

where $e_i \in \mathbb{R}^d$ denotes the vector that has a one in the i-th position and zeros otherwise. In the last equation we use that $f * \partial_i\alpha_n$ is continuous by Lemma A.1 and $x + \xi_h e_i \overset{h\to 0}{\longrightarrow} x$. So, by Lemma A.1 and the calculations above $f * \alpha_n$ is partially differentiable in direction $i \in \{1, \dots, d\}$ with continuous partial derivatives $f * \partial_i\alpha_n$. For $k \in \mathbb{N}$ and $j = (j_1, \dots, j_k) \in \{1, \dots, d\}^k$ we define the operator $\Delta_j := \frac{\partial^k}{\partial_{j_1} \dots \partial_{j_k}}$. Iterating the calulation above then yields $\Delta_j(f * \alpha_n) = f * (\Delta_j\alpha_n)$.

(4) To show the boundedness consider

$$\begin{aligned}|\partial_i(f*\alpha_n)(x)| &= \lim_{h\to 0}\left|\frac{1}{h}\big((f*\alpha_n)(x+he_i)-(f*\alpha_n)(x)\big)\right| \\ &= \lim_{h\to 0}\left|\frac{1}{h}\left(\int_{\mathbb{R}^d} f(x+he_i-y)\alpha_n(y)\,\mathrm{d}y-\int_{\mathbb{R}^d} f(x-y)\alpha_n(y)\,\mathrm{d}y\right)\right| \\ &\le \lim_{h\to 0}\int_{\mathbb{R}^d}\left|\frac{1}{h}\big(f(x+he_i-y)-f(x-y)\big)\right|\alpha_n(y)\,\mathrm{d}y \\ &\le \gamma\int_{\mathbb{R}^d}\alpha_n(y)\,\mathrm{d}y=\gamma,\end{aligned}$$

for all $i\in I$. Replacing α_n by $\Delta_j\alpha_n$, where $j=(j_1,\dots,j_k)\in I^k$ in the calculation above yields

$$|\partial_i(\Delta_j(f*\alpha_n))(x)|\le\gamma\int_{\mathbb{R}^d}|\Delta_j\alpha_n(y)|\,\mathrm{d}y<\infty,$$

for all $i\in I$.

(5) First note that a function that is locally Lipschitz continuous in direction I is Lipschitz continuous in direction I on every compact set. Let $x\in\mathbb{R}^d$ be arbitrary but fixed and $i\in I$. In the same way as in (4) we obtain

$$\partial_i(f*\alpha_n)(x)=\lim_{h\to 0}\int_{\mathbb{R}^d}\frac{1}{h}\big(f(x+he_i-y)-f(x-y)\big)\alpha_n(y)\,\mathrm{d}y$$

For $\|y\|>1/n$ the integrand is zero and for $\|y\|\le 1/n$ (and assuming $h<1$) we have that $x+he_i-y, x-y\in B_2(x)$. Since f is locally Lipschitz in direction I, f is Lipschitz continuous in direction I on $\overline{B_2(x)}$ with some Lipschitz constant $\gamma(x)\ge 0$. It follows

$$\left|\frac{1}{h}\big(f(x+he_i-y)-f(x-y)\big)\right|\alpha_n(y)\le\gamma(x)\alpha_n(y),$$

where the right-hand side is integrable with respect to y and independent of h. By Stepanov's Theorem (a consequence of Rademacher's Theorem, compare [4] Theorem 3.1.9) $\partial_i f$ exists almost everywhere and we obtain by dominated convergence

$$\begin{aligned}&\lim_{h\to 0}\int_{\mathbb{R}^d}\frac{1}{h}\big(f(x+he_i-y)-f(x-y)\big)\alpha_n(y)\,\mathrm{d}y \\ &\quad=\int_{\mathbb{R}^d}\lim_{h\to 0}\frac{1}{h}\big(f(x+he_i-y)-f(x-y)\big)\alpha_n(y)\,\mathrm{d}y \\ &\quad=\int_{\mathbb{R}^d}\partial_i f(x-y)\alpha_n(y)\,\mathrm{d}y=(\partial_i f*\alpha_n)(x).\end{aligned}$$

□

Lemma A.2. *Let $(\Omega,\mathfrak{A},\mathcal{P})$ be a probability space and $Y\in L^1(\Omega,\mathfrak{A},\mathcal{P})$. Further, we assume that Π is a π-system, i.e. a non-empty family of subsets of Ω that is closed under finite intersection, with $\sigma(\pi)=\mathfrak{A}$. If $\mathbb{E}[\mathbf{1}_A Y]=0$ for all $A\in\Pi$, then $Y=0$ almost surely.*

Proof. Let $Y_+ = Y\mathbf{1}_{\{Y\geq 0\}}$ and $Y_- = -Y\mathbf{1}_{\{Y<0\}}$. Then $Y = Y_+ - Y_-$ and we define measures ν_1, ν_2 on Π as $\nu_1(A) = \mathbb{E}[\mathbf{1}_A Y_+]$ and $\nu_2(A) = \mathbb{E}[\mathbf{1}_A Y_-]$ for $A \in \Pi$. For any $A \in \Pi$ we have $0 = \mathbb{E}[\mathbf{1}_A Y] = \nu_1(A) - \nu_2(A)$ and therefore ν_1 and ν_2 coincide on a π-system that generates the σ-algebra $\mathfrak{A}$. It follows that $\nu_1 = \nu_2$ on $\mathfrak{A}$ (see e.g. Lemma 1.42 in [7]). Thus, we have $\mathbb{E}[\mathbf{1}_B Y] = \nu_1(B) - \nu_2(B) = 0$ for any $B \in \mathfrak{A}$. Plugging in $B = \{Y \geq 0\} \in \mathfrak{A}$ and $B = \{Y < 0\} \in \mathfrak{A}$ gives us the assertion. □

As in the main text, we denote by $I_p(g)$ the multiple stochastic Wiener integral over $g \in L^2([0,T]^p)$. In what follows let $\cdot$ and $\bullet$ be placeholders for disjunct variables. In iterated integrals we always integrate over the variables represented by $\cdot$ and never over those represented by $\bullet$. To simplify notation in the following lemma we set $L^2([0,T]^0) := \mathbb{R}$ and I_0 the identity fucntion on $\mathbb{R}$.

Lemma A.3. *Let $p, q \in \mathbb{N}$ and $g \in L^2([0,T]^p)$, $f \in L^2([0,T]^q)$. Then we have*

$$\left\| \int_0^T I_{p-1}(g(t,\cdot)) f(t,\bullet)\,\mathrm{d}t - I_{p-1}\Big(\int_0^T g(t,\cdot) f(t,\bullet)\,\mathrm{d}t\Big)\right\|_{L^2([0,T]^{q-1})} = 0 \quad a.s.$$

Proof. We write L^2_m for $L^2([0,T]^m)$. Let $(g^n)_{n\in\mathbb{N}}$ $((f^n)_{n\in\mathbb{N}})$ be a sequence of bounded, continuous functions in approximating g (f) in L^2_p (L^2_q) with $|g^n(x)| \leq |g(x)|$ for all $x \in [0,T]^p$ $(|f^n(x)| \leq |f(x)|$ for all $x \in [0,T]^q)$ and all $n \in \mathbb{N}$. Stochastic Fubini (e.g. Theorem 64 in [10], p.210) yields that, for fixed $t \in [0,T]$ and $n \in \mathbb{N}$,

$$\int_0^T I_{p-1}(g^n(t,\cdot)) f^n(t,\bullet)\,\mathrm{d}t = I_{p-1}\Big(\int_0^T g^n(t,\cdot) f^n(t,\bullet)\,\mathrm{d}t\Big) \tag{A.1}$$

almost surely. The continuity of f^n, g^n together with a density argument yields that the null set for which (A.1) does not hold can be chosen simultaneously for all $t \in [0,T]$. It follows that

$$\left\| \int_0^T I_{p-1}(g^n(t,\cdot)) f^n(t,\bullet)\,\mathrm{d}t - I_{p-1}\Big(\int_0^T g^n(t,\cdot) f^n(t,\bullet)\,\mathrm{d}t\Big)\right\|_{L^2_{q-1}} = 0$$

almost surely for all $n \in \mathbb{N}$. By dominated convergence along a suitable subsequence, we obtain

$$\left\| \int_0^T g^n(t,\cdot) f^n(t,\bullet)\,\mathrm{d}t - \int_0^T g(t,\cdot) f(t,\bullet)\,\mathrm{d}t\right\|_{L^2_{p+q-2}} \overset{n\to 0}{\longrightarrow} 0. \tag{A.2}$$

This implies with the help of the standard Fubini theorem that

$$\begin{aligned}
&\mathbb{E}\Big[\Big\| I_{p-1}\Big(\int_0^T g^n(t,\cdot) f^n(t,\bullet)\,\mathrm{d}t\Big) - I_{p-1}\Big(\int_0^T g(t,\cdot) f(t,\bullet)\,\mathrm{d}t\Big)\Big\|^2_{L^2_{q-1}}\Big] \\
&\quad = \frac{1}{(p-1)!}\Big\| \int_0^T g^n(t,\cdot) f^n(t,\bullet)\,\mathrm{d}t - \int_0^T g(t,\cdot) f(t,\bullet)\,\mathrm{d}t\Big\|^2_{L^2_{p+q-2}} \overset{n\to 0}{\longrightarrow} 0,
\end{aligned}$$

and thus

$$\Big\| I_{p-1}\Big(\int_0^T g^n(t,\cdot) f^n(t,\bullet)\,\mathrm{d}t\Big) - I_{p-1}\Big(\int_0^T g(t,\cdot) f(t,\bullet)\,\mathrm{d}t\Big)\Big\|_{L^2_{q-1}} \overset{n\to 0}{\longrightarrow} 0 \tag{A.3}$$

almost surely.

It is easy to see that

$$I_{p-1}(g^n(t,\cdot)) \xrightarrow{L_1^2} I_{p-1}(g(t,\cdot)) \quad a.s.$$

as $n \to \infty$ and therefore

$$\int_0^T I_{p-1}(g^n(t,\cdot))f^n(t,\bullet)\,\mathrm{d}t \xrightarrow{L_{q-1}^2} \int_0^T I_{p-1}(g(t,\cdot))f(t,\bullet)\,\mathrm{d}t \quad a.s. \tag{A.4}$$

as $n \to \infty$. Putting (A.2) – (A.4) together yields

$$\begin{aligned} &\Big\| \int_0^T I_{p-1}(g(t,\cdot))f(t,\bullet)\,\mathrm{d}t - I_{p-1}\Big(\int_0^T g(t,\cdot)f(t,\bullet)\,\mathrm{d}t\Big)\Big\|_{L^2([0,T]^{q-1})} \\ &\quad = \lim_{n\to\infty} \Big\| \int_0^T I_{p-1}(g^n(t,\cdot))f^n(t,\bullet)\,\mathrm{d}t - I_{p-1}\Big(\int_0^T g^n(t,\cdot)f^n(t,\bullet)\,\mathrm{d}t\Big)\Big\|_{L_{q-1}^2} \\ &\quad = 0 \quad a.s. \end{aligned}$$

□

Acknowledgment. The author wants to thank his supervisor Andreas Neuenkirch for his guidance, his careful reading of the manuscript and numerous fruitful discussions on the topic. This research is supported by the DFG-RTG 1953 "Statistical Modeling of Complex Systems and Processes."

References

1. Altmayer, M.: *Quadrature of discontinuous SDE functionals using Malliavin integration by parts*, PhD thesis, University of Mannheim, 2015.
2. Altmayer, M. and Neuenkirch, A.: Multilevel Monte Carlo quadrature of discontinuous payoffs in the generalized Heston model using Malliavin integration by parts, *SIAM J. Financial Math.* **6** (2015), no. 1, 22–52.
3. Deya, A. and Tindel, S.: Malliavin calculus for fractional heat equation, in: *Malliavin Calculus and Stochastic Analysis* (2013) 361–384, Springer.
4. Federer, H.: *Geometric Measure Theory*, Springer, 1969.
5. Folland, G.: *Real Analysis: Modern Techniques and Their Applications*, Wiley, 2nd edition, 1999.
6. Imkeller, P., dos Reis, G., Salkeld, W., and Smith, G.: Differentiability of SDEs with drifts of super-linear growth, *eprint arXiv:1803.06947* (2018).
7. Klenke, A.: *Probability Theory*, Springer, 2nd edition, 2014.
8. Nualart, D.: *The Malliavin Calculus and Related Topics*, Springer, 2nd edition, 2006.
9. Nualart, D. and Pardoux, E.: Stochastic calculus with anticipating integrands, *Probab. Theory and Relat. Fields* **78** (1988), 535–581.
10. Protter, P.: *Stochastic Integration and Differential Equations*, Springer, 2. ed., corr. 3. print, 2005.
11. Üstünel, A. and Zakai, M.: On independence and conditioning on Wiener space, *Ann. Probab.* **17** (1989), no. 4, 1441–1453.

Stefan Koch: Institute of Mathematics, University of Mannheim, Mannheim, Germany

E-mail address: stefan.koch@uni-mannheim.de