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ON THE VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS ON ANNULI

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Abstract: *The main purpose of this paper is to establish analogous of Milloux inequality and Hayman's alternative on annuli. As an application of our results, we deduce some interesting analogous results on annuli.*

Keywords: *Value Distribution Theory; Nevanlinna theory; the annuli.*

Subject Classification: *30D35*

1. INTRODUCTION

In 2005, Khrystiyanyn and Kondratyuk [6,7] gave an extension of the Nevanlinna value distribution theory for meromorphic functions in annuli. In their extension the main characteristics of meromorphic function are one-parameter and posses the same properties as in the classical case of a simply connected domain. In [6] and [7], we can get the analogous of the Jensen's formula, the first fundamental theorem, the lemma of the logarithmic derivative and the second fundamental theorem of Nevanlinna theory for meromorphic function in annuli. After [6,7], Fernandez [5] study the value distribution of meromorphic functions in the punctured plane, Cao, Deng, Yi and Xu [1]-[3] study the uniqueness of meromorphic functions in annuli, Chen and Wu [4] study the exceptional values of meromorphic functions and its derivatives in annuli. In the following, we introduce the definitions, notations and basic results of [4] and [6,7] which will be used in this paper.

2. DEFINITIONS AND MAIN RESULTS

Let $f(z)$ be a meromorphic function in the annulus $\mathrm{A}(R_0) = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 < +\infty$. Denote

$$m\left(R, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(Re^{i\theta} - a)|} d\theta,$$

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$$m(R, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ | f(Re^{i\theta} | d\theta,$$

where $a \in \mathbb{C}$ and $1/R_0 <| z |< R_0$. Let

$$m_0(R, a, f) = m_0\left(R, \frac{1}{f-a}\right) = m\left(R, \frac{1}{f-a}\right) + m\left(\frac{1}{R}, \frac{1}{f-a}\right),\ 1 < R < R_0$$

and

$$m_0(R, \infty, f) = m_0(R, f) = m(R, f) + m\left(\frac{1}{R}, f\right),\ 1 < R < R_0.$$

Put

$$N_1\left(R, \frac{1}{f-a}\right) = \int_{\frac{1}{R}}^{1} \frac{n_1\left(t, \frac{1}{f-a}\right)}{t} dt, \quad N_2\left(R, \frac{1}{f-a}\right) = \int_1^R \frac{n_2\left(t, \frac{1}{f-a}\right)}{t} dt$$

where $1 < R < R_0$, $n_1\left(t, \frac{1}{f-a}\right)$ is the counting function of zeros of the function $f - a$ in $\{z : t <| z |\le 1\}$ and $n_2\left(t, \frac{1}{f-a}\right)$ is the counting function of zeros of the function $f - a$ in $\{z : 1 <| z |\le t\}$. Denote also

$$N_1(R, f) = \int_{\frac{1}{R}}^{1} \frac{n_1(t, f)}{t} dt, \quad N_2(R, f) = \int_1^R \frac{n_2(t, f)}{t} dt$$

where $1 < R < R_0$, $n_1(t, f)$ is the counting function of poles of the function f in $\{z : t <| z |\le 1\}$ and $n_2(t, f)$ is the counting function of poles of the function f in $\{z : 1 <| z |\le t\}$. Let

$$N_0(R, a, f) = N_0\left(R, \frac{1}{f-a}\right) = N_1\left(R, \frac{1}{f-a}\right) + N_2\left(R, \frac{1}{f-a}\right).$$

$$N_0(R,\infty, f) = N_0(R, f) = N_1(R, f) + N_2(R, f).$$

Finally, we define the Nevanlinna characteristic of f in $\mathbb{A}(R_0)$ by

$$T_0(R, f) = m_0(R, f) - 2m(1, f) + N_0(R, f),\ 1 < R < R_0 \tag{2.1}$$

where $R_0 \le +\infty$. Suppose that f, g are two meromorphic functions in $\mathbb{A}(R_0)$, where $R_0 \le +\infty$. Then

$$m_0(R, fg) \le m_0(R, f) + m_0(R, g) + O(1). \tag{2.2}$$

Definition 2.1[4] *Let $f(z)$ be a non-constant meromorphic function on the annulus $\mathbb{A}(R_0) = \{z : 1/R_0 <|z|< R_0\}$, where $1 < R_0 < +\infty$. The function f is called a transcedental or admissible meromorphic function on the annulus $\mathbb{A}(R_0)$ provided that*

$$\lim_{R\to\infty} sup \frac{T_0(R, f)}{\log R} = \infty,\ 1 < R < R_0 = +\infty \tag{2.3}$$

or

$$\lim_{R\to R_0} sup \frac{T_0(R, f)}{-\log(R_0 - R)} = \infty,\ 1 < R < R_0 < +\infty \tag{2.4}$$

respectively.

Thus for a transcedental or admissible meromorphic function on the annulus $\mathbb{A}$, $S(R, f) = o(T_0(R, f))$ holds for all $1 < R < R_0$ except for the set Δ_R or the set Δ'_R mentioned in Theorem 2.B, respectively.

Definition 2.2[4] *Let f be a non-constant meromorphic function on the annulus $\mathbb{A}(R_0)$, where $1 < R_0 < +\infty$. Then the order of $f(z)$ is defined by*

$$\rho(f) = \lim_{R\to\infty} sup \frac{T_0(R, f)}{\log R},\ 1 < R < R_0 = +\infty \tag{2.5}$$

or

$$\rho(f) = \lim_{R\to R_0} sup \frac{T_0(R, f)}{-\log(R_0 - R)}, \; 1 < R < R_0 < +\infty. \tag{2.6}$$

Theorem A. [6] *(The First Fundamental Therorem). Let f be a non-constant meromorphic function in* $\mathrm{A}(R_0)$, *where* $1 < R_0 \le +\infty$. *Then*

$$T_0\left(R, \frac{1}{f-a}\right) = T_0(R, f) + O(1) \tag{2.7}$$

for any fixed $a \in \mathrm{C}$.

Theorem B. [7] *(Lemma on the Logarithmic Derivative). Let f be a non-constant meromorphic function in* $\mathrm{A}(R_0)$, *where* $1 < R_0 \le +\infty$ *and* $\alpha \ge 0$. *Then*

1. In the case, $R_0 = +\infty$,

$$m_0\left(R, \frac{f'}{f}\right) = O(\log(RT_0(R, f))) \tag{2.8}$$

for $R \in (1, +\infty)$ except for the set Δ_R such that $\int_{\Delta_R} R^{\alpha-1} dR < +\infty$;

2. In the case, $R_0 < +\infty$,

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log\left(\frac{T_0(R, f)}{R_0 - R}\right)\right) \tag{2.9}$$

for $R \in (1, R_0)$ except for the set Δ'_R such that $\int_{\Delta'_R} \frac{dR}{(R_0 - R^{\alpha-1})} < +\infty$.

Theorem C. [7] *(The Second Fundamental Theorem) Let f be a non-constant meromorphic function in* $\mathrm{A}(R_0)$, *where* $1 < R_0 \le +\infty$. *Let* $a_1, a_2, a_3, ..., a_p$ *be p distinct finite complex numbers and* $\alpha \ge 0$. *Then*

$$m_0(R, f) + \sum_{v=1}^{p} m_0\left(R, \frac{1}{f-a_j}\right) \le 2T_0(R, f) - N_0^{(1)}(R, f) + S(R, f) \tag{2.10}$$

where

$$N_0^{(1)}(R, f) = 2N_0(R, f) - N_0(R, f') + N_0\left(R, \frac{1}{f'}\right)$$

and

1. In the case, $R_0 = +\infty$,

$$S(R, f) = O(\log(RT_0(R, f))) \tag{2.11}$$

for $R \in (1, +\infty)$ except for the set Δ_R such that $\int_{\Delta_R} R^{\alpha-1} dR < +\infty$;

2. In the case, $R_0 < +\infty$,

$$S(R, f) = O\left(\log\left(\frac{T_0(R, f)}{R_0 - R}\right)\right) \tag{2.12}$$

for $R \in (1, R_0)$ except for the set Δ'_R such that $\int_{\Delta'_R} \frac{dR}{(R_0 - R^{\alpha-1})} < +\infty.$

Theorem D.[7,8] *(Lemma on the Logarithmic Derivative). Let f be a non-constant meromorphic function in* $A(R_0)$, *where* $1 < R_0 \le +\infty$ *and* $\alpha \ge 0$. *Then*

$$m_0\left(R, \frac{f^{(k)}}{f}\right) = S(R, f)$$

holds for every positive integer *k*.

In the value distribution theory, it is very important to introduce and study the derivative of a given function. It is natural to ask whether can we establish the analogous of Milloux inequality and Hayman's alternative in annulus.

In this paper, we prove the following theorems and establish an interesting and remarkable result of the Milloux inequality and Heyman's alternative in annulus.

Theorem 2.1. *Let* f *be a function that is meromorphic and admissible in* $A(R_0)$, *where* $1 < R_0 \le +\infty$. *Let*

$$\Theta(z) = \sum_{l=0}^{k} a_l f^{(l)}(z) \tag{2.13}$$

for any positive integer k.Where $a_0, a_1, a_2, a_3, \ldots, a_k$ are small functions of f. Then

$$m_0\left(R, \frac{\Theta}{f}\right) = S(R, f) \tag{2.14}$$

and

$$T_0(R, \Theta) \leq (k+1)T_0(R, f) + S(R, f). \tag{2.15}$$

Theorem 2.2. *Let f(z) be a non-constant meromorphic function and admissible in* $A(R_0)$, *where* $1 < R_0 \leq +\infty$ *and* $\Theta(z)$ *be the function defined by* (2.13). *If* $\Theta(z)$ *is not a constant, then*

$$T_0(R, f) < \overline{N}_0(R, f) + N_0\left(R, \frac{1}{f}\right) + \overline{N}_0\left(R, \frac{1}{\Theta - a}\right) - N_0^{(0)}\left(R, \frac{1}{\Theta'}\right) + S(R, f) \tag{2.16}$$

where $(a \neq 0, \infty)$ and $N_0^{(0)}\left(R, \frac{1}{\Theta'}\right)$ counts only zeros of Θ' but not the repeated roots of $\Theta = a$ in $A(R_0)$.

Theorem 2.3. *Let* f *be a transcedental meromorphic function and admissible in* $A(R_0)$, *where* $1 < R_0 \leq +\infty$. $\Theta = f^{(k)}$ *and* $N_0^{(0)}\left(R, \frac{1}{\Theta'}\right)$ *be defined as in Theorem 2.2. Then*

$$kN_0^1(R, f) \leq \overline{N}_0^{(2}(R, f) + \overline{N}_0\left(R, \frac{1}{\Theta - a}\right) + N_0^{(0)}\left(R, \frac{1}{\Theta'}\right) + S(R, f) \tag{2.17}$$

where $N_0^1(R, f)$ counts the simple poles of f and $\overline{N}_0^{(2}(R, f)$ counts the multiple poles of *f*, not including multiplicity in $A(R_0)$.

Theorem 2.4. *Let f be a transcedental meromorphic function and admissible in $\mathbb{A}(R_0)$, where $1 < R_0 \le +\infty$. Then*

$$T_0(R, f) \le \left(2+\frac{1}{k}\right)N_0\left(R, \frac{1}{f}\right)+\left(2+\frac{2}{k}\right)\overline{N}_0\left(R, \frac{1}{\Theta - a}\right)+S(R, f). \quad (2.18)$$

By replacing $\Theta = f^{(k)}$ in the Theorem 2.2, we get the following result

Corollary 2.1. *Suppose $f(z)$ is a non-constant meromorphic function and admissible in $\mathbb{A}(R_0)$, where $1 < R_0 \le +\infty$ and k is any positive integer. Then*

$$T_0(R, f) \le \overline{N}_0(R, f)+ N_0\left(R, \frac{1}{f}\right)+\overline{N}_0\left(R, \frac{1}{f^{(k)} - a}\right)- N_0^{(0)}\left(R, \frac{1}{f^{(k+1)}}\right)+S(R, f).$$

By Theorem 2.2, we get the following Corollary

Corollary 2.2. *Let f be a non constant transcedental meromorphic function and admissible in $\mathbb{A}(R_0)$, where $1 < R_0 \le +\infty$, with only a finite number of zeros and poles. Then every function Θ as defined in* (2.13) *assumes every finite complex value, except possibly zero, infinitely often or else is identically constant.*

By replacing $\Theta = f^{(k)}$ in the Theorem 2.4, we get the following result

Corollary 2.3 *Let f be a transcedental meromorphic function and admissible in $\mathbb{A}(R_0)$, where $1 < R_0 \le +\infty$. Then*

$$T_0(R, f) \le \left(2+\frac{1}{k}\right)N_0\left(R, \frac{1}{f}\right)+\left(2+\frac{2}{k}\right)\overline{N}_0\left(R, \frac{1}{f^{(k)} - a}\right)+S(R, f).$$

By replacing the value of $F = \dfrac{f - \omega_1}{\omega_2}$, where ω_1 and ω_2 be complex numbers $\omega_2 \ne 0$ and $T_0(R, F) = T_0(R, f)+O(1)$ in Theorem 2.4. Then we get the following result.

Corollary 2.4 *(Hayman's Alternative in annuli.) Let f be a transcedental meromorphic function and admissible in $\mathbb{A}(R_0)$, where $1 < R_0 \le +\infty$. Then either f*

assumes every finite value infinitely often or $f^{(k)}$ *assumes every finite value except possibly zero infinitely often in* $\mathrm{A}(R_0)$.

3. PROOF OF THEOREMS

3.1. Proof of Theorem 2.1

First of all, we prove the Theorem 2.1 for the case $\Theta(z) = f^{(k)}$ using induction on the number k and then deduce the conclusion of the Theorem 2.1 for the general case.

By Theorem B, we have

$$T_0(R, f') = T_0\left(R, f\frac{f'}{f}\right) \le T_0(R, f) + T_0\left(R, \frac{f'}{f}\right) + O(1)$$

$$= T_0(R, f) + m_0\left(R, \frac{f'}{f}\right) + N_0\left(R, \frac{f'}{f}\right) - 2m\left(1, \frac{f'}{f}\right) + O(1)$$

$$\le T_0(R, f) + \overline{N}_0(R, f) + S(R, f)$$

$$\le 2T_0(R, f) + S(R, f).$$

Hence the result is true for $k = 1$.

Suppose that the theorem is true for $k = n.$ Then by assumption, we have

$$m_0\left(R, \frac{f^{(n)}}{f}\right) = S(R, f) \tag{3.1}$$

and

$$T_0(R, f^{(n)}) \le (n+1)T_0(R, f) + S(R, f). \tag{3.2}$$

Also we have,

$$m_0\left(R, f^{(n+1)}\right) = m_0\left(R, f^{(n)}\right) + m_0\left(R, \frac{f^{(n+1)}}{f^{(n)}}\right) \tag{3.3}$$

and

$$\begin{aligned} N_0(R, f^{(n+1)}) &= N_0(R, f^{(n)}) + \overline{N}_0(R, f^{(n)}) \\ &= N_0(R, f^{(n)}) + \overline{N}_0(R, f) \\ &\le N_0(R, f^{(n)}) + N_0(R, f). \end{aligned} \tag{3.4}$$

By Theorem D, we have

$$m_0\left(R, \frac{f^{(n+1)}}{f}\right) \le m_0\left(R, \frac{f^{(n+1)}}{f^{(n)}}\right) + m_0\left(R, \frac{f^{(n)}}{f}\right)$$

$$\le S(R, f^{(n)}) + S(R, f)$$

$$\le S(R, f) \tag{3.5}$$

and

$$T_0(R, f^{(n+1)}) = m_0\left(R, f^{(n+1)}\right) + N_0(R, f^{(n+1)}) - 2m(1, f^{(n+1)})$$

$$\le m_0(R, f^{(n)}) + m_0\left(R, \frac{f^{(n+1)}}{f^{(n)}}\right) + N_0(R, f^{(n)}) + N_0(R, f) + O(1)$$

$$\le T_0(R, f^{(n)}) + N_0(R, f) + S(R, f)$$

$$\le (n+1)T_0(R, f) + T_0(R, f) + S(R, f)$$

$$\le (n+2)T_0(R, f) + S(R, f). \tag{3.6}$$

Hence the result is true for all positive integer k.

In the following, we consider the general case.

By above case, it is obvious that

$$m_0\left(R, \frac{\Theta}{f}\right) \le \sum_{l=0}^{k} m_0\left(R, \frac{a_l f^{(l)}}{f}\right) + log(k+1)$$

$$\le \sum_{l=0}^{k}\left[m_0\left(R, a_l\right) + m_0\left(R, \frac{f^{(l)}}{f}\right)\right] + log(k+1)$$

$$\le S(R, f). \tag{3.7}$$

Thus, we have

$$m_0(R,\Theta) \le m_0\left(R,\frac{\Theta}{f}\right) + m_0(R,f) \le m_0(R,f) + S(R,f). \tag{3.8}$$

On the other hand, we have

$$N_0(R,\Theta) \le N_0\left(R,f^{(k)}\right) \le N_0(R,f) + k\overline{N}_0(R,f). \tag{3.9}$$

it follows from (3.8) and (3.9) that

$$T_0(R,\Theta) = m_0(R,\Theta) + N_0(R,\Theta) - 2m(1,\Theta)$$

$$\le m_0(R,f) + N_0(R,f) + k\overline{N}_0(R,f) + S(R,f)$$

$$\le T_0(R,f) + k\overline{N}_0(R,f) + S(R,f)$$

$$\le (k+1)T_0(R,f) + S(R,f)$$

$$T_0(R,\Theta) \le (k+1)T_0(R,f) + S(R,f)$$

which completes the proof of Theorem 2.1.

3.2. Proof of Theorem 2.2

By the Second Fundamental theorem in annulus, we have

$$m_0(R,\Theta) + m_0\left(R,\frac{1}{\Theta}\right) + m_0\left(R,\frac{1}{\Theta - a}\right) \le 2T_0(R,\Theta) - N_0^{(1)}(R,f) + S(R,\Theta). \tag{3.10}$$

By the First Fundamental theorem in annulus, we have

$$2T_0(R,\Theta) - N_0^{(1)}(R,f) = m_0(R,\Theta) + m_0(R,a,\Theta) + N_0(R,\Theta) + N_0(R,a,\Theta)$$

$$-\left[2N_0(R,\Theta) - N_0(R,\Theta') + N_0\left(R,\frac{1}{\Theta'}\right)\right]$$

$$= m_0(R,\Theta) + m_0(R,a,\Theta) + N_0(R,a,\Theta)$$

$$-N_0\left(R,\frac{1}{\Theta'}\right) + N_0(R,\Theta') - N_0(R,\Theta). \tag{3.11}$$

It is obvious that

$$N_0(R,\Theta')-N_0(R,\Theta)\leq \overline{N}_0(R,f) \tag{3.12}$$

and

$$N_0\left(R,\frac{1}{\Theta-a}\right)-N_0\left(R,\frac{1}{\Theta'}\right)=\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)-N_0^{(0)}\left(R,\frac{1}{\Theta'}\right). \tag{3.13}$$

Hence it follows from (3.10), (3.11), (3.12) and (3.13) that

$$m_0\left(R,\frac{1}{\Theta}\right)\leq \overline{N}_0(R,f)+\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)-N_0^{0}\left(R,\frac{1}{\Theta'}\right)+S(R,\Theta). \tag{3.14}$$

From (2.15), we have

$$S(R,\Theta)=S(R,f).$$

By First fundamental Theorem in annulus, we have

$$T_0(R,f)=m_0\left(R,\frac{1}{f}\right)+N_0\left(R,\frac{1}{f}\right)+O(1)$$

$$\leq m_0\left(R,\frac{1}{\Theta}\right)+m_0\left(R,\frac{\Theta}{f}\right)+N_0\left(R,\frac{1}{f}\right)+O(1)$$

$$\leq m_0\left(R,\frac{1}{\Theta}\right)+N_0\left(R,\frac{1}{f}\right)+S(R,f). \tag{3.15}$$

From (3.14) and (3.15), we have

$$T_0(R,f)\leq \overline{N}_0(R,f)+N_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)-N_0^{(0)}\left(R,\frac{1}{\Theta'}\right)+S(R,f)$$

which completes the Proof of Theorem 2.2.

3.3. Proof of Theorem 2.3

We first define the function

$$g=\frac{\left(f^{(k+1)}\right)^{k+1}}{\left(a-f^{(k)}\right)^{k+2}}=\frac{(\Theta')^{k+1}}{(a-\Theta)^{k+2}}. \tag{3.16}$$

Suppose f has a simple pole at z_0, i,e $f(z) = b(z - z_0)^{-1} + O(1)$ for some $b \neq 0$. Then differentiating k times,

$$f^{(k)}(z) = \frac{(-1)^k ak!}{(z - z_0)^{k+1}}\left(1 + O((z - z_0)^{k+1})\right).$$

Differentiating again and then substituting it into g, we find that

$$g = \frac{(-1)^k (k+1)^{k+1}}{ak!}\left(1 + O((z - z_0)^{k+1})\right).$$

Thus, at a simple pole of $f, g \neq 0, \infty$, but g' has a zero of order at least k. Now we apply First Fundamental Theorem in annulus to $\frac{g'}{g}$, assuming g to be non constant, giving

$$m_0\left(R, \frac{g'}{g}\right) - m_0\left(R, \frac{g}{g'}\right) + O(1)$$

$$= N_0\left(R, \frac{g}{g'}\right) - N_0\left(R, \frac{g'}{g}\right)$$

$$= N_0(R, g) + N_0\left(R, \frac{1}{g'}\right) - N_0(R, g') - N_0\left(R, \frac{1}{g}\right)$$

$$= N_0\left(R, \frac{1}{g'}\right) - N_0\left(R, \frac{1}{g}\right) - \overline{N}_0(R, g)$$

$$= N_0^{(0)}\left(R, \frac{1}{g'}\right) - \overline{N}_0\left(R, \frac{1}{g}\right) - \overline{N}_0(R, g). \tag{3.17}$$

Thus using (3.17), Theorem B and the property that $m_0\left(R, \frac{g}{g'}\right)$ is non negative, we have

$$kN_0^1(R,f)\le N_0^{(0)}\left(R,\frac{1}{g'}\right)\le \overline{N}_0\left(R,\frac{1}{g}\right)+\overline{N}_0(R,g)+m_0\left(R,\frac{g'}{g}\right)+O(1)$$

$$\le \overline{N}_0\left(R,\frac{1}{g}\right)+\overline{N}_0(R,g)+S(R,g). \qquad (3.18)$$

By (3.18) and zeros and poles of g can only occur at multiple poles of *f*, a-points of Θ or zeros of Θ' which are not a-points of Θ and so

$$\overline{N}_0\left(R,\frac{1}{g}\right)+\overline{N}_0(R,g)\le \overline{N}_0\left(R,\frac{1}{\Theta-a}\right)+\overline{N}_0^{(2}(R,f)+N_0^{(0)}\left(R,\frac{1}{\Theta'}\right).$$

Hence by (3.15), we have

$$kN_0^1(R,f)\le \overline{N}_0^{(2}(R,f)+\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)+N_0^{(0)}\left(R,\frac{1}{\Theta'}\right)+S(R,f).$$

3.4. Proof of Theorem 2.4

We start by noting that in $N_0(R,f)$, multiple poles are counted at least twice and then apply (2.16)

$$N_0^1(R,f)+2\overline{N}_0^{(2}(R,f)\le T_0(R,f)$$

$$\le \overline{N}_0(R,f)+N_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)-N_0^{(0)}\left(R,\frac{1}{\Theta'}\right)+S(R,f). \qquad (3.19)$$

Since $\overline{N}_0(R,f)=N_0^1(R,f)+\overline{N}_0^{(2}(R,f)$, hence by (3.20), we get

$$\overline{N}_0^{(2}(R,f)\le N_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)-N_0^{(0)}\left(R,\frac{1}{\Theta'}\right)+S(R,f). \qquad (3.20)$$

By (3.20) and (2.17), we get

$$kN_0^1(R,f)\le N_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)-N_0^{(0)}\left(R,\frac{1}{\Theta'}\right)+\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)$$

$$+N_0^{(0)}\left(R,\frac{1}{\Theta'}\right)+S(R,f)$$

$$kN_0^1(R,f)\le N_0\left(R,\frac{1}{f}\right)+2\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)+S(R,f). \qquad (3.21)$$

By (3.20) and (3.21), we can write

$$\overline{N}_0(R,f)=N_0^1(R,f)+\overline{N}_0^{(2}(R,f)$$

$$\le\frac{1}{k}N_0\left(R,\frac{1}{f}\right)+\frac{2}{k}\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)+N_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)$$

$$-N_0^{(0)}\left(R,\frac{1}{\Theta'}\right)+S(R,f)$$

$$\overline{N}_0(R,f)\le\left(1+\frac{1}{k}\right)N_0\left(R,\frac{1}{f}\right)+\left(1+\frac{2}{k}\right)\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)-N_0^{(0)}\left(R,\frac{1}{\Theta'}\right)+S(R,f). \qquad (3.22)$$

Since $N_0^{(0)}\left(R,\frac{1}{\Theta'}\right)\ge 0,$ we substitute this and (3.22) into $(2.16),$ we get

$$T_0(R,f)\le\left(2+\frac{1}{k}\right)N_0\left(R,\frac{1}{f}\right)+\left(2+\frac{2}{k}\right)\overline{N}_0\left(R,\frac{1}{\Theta-a}\right)+S(R,f).$$

4. APPLICATIONS

We can use Milloux inequality and Hayman's alternative in annulus to prove results related to uniqueness and sharing of two meromorphic functions in annulus.

5. OPEN QUESTIONS

Can we establish Milloux inequality and Hayman's alternative for more general differential polynomials in annulus and use those to prove results related to sharing of two differential polynomials of meromorphic functions in annulus.

Conflict of Interest

The authors declares that there is no conflict of interest regarding the publication of this paper.

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