Communications on Stochastic Analysis
Vol. 12, No. 2 (2018) 109-127

Serials Publications
www.serialspublications.com

NONLOCAL DIFFUSIONS AND THE QUANTUM BLACK-SCHOLES EQUATION: MODELLING THE MARKET FEAR FACTOR

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Abstract. In this paper, we establish a link between quantum stochastic processes, and nonlocal diffusions. We demonstrate how the non-commutative Black-Scholes equation of Accardi & Boukas (cf [1]) can be written in integral form. This enables the application of the Monte-Carlo methods adapted to McKean stochastic differential equations (cf [18]) for the simulation of solutions. We show how unitary transformations can be applied to classical Black-Scholes systems to introduce novel quantum effects. These have a simple economic interpretation as a market 'fear factor', whereby recent market turbulence causes an increase in volatility going forward, that is not linked to either the local volatility function or an additional stochastic variable. Lastly, we extend this system to 2 variables, and consider Quantum models for bid-offer spread dynamics.

1. Introduction

The link between the classical Black-Scholes equation and quantum mechanics and the application of quantum formalism to Mathematical Finance has been investigated by several authors. For example: [1]-[5], [10]-[12], [15], and [16]-[21]. In particular, the approach of modelling derivative prices using self-adjoint operators on a Hilbert space was suggested by Segal & Segal in [21]. In this paper the authors noted that, in the real world, the market operates with imperfect information and that different observables, such as underlying price and option delta, are usually not simultaneously observable. This fact makes the non-commutative extension of the Black-Scholes framework a natural step. The authors point out that this approach addresses some of the limitations of the classical Black-Scholes model, such as the underestimation of the probability of extreme events - so called "fat tails". In this sense, non-commutative Quantum models present an alternative means to capture complex market dynamics, without the addition of new stochastic variables.

In [1], Accardi & Boukas derive a general form for the Quantum Black-Scholes equation based on the Hudson-Parthasarathy calculus (cf [13]) and show that a commutative unitary time development operator acting on the market state, leads

Received 2018-6-11; Communicated by A. Boukas.

2010 *Mathematics Subject Classification.* Primary 81S25; Secondary 91B70.

Key words and phrases. Quantum Black-Scholes, Hudson-Parthasarathy quantum stochastic calculus, nonlocal diffusion, particle method.

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to a classical Black-Scholes system. Further they give the quantum stochastic differential equation governing the time development operator, and demonstrate how unitary transformations can lead to non-commutativity.

An example of a non-commutative Quantum Black-Scholes partial differential equation is derived, although the authors work in an abstract setting and do not discuss specific unitary transformations and Hilbert space representations of financial markets.

Therefore, one objective of this work is to use the Accardi-Boukas framework to look at how different unitary transformations can be used to transform the classical Black-Scholes equation, and to understand how quantum effects become apparent. We then go on to explore an example application of the approach in the modelling of bid-offer spread dynamics. The final objective of the current work is to identify suitable Monte-Carlo methods, which can be used for the simulation of solutions.

In section 2, we give an overview of the Accardi-Boukas derivation of the general form for the Quantum Black-Scholes equation, from [1]. With the objective of looking at "near classical" Black-Scholes worlds, we then derive specific forms for the resulting partial differential equations that result from small translations, and rotations. This in turn involves the extension of the Accardi-Boukas equation to systems with more than one underlying variable. We go on to discuss how this approach can be applied to the modelling of bid-offer dynamics.

In section 3, we show how this can be linked to the nonlocal diffusion processes discussed by Luczka, Hänggi and Gadomski in [14]. Here the impact of the diffusion differential operator is spread out through the convolution with a "blurring" function. The Kramers-Moyal expansion of the nonlocal Fokker-Planck equations allows us to derive the moments of the blurring function for the "near classical" quantum system.

This approach allows a natural route to the visualisation of the quantum effects on the system using McKean SDEs (cf [18]). The Monte-Carlo methods, developed by Guyon, and Henry-Labordère in [8], can then be adapted to the simulation of solutions. This is discussed in section 4, where we present numerical results and show how, by introducing small transformations to the system, the stochastic process now reacts to a market downturn by returning higher volatility. This effect is observed even where there is a single static Black-Scholes type volatility.

2. Quantum Black-Scholes Equation

In this section we follow the notation given, by Accardi & Boukas, in [1]. The current market is represented by a vector in a Hilbert space: $\mathbb{H}$, which contains all relevant information about the state of the market at an instant in time. The tradeable price for a security is represented by an self-adjoint operator on $\mathbb{H}$: X, and the spectrum of X represents possible prices.

Let $L^2[\mathbb{R}^+;\mathbb{H}]$ represent square integrable functions from the positive real axis (time) to the Hilbert space $\mathbb{H}$. Then the random behaviour of tradeable securities can be modelled using the tensor product of $\mathbb{H}$ with the bosonic Fock space: $\mathbb{H}\otimes\Gamma(L^2[\mathbb{R}^+;\mathbb{H}])$. We term this the "market space". The operator that returns the current price becomes $X \otimes \mathbb{I}$, where $\mathbb{I}$ represents the identity operator. The time development of $X \otimes \mathbb{I}$ into the future is modelled by:

$j_t(X) = U_t^* X \otimes \mathbb{I} U_t$

$\mathbb{H}$ carries the initial state of the market and U_t acts by introducing random fluctuations that fill up the empty states in: $\Gamma(L^2[\mathbb{R}^+;\mathbb{H}])$. The functional form for U_t is derived by Hudson & Parthasarathy in [13], and is given by:

$$dU_t = -\left(\left(iH + \tfrac{1}{2}L^*L\right)dt + L^*SdA_t - LdA_t^\dagger + \left(1 - S\right)d\Lambda_t\right)U_t$$

$dA_t^\dagger, dA_t, d\Lambda_t$ represent the standard creation, annihilation, and Poisson operators of quantum stochastic calculus. H, S and L also operate on the market space, with S unitary, and H self-adjoint. The multiplication rules of the Hudson-Parthasarathy calculus are given below (cf [13]):

-	$dA_t^\dagger$	$d\Lambda_t$	dA_t	dt
$dA_t^\dagger$	0	0	0	0
$d\Lambda_t$	$dA_t^\dagger$	$d\Lambda_t$	0	0
dA_t	dt	dA_t	0	0
dt	0	0	0	0

The first thing to note is that, for $S \neq 1$, there is a non-zero Poisson term and the time development operator is non-commutative.

The next thing to note is that, where $S = 1$, the Poisson term disappears. The model can be written using the Ito calculus in place of the more general Hudson-Parthasarathy framework. The Wiener process dW_t can be modelled using: $dA_t + dA_t^\dagger$.

Let $V_T = j_T(X - K)^+$, represent the option price process as at final expiry T, and K the operator given by multiplying by the strike. Further, for $V_t = j_t(X - K)^+$ the following expansion is assumed:

$V_t = F(t, x) = \sum_{n,k} a_{n,k}(t - t_0)^n (x - x_0)^k$

The Hudson-Parthasarathy multiplication rules can be applied to this expansion to give a quantum stochastic differential equation for V_t, that corresponds to the usual Ito expansion used in the derivation of the classical Black-Scholes. By assuming one can construct a hedge portfolio, Accardi & Boukas are able to derive the general form of the Quantum Black Scholes equation using the assumption that any portfolio must be self financing. Proposition 1, from [1] gives the full Quantum Black-Scholes equation:

$$\begin{aligned} a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X))j_t(\theta) + \sum_{k=2}^{\infty} a_{0,k}(t, j_t(X))j_t(\alpha\lambda^{k-2}\alpha^\dagger) \\ = a_t j_t(\theta) + V_t r - a_t j_t(X) r \end{aligned} \tag{2.1}$$

Here, a_t represents the holding in the underlying asset and is given by the boundary conditions:

$\sum_{k=1}^{\infty} a_{0,k}(t, j_t(X)) j_t(\lambda^{k-1}\alpha^{\dagger}) = a_t j_t(\alpha^{\dagger})$

$\sum_{k=1}^{\infty} a_{0,k}(t, j_t(X)) j_t(\alpha\lambda^{k-1}) = a_t j_t(\alpha)$

$\sum_{k=1}^{\infty} a_{0,k}(t, j_t(X)) j_t(\lambda^{k}) = a_t j_t(\lambda)$

Further, θ, α and λ are given by:

$\alpha = [L^*, X]S$, $\lambda = S^*XS - X$, $\theta = i[H, X] - \frac{1}{2}\{L^*LX + XL^*L + 2L^*XL\}$.

In this case the boundary conditions arise because when the Poisson term: $d\Lambda$ is non-zero, unlike Ito calculus where expansion terms with order above 2 can be ignored, higher order terms still contain non-vanishing contribution.

2.1. Translation. The natural Hilbert space for an equity price (say the FTSE price) is: $\mathbb{H} = L^2[\mathbb{R}]$. In this case, the only unitary transactions we can use are the translations:

$T_{\epsilon} : f(x) \to f(x - \varepsilon)$

Here we have, for a translation invariant Lebesgue measure μ:

$\langle T_{\varepsilon} f | T_{\varepsilon} g\rangle = \int_{\mathbb{R}} \overline{f(x-\varepsilon)} g(x-\varepsilon) d\mu(x) = \int_{\mathbb{R}} \overline{f(x)} g(x) d\mu(x) = \langle f|g\rangle$

So S is unitary in this case. Therefore, translating by ε we get:

$\lambda = T_{-\varepsilon} X T_{\varepsilon} f(x) - Xf(x) = T_{-\varepsilon} x f(x-\varepsilon) - xf(x) = (x+\varepsilon) f(x) - xf(x) = \varepsilon f(x)$

So we have $\lambda = \varepsilon$, and it is clear the example given in [1] relates to a translation by $\varepsilon = 1$. Following the key steps from [1] Proposition 3, and inserting this back into equation 2.1, we get the following Quantum Black-Scholes partial differential equation for this system:

Lemma 2.1. *Let $u(t, x)$ represent the price at time t, of a derivative contract in the system described above under small translation ε, and with interest rate r. Then the quantum Black-Scholes equation becomes:*

$$\frac{\partial u(t,x)}{\partial t} = rx\frac{\partial u(t,x)}{\partial x} - u(t,x)r + \sum_{k=2}^{\infty} \frac{\varepsilon^{k-2}}{k!} \frac{\partial^k u(t,x)}{\partial x^k} g(x) \tag{2.2}$$

Proof. The proof follows the same steps Accardi & Boukas outline in [1] proposition 3, with small modifications. □

For $\varepsilon = 0$, the sum over $k \geq 3$ drops out, and the equation reverts to the classical Black-Scholes. We investigate the impact of non-zero ε in section 4.

2.2. Rotation. The true current state of the financial market contains a much richer variety of information than just a single price, and by increasing the dimensionality of the market space accordingly we introduce a wider variety of unitary transformations, that can introduce non-commutativity. For example, let x represent the FTSE mid-price, and ϵ half of the bid-offer spread so that $(x+\epsilon)$ represents the best offer-price and $(x-\epsilon)$ the best bid-price. Now the market is represented by the Hilbert space: $\mathbb{H} = L^2[\mathbb{R}^2]$, and we can apply rotations, in addition to translations.

We make the simplifying assumption that market participants can trade the mid-price: x (for example during the end of day auction process) and that the market has sufficient liquidity to enable participants to alternatively act as market makers (receiving bid-offer spread) or as hedgers (crossing bid-offer spread) and therefore trade the bid-offer spread: ϵ. Therefore we make the following assumption:

Remark 2.2. We make the assumption that, for any derivative payout $V(x_T, \epsilon_T)$, we can construct a hedged portfolio, and can proceed with the derivation of the Quantum Black Scholes equation following the basic methodology from [1].

We now have separate creation, annihilation and Poisson operators, for x and ϵ; dA_x, dA_ϵ etc. These can be combined using the multiplication table ([13], Theorem 4.5), by making the assumption that the bid-offer is uncorrelated with the equity price. This corresponds to assumption 2.3:

Remark 2.3. We also assume:
$dA_x d\Lambda_\epsilon = dA_\epsilon d\Lambda_x = d\Lambda_x d\Lambda_\epsilon = d\Lambda_\epsilon d\Lambda_x = dA_x dA_\epsilon^\dagger = dA_\epsilon dA_x^\dagger = d\Lambda_x dA_\epsilon^\dagger = d\Lambda_\epsilon dA_x^\dagger = 0$.

Lastly, we make the assumption that we can expand the derivative payout as before:

Remark 2.4. $V_t = F(t, x, \epsilon) = \sum_{n,k,l} a_{n,l,k}(t-t_0)^n (x-x_0)^k (\epsilon-\epsilon_0)^l$

We can now derive the relevant Quantum Black-Scholes equation.

Proposition 2.5. *Let $\mathbb{H} = L^2[\mathbb{R}^2]$, and let $X \otimes 1$ and $\epsilon \otimes 1$ operate on the market space: $\mathbb{H} \otimes \Gamma(L^2[\mathbb{R}^+; \mathbb{H}])$, to return the mid-price, and bid-offer spread for a tradeable security respectively. Further, let the notation from* [1]*, and the above assumptions apply. Then the Quantum Black-Scholes equation in this case is given by:*

$$\begin{aligned} & a_{1,0,0}(t, j_t(X), j_t(\epsilon)) + a_{0,1,0}(t, j_t(X), j_t(\epsilon)) j_t(\theta_x) + a_{0,0,1}(t, j_t(X), j_t(\epsilon)) j_t(\theta_\epsilon) \\ & + \sum_{k=2}^{\infty} a_{0,k,0}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_x \lambda_x^{k-2} \alpha_x^\dagger) + \sum_{l=2}^{\infty} a_{0,0,l}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_\epsilon \lambda_\epsilon^{l-2} \alpha_\epsilon^\dagger) \\ & = a_{x,t} j_t(\theta_x) + a_{\epsilon,t} j_t(\theta_\epsilon) + V_t r - a_{x,t} j_t(X) r - a_{\epsilon,t} j_t(\epsilon) r \end{aligned} \tag{2.3}$$

where for $j_t(X)$ we have:

$\sum_{k=1}^{\infty} a_{0,k,0}(t, j_t(X), j_t(\epsilon)) j_t(\lambda_x^{k-1} \alpha_x^\dagger) = a_{x,t} j_t(\alpha_x^\dagger)$

$\sum_{k=1}^{\infty} a_{0,k,0}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_x \lambda_x^{k-1}) = a_{x,t} j_t(\alpha_x)$
$\sum_{k=1}^{\infty} a_{0,k,0}(t, j_t(X), j_t(\epsilon)) j_t(\lambda_x^{k}) = a_{x,t} j_t(\lambda_x)$

and for $j_t(\epsilon)$ *we have:*

$\sum_{l=1}^{\infty} a_{0,0,l}(t, j_t(X), j_t(\epsilon)) j_t(\lambda_\epsilon^{l-1} \alpha_\epsilon^\dagger) = a_{\epsilon,t} j_t(\alpha_\epsilon^\dagger)$
$\sum_{l=1}^{\infty} a_{0,0,l}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_\epsilon \lambda_\epsilon^{l-1}) = a_{\epsilon,t} j_t(\alpha_\epsilon)$
$\sum_{l=1}^{\infty} a_{0,0,l}(t, j_t(X), j_t(\epsilon)) j_t(\lambda_\epsilon^{l}) = a_{\epsilon,t} j_t(\lambda_\epsilon)$

Proof. First, the equations for time-development operators for $X \otimes 1$, and $\epsilon \otimes 1$ become:

$$dU_{x,t} = -\Bigg(\bigg(iH + \tfrac{1}{2} L_x^* L_x\bigg) dt + L_x^* S dA_x - L_x dA_x^\dagger + \bigg(1 - S\bigg) d\Lambda_x$$

$$dU_{\epsilon,t} = -\Bigg(\bigg(iH + \tfrac{1}{2} L_\epsilon^* L_\epsilon\bigg) dt + L_\epsilon^* S dA_\epsilon - L_\epsilon dA_\epsilon^\dagger + \bigg(1 - S\bigg) d\Lambda_\epsilon$$

Then, applying the Hudson-Parthasarathy multiplication rules to the expansion given in assumption 2.4 gives:

$$\begin{aligned}
dV_t = \Bigg(& a_{1,0,0}(t, j_t(x), j_t(\epsilon)) + a_{0,1,0}(t, j_t(x), j_t(\epsilon)) j_t(\theta_x) \\
& + a_{0,0,1}(t, j_t(x), j_t(\epsilon)) j_t(\theta_\epsilon) \\
& + \sum_{k=2}^{\infty} a_{0,k,0}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_x \lambda_x^{k-2} \alpha_x^\dagger) \\
& + \sum_{l=2}^{\infty} a_{0,0,l}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_\epsilon \lambda_\epsilon^{l-2} \alpha_\epsilon^\dagger)\Bigg) dt \\
& + \Bigg(a_{0,1,0}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_x) + \sum_{k=2}^{\infty} a_{0,k,0}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_x \lambda_x^{k-1})\Bigg) dA_x \\
& + \Bigg(a_{0,0,1}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_\epsilon) + \sum_{l=2}^{\infty} a_{0,0,l}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_\epsilon \lambda_\epsilon^{k-1})\Bigg) dA_\epsilon \\
& + \Bigg(a_{0,1,0}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_x^\dagger) + \sum_{k=2}^{\infty} a_{0,k,0}(t, j_t(X), j_t(\epsilon)) j_t(\lambda_x^{k-1} \alpha_x^\dagger)\Bigg) dA_x^\dagger \\
& + \Bigg(a_{0,0,1}(t, j_t(X), j_t(\epsilon)) j_t(\alpha_\epsilon^\dagger) + \sum_{l=2}^{\infty} a_{0,0,l}(t, j_t(X), j_t(\epsilon)) j_t(\lambda_\epsilon^{k-1} \alpha_\epsilon^\dagger)\Bigg) dA_\epsilon^\dagger
\end{aligned} \tag{2.4}$$

where $\theta_x, \theta_\epsilon$ are given by:

$$\theta_x = i[H, X] - \tfrac{1}{2}\bigg(L_x^* L_x X + X L_x^* L_x - 2 L_x^* X L_x\bigg)$$

$$\theta_\epsilon = i[H, \epsilon] - \tfrac{1}{2}\bigg(L_\epsilon^* L_\epsilon \epsilon + \epsilon L_\epsilon^* L_\epsilon - 2 L_\epsilon^* \epsilon L_\epsilon\bigg)$$

$\alpha_x, \alpha_\epsilon$ are given by:
$\alpha_x = [L_x^*, X]S$
$\alpha_\epsilon = [L_\epsilon^*, \epsilon]S$

and finally $\lambda_x, \lambda_\epsilon$ are given by:
$\lambda_x = S^*XS - X$
$\lambda_\epsilon = S^*\epsilon S - \epsilon$

By assumption 2.2 we can form a hedge portfolio which we now use:
$V_t = a_{x,t}j_t(X) + a_{\epsilon,t}j_t(\epsilon) + b_t\beta$, for risk free numeraire asset β.
$dV_t = a_{x,t}dj_t(X) + a_{\epsilon,t}dj_t(\epsilon) + b_t\beta r dt$

Applying the unitary time development operators for ϵ and x we have:

$$\begin{aligned} dV_t = a_{x,t}\big(j_t(\alpha_x^\dagger)dA_x^\dagger + j_t(\lambda_x)d\Lambda_x + j_t(\alpha_x)dA_x\big) \\ +a_{\epsilon,t}\big(j_t(\alpha_\epsilon^\dagger)dA_\epsilon^\dagger + j_t(\lambda_\epsilon)d\Lambda_\epsilon + j_t(\alpha_\epsilon)dA_\epsilon\big) \\ +\big(j_t(\theta_x) + (V_t - a_{x,t}j_t(X) - a_{\epsilon,t}j_t(\epsilon))r\big)dt \end{aligned} \tag{2.5}$$

Equating the risky terms between equations (2.4), and (2.5) leads to the boundary conditions on $a_{x,t}$ and $a_{\epsilon,t}$. Similarly, equating the dt terms, leads to the Quantum Black-Scholes equation for this system: equation (2.3). □

Now, let $f\big(x,\epsilon\big)$ represent a vector in $\mathbb{H}$, and define the following operator X as multiplication by x: $Xf\big(x,\epsilon\big) = xf\big(x,\epsilon\big)$ and define operator S by:

$$Sf\big(x,\epsilon\big) = f\left(S.\begin{pmatrix} x \\ \epsilon \end{pmatrix}\right)$$

where S is given by:

$$S = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

Then we have:
$Sf\big(x,\epsilon\big) = f\big(\cos(\phi)x - \sin(\phi)\epsilon, \cos(\phi)\epsilon + \sin(\phi)x\big)$

$XSf = xf\big(\cos(\phi)x - \sin(\phi)\epsilon, \cos(\phi)\epsilon + \sin(\phi)x\big)$

$S^*XSf = \big(\cos(\phi)x + \sin(\phi)\epsilon\big)f(x,\epsilon)$

So, we end up with:
$\lambda_x = \Big(\big(\cos(\phi) - 1\big)x + \sin(\phi)\epsilon\Big)$, $\lambda_\epsilon = \Big(\big(\cos(\phi) - 1\big)\epsilon - \sin(\phi)x\Big)$.
Finally, inserting this back into equation (2.3), we get the Black-Scholes equation for the system (following notation from [1]):

Proposition 2.6. *Let $u(t,x,\epsilon)$ represent the price at time t, of a derivative contract in the system described above under rotation ϕ, and with interest rate r. Then the quantum Black-Scholes equation becomes:*

$$\begin{aligned}\frac{\partial u(t,x,\epsilon)}{\partial t} &= rx\frac{\partial u(t,x,\epsilon)}{\partial x} + r\epsilon\frac{\partial u(t,x,\epsilon)}{\partial \epsilon} - u(t,x,\epsilon)r \\ &+\sum_{k=2}^{\infty}\frac{((\cos(\phi)-1)x+\sin(\phi)\epsilon)^{k-2}}{k!}\frac{\partial^k u(t,x,\epsilon)}{\partial x^k}g_1(x,\epsilon) \\ &+\sum_{l=2}^{\infty}\frac{((\cos(\phi)-1)\epsilon-\sin(\phi)x)^{l-2}}{l!}\frac{\partial^l u(t,x,\epsilon)}{\partial \epsilon^l}g_2(x,\epsilon)\end{aligned} \tag{2.6}$$

Proof. We assume that the operators $L_x, L_x^*, L_\epsilon, L_\epsilon^*$ involve multiplication by a polynomial in x, ϵ, and therefore commute with $\lambda_x, \lambda_\epsilon$. Therefore, from the boundary conditions we have:
$\sum_{k=1}^{\infty} a_{0,k,0}(t, j_t(X), j_t(\epsilon))j_t(\lambda_x^{k-1}) = a_{x,t}$
$\sum_{l=1}^{\infty} a_{0,0,l}(t, j_t(X), j_t(\epsilon))j_t(\lambda_\epsilon^{l-1}) = a_{\epsilon,t}$

Inserting this into 2.3 gives:

$$\begin{aligned}a_{1,0,0}(t,j_t(X),j_t(\epsilon)) + a_{0,1,0}(t,j_t(X),j_t(\epsilon))j_t(X)r + a_{0,0,1}(t,j_t(X),j_t(\epsilon))j_t(\epsilon)r \\ +\sum_{k=2}^{\infty}a_{0,k,0}(t,j_t(X),j_t(\epsilon))j_t(\lambda_x^{k-2}(\alpha_x\alpha_x^* - \lambda_x(\theta_x - xr))) \\ +\sum_{l=2}^{\infty}a_{0,0,l}(t,j_t(X),j_t(\epsilon))j_t(\lambda_\epsilon^{l-2}(\alpha_\epsilon\alpha_\epsilon^* - \lambda_\epsilon(\theta_\epsilon - \epsilon r))) \\ = V_t r\end{aligned} \tag{2.7}$$

Now writing $g_1(x,\epsilon) = j_t(\alpha_x\alpha_x^* - \lambda_x(\theta_x - xr))$, $g_2(x,\epsilon) = j_t(\alpha_\epsilon\alpha_\epsilon^* - \lambda_\epsilon(\theta_\epsilon - \epsilon r))$, and $a_{0,k,0}(t,j_t(X),j_t(\epsilon)) = \frac{1}{k!}\frac{\partial^k u}{\partial x^k}, a_{0,0,l}(t,j_t(X),j_t(\epsilon)) = \frac{1}{l!}\frac{\partial^l u}{\partial \epsilon^l}$, we have the result given. □

For small rotations, we have $\cos(\varepsilon) = 1 - \frac{\varepsilon^2}{2} + o(\varepsilon^2)$, and $\sin(\phi) = \varepsilon + o(\varepsilon^2)$. Inserting this into equation (2.6), we have a new partial differential equation, where the coefficient of the kth partial derivative, for $k \geq 3$, with respect to x, ϵ, is correct to $o(\varepsilon^{2(k-2)})$. This form for small rotations is more amenable to the methods we apply in section 3.

$$\begin{aligned}\frac{\partial u(t,x,\epsilon)}{\partial t} &= rx\frac{\partial u(t,x,\epsilon)}{\partial x} + r\epsilon\frac{\partial u(t,x,\epsilon)}{\partial \epsilon} - u(t,x,\epsilon)r \\ &+\sum_{k=2}^{\infty}\frac{(\varepsilon\epsilon-(\varepsilon^2/2)x)^{k-2}}{k!}\frac{\partial^k u(t,x,\epsilon)}{\partial x^k}g_1(x,\epsilon) \\ &+\sum_{l=2}^{\infty}\frac{(-\varepsilon x-(\varepsilon^2/2)\epsilon)^{l-2}}{l!}\frac{\partial^l u(t,x,\epsilon)}{\partial \epsilon^l}g_2(x,\epsilon)\end{aligned} \tag{2.8}$$

As is the case for equation (2.2), this reduces to the classical Black-Scholes for 2 uncorrelated random variables (in this case price: x, and bid-offer spread: ϵ) when $\varepsilon = 0$.

For the classical case, the addition of the bid-offer spread is in some ways unnecessary when using the model for derivative pricing. For derivative contracts depending on the close price, one can usually hedge daily at the closing price during the end of day auction process. For many trading desks this may be sufficient in practice, and terms involving the bid-offer spread will drop out of the model. In the quantum case, examination of equations (2.6) and (2.8) shows that we expect interference between the bid-offer spread dynamics and the price dynamics. For small rotations, these equations are singular PDEs, and we expect the behaviour in most regions to approximate classical behaviour. However, when the higher derivative terms are larger, quantum interference may be significant. We discuss this more in sections 3 and 4.

3. Nonlocal Diffusions

In this section, we derive the Fokker-Planck equations associated to the Quantum Black-Scholes equations: (2.2), and (2.8). We show how these can be written in integral form, by using the Kramers-Moyal expansion (see for example [7]). This enables us to link the Quantum Black-Scholes models of the previous section to nonlocal diffusions (see for example the paper by Luczka, Hänggi and Gadomski: [14]). We assume zero interest rates in this section to help clarify the notation without changing the key dynamics. The integral form for the Fokker-Planck equations is given by:

$$\begin{aligned}\frac{\partial p(x,\epsilon,t)}{\partial t} &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(H(y_x,y_\epsilon|x,\epsilon)g_1(x,\epsilon)p(x-y_x,\epsilon-y_\epsilon,t)\big)dy_x dy_\epsilon\right)\\ &+\frac{1}{2}\frac{\partial^2}{\partial \epsilon^2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(H(y_x,y_\epsilon|x,\epsilon)g_2(x,\epsilon)p(x-y_x,\epsilon-y_\epsilon,t)\big)dy_x dy_\epsilon\right)\end{aligned} \tag{3.1}$$

The function $H(y_x, y_\epsilon|x, \epsilon)$ has the effect of "blurring" the impact of the diffusion operator. In the case that $H(y_x, y_\epsilon|x, \epsilon)$ is a Dirac delta function, the diffusion operator is localised as usual, and the associated Fokker-Planck equation reduces to the standard Kolmogorov forward equation associated with the classical Black-Scholes.

We start with the following general form for equations (2.2) and (2.8):

$$\begin{aligned}\frac{\partial u(t,x,\epsilon)}{\partial t} &= g_1(x,\epsilon)\sum_{k=2}^{\infty}\frac{f_1(x,\epsilon,\varepsilon)^{k-2}}{k!}\frac{\partial^k u(t,x,\epsilon)}{\partial x^k}\\ &+g_2(x,\epsilon)\sum_{l=2}^{\infty}\frac{f_2(x,\epsilon,\varepsilon)^{l-2}}{l!}\frac{\partial^l u(t,x,\epsilon)}{\partial \epsilon^l}\end{aligned} \tag{3.2}$$

Proposition 3.1. *The Fokker-Planck equation associated to equation (3.2), with $r = 0$ is given by:*

$$\begin{aligned}\frac{\partial p(x,\epsilon,t)}{\partial t} &= \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k \big(g_1(x,\epsilon) f_1(x,\epsilon,\varepsilon)^{k-2} p(x,\epsilon,t)\big)}{\partial x^k} \\ &+ \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \frac{\partial^l \big(g_2(x,\epsilon) f_2(x,\epsilon,\varepsilon)^{l-2} p(x,\epsilon,t)\big)}{\partial \epsilon^l}\end{aligned} \tag{3.3}$$

Proof. For a derivative payout $h(x,\epsilon)$, with zero interest rates, we have the following price in risk neutral measure Q:

$$u(x_t,\epsilon_t,t) = E^Q\big[h(x_T,\epsilon_T)\big] = \int_{\mathbb{R}^2} h(y_x,y_\epsilon) p(y_x,y_\epsilon|x,\epsilon,t) dy_x dy_\epsilon$$

where $p(y_x,y_\epsilon|x,\epsilon,t)$ represents the risk neutral probability density for the variables observed at time T, conditional on the values at time t. $h(x,\epsilon)$ represents a derivative payout at T. We then write the right hand integral as:

$$\int_{\mathbb{R}^2} g(y_x,y_\epsilon) p(y_x,y_\epsilon|x,\epsilon,t) dy_x dy_\epsilon = \int_0^t \int_{\mathbb{R}^2} Lh(y_x,y_\epsilon) p(y_x,y_\epsilon|x,\epsilon,s) dy_x dy_\epsilon ds$$

where L represents the operator:

$$Lh(x,\epsilon) = \left(g_1(x,\epsilon) \sum_{k=2}^{\infty} \frac{f_1(x,\epsilon,\varepsilon)^{k-2}}{k!} \frac{\partial^k}{\partial x^k} + g_2(x,\epsilon) \sum_{l=2}^{\infty} \frac{f_2(x,\epsilon,\varepsilon)^{l-2}}{l!} \frac{\partial^l}{\partial \epsilon^l} \right) h(x,\epsilon)$$

The Fokker-Planck equation, is given by the adjoint operator L^*. Therefore, since:

$$\int_0^t \int_{\mathbb{R}^2} Lh(y_x,y_\epsilon) p(y_x,y_\epsilon|x,\epsilon,s) dy_x dy_\epsilon ds = \int_0^t \int_{\mathbb{R}^2} h(y_x,y_\epsilon) L^* p(y_x,y_\epsilon|x,\epsilon,s) dy_x dy_\epsilon ds,$$

if we truncate equation (3.2) at a certain order for the derivative: N, the result follows by integrating by parts N times. Proceeding with higher and higher N, we can match the derivative terms of any arbitary order, and the result follows. □

The objective now, is to write equation (3.3) in the form of (3.1). To do this we can follow a Moment Matching algorithm. We use the following expansion:

$$g(x,\epsilon) p(x-y_x,\epsilon-y_\epsilon,t) = \sum_{i,j=0}^{\infty} \frac{(-1)^{(i+j)}}{(i+j)!} y_x^i y_\epsilon^j \frac{\partial^{i+j}(g(x,\epsilon)p(x,\epsilon,t))}{\partial x^i \partial \epsilon^j}$$

Inserting this into equation (3.1) gives:

$$\begin{aligned}&\frac{\partial p(x,\epsilon,t)}{\partial t} = \\ &\frac{1}{2}\frac{\partial^2}{\partial x^2}\Bigg(\sum_{i,j=0}^{\infty} \frac{(-1)^{(i+j)}}{(i+j)!} \frac{\partial^{i+j}(g_1(x,\epsilon)p(x,\epsilon,t))}{\partial x^i \partial \epsilon^j} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(y_x,y_\epsilon|x,\epsilon) y_x^i y_\epsilon^j dy_x dy_\epsilon \Bigg) \\ &+\frac{1}{2}\frac{\partial^2}{\partial \epsilon^2}\Bigg(\sum_{i,j=0}^{\infty} \frac{(-1)^{(i+j)}}{(i+j)!} \frac{\partial^{i+j}(g_2(x,\epsilon)p(x,\epsilon,t))}{\partial x^i \partial \epsilon^j} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(y_x,y_\epsilon|x,\epsilon) y_x^i y_\epsilon^j dy_x dy_\epsilon \Bigg)\end{aligned} \tag{3.4}$$

Now by equating the coefficients of the derivatives with respect to x and ϵ, between equations (3.4) and (3.3) one can calculate the moments of the "blurring" function $H(y_x, y_\epsilon|x, \epsilon)$. For the translation case, $g_2(x, \epsilon) = 0$, and the probability density is a function of x only.

3.1. Moment Matching: Translation Case. In the translation case, of section 2.1, since the coefficients of each differential term in equation (2.2) is a constant multiplied by $g(x)$, the moments of the "blurring" function $H(y)$ will not depend of x. Equation (3.4) becomes:

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}\left(\sum_{j=0}^{\infty}\frac{(-1)^{(j)}}{j!}\frac{d^{(j+2)}(g(x)p(x,t))}{dx^{(j+2)}}\int_{-\infty}^{\infty}H(y)y^j\,dy\right) \tag{3.5}$$

Similarly, the Fokker-Planck associated with equation (2.2), with $r = 0$, is given by:

$$\frac{\partial p(x,t)}{\partial t} = \sum_{k=2}^{\infty}\frac{(-1)^k\varepsilon^{k-2}}{k!}\frac{\partial^k(g(x)p(x,t))}{\partial x^k} \tag{3.6}$$

Now the moments of the "blurring" function can be matched by equating directly equations (3.5) and (3.6):

Proposition 3.2. *Let H_i represent the i^{th} moment of $H(y)$, for the Fokker-Planck equation (3.3), relating to the translation case described in section 2.1. Then, H_i is given by:*
$H_i = \frac{2(\varepsilon)^i}{(i+1)(i+2)}$

Proof. H_i follows (for $i \geq 0$) by equating the coefficients for: $\frac{\partial^{(i+2)}}{\partial x^{(i+2)}}$, between equations (3.5) and (3.6). Comparing the coefficients of the 2nd partial derivative with respect to x gives the zeroth moment. From equations (3.5) and (3.5), we have:

$\frac{1}{2}\frac{\partial^2(g(x)p(x,t))}{\partial x^2} = \frac{1}{2}\frac{\partial^2(g(x)p(x,t))}{\partial x^2}H_0$

So $H_0 = 1$. Similarly, for the $(i+2)th$ partial derivative, we have:

$\frac{(-1)^{i+2}}{i+2!}\frac{\varepsilon^i\partial^{i+2}(g(x)p(x,t))}{\partial x^{i+2}} = \frac{(-1)^i}{i!}\frac{\partial^{i+2}(g(x)p(x,t))}{\partial x^{i+2}}H_i$

So we get: $H_i = \frac{2(\varepsilon)^i}{(i+1)(i+2)}$ as required. □

We find that, in this case, $H(y)$ is a normalised function that tends to a Dirac function as ε tends to zero, and for $\varepsilon = 0$ we end up with classical 2nd order Fokker-Planck equation. This is discussed further in section 4.

3.2. Moment Matching: Rotation Case. In the rotation case of section 2.2, the coefficients of each differential term in equation (3.3) are functions of x and ϵ. Therefore, we require the moments for the "blurring" function also to be functions of x, and ϵ: $H(y_x, y_\epsilon|x,\epsilon)$. Once we have calculated the coefficients for the differential terms, we can use these to form an inhomogeneous 2nd order differential equation for the moments of $H(y_x, y_\epsilon|x,\epsilon)$.

In this case, from equation (3.3) we have: $f_1(x,\epsilon) = \varepsilon\epsilon - (\varepsilon^2/2)x$, and $f_2(x,\epsilon) = -\varepsilon x - (\varepsilon^2/2)\epsilon$. Therefore, the Fokker-Planck equation associated with equation (2.8), with $r=0$, is given by:

$$\begin{aligned}\frac{\partial p(x,\epsilon,t)}{\partial t} &= \sum_{k=2}^{\infty}\frac{1}{k!}\frac{\partial^k\Big(\big((\varepsilon^2/2)x-\varepsilon\epsilon\big)^{k-2}g_1(x,\epsilon)p(x,\epsilon,t)\Big)}{\partial x^k} \\ &+\sum_{l=2}^{\infty}\frac{1}{l!}\frac{\partial^l\Big(\big(\varepsilon x+(\varepsilon^2/2)\epsilon\big)^{l-2}g_2(x,\epsilon)p(x,\epsilon,t)\Big)}{\partial \epsilon^l}\end{aligned} \tag{3.7}$$

The moments of the "blurring" function will now follow by equating coefficients for the differential terms between equations (3.4), and (3.7).

Proposition 3.3. *Assume the moments of the "blurring" function: $H(y_x, y_\epsilon|x,\epsilon)$ are given by:*

$a_x^i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(y_x, y_\epsilon|x,\epsilon)y_x^i dy_x dy_\epsilon$
$a_\epsilon^j = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(y_x, y_\epsilon|x,\epsilon)y_\epsilon^j dy_x dy_\epsilon$

Further, define a^0, a_x^1, a_ϵ^1 as: $a^0 = 1, a_x^1, a_\epsilon^1 = 0$. Then for the higher moments we have, for $n \geq 2$:

$$\frac{(-1)^n a_x^{n-2} + 2n\frac{\partial a_x^{n-1}}{\partial x} + n(n-1)\frac{\partial^2 a_x^n}{\partial x^2}}{n!} = \frac{((\varepsilon^2/2)x - \varepsilon\epsilon)^{n-2}}{n(n-1)(1-(\varepsilon^2/2))^{(n-1)}} \tag{3.8}$$

$$\frac{(-1)^n a_\epsilon^{n-2} + 2n\frac{\partial a_\epsilon^{n-1}}{\partial \epsilon} + n(n-1)\frac{\partial^2 a_\epsilon^n}{\partial \epsilon^2}}{n!} = \frac{((\varepsilon^2/2)\epsilon + \varepsilon x)^{n-2}}{n(n-1)(1-(\varepsilon^2/2))^{(n-1)}} \tag{3.9}$$

Proof. We first calculate the coefficients for $\frac{\partial^n(g_1(x,\epsilon)p(x,\epsilon))}{\partial x^n}$ from equation (3.7). The 2nd order coefficient is given by:

$\sum_{i\geq 2}\frac{(i-2)!(\varepsilon^2/2)^{i-2}\binom{i}{2}}{i!} = \frac{1}{2}\sum_{i\geq 0}(\varepsilon^2/2)^i = \frac{1}{2(1-(\varepsilon^2/2))}$

Similarly, the 3rd order coefficient is given by:

$\sum_{i\geq 3}\frac{(i-2)!(\varepsilon^2/2)^{(i-2)}\binom{i}{3}((\varepsilon^2/2)x-\varepsilon\epsilon)}{i!} = \frac{((\varepsilon^2/2)x-\varepsilon\epsilon)}{3!}\sum_{i\geq 0}(i+1)(\varepsilon^2/2)^i$
$= \frac{((\varepsilon^2/2)x-\varepsilon\epsilon)}{3!(1-(\varepsilon^2/2))^2}$

In general, the nth order coefficient is given by:

$\sum_{i\geq n} \frac{(i-2)!(\varepsilon^2/2)^{(i-2)}\binom{i}{n}((\varepsilon^2/2)x-\varepsilon\epsilon)^{n-2}}{i!(n-2)!}$
$= \frac{((\varepsilon^2/2)x-\varepsilon\epsilon)^{(n-2)}}{n!}\sum_{i\geq 0}(i+1)(i+2)...(i+n-2)(\varepsilon^2/2)^i$

The final summation can be calculated by differentiating $(n-2)$ times, the infinite sum $1/(1-v)$, where $v = (\varepsilon^2/2)$.

Therefore, the coefficient for $n \geq 2$ is given by:

$$\frac{((\varepsilon^2/2)x-\varepsilon\epsilon)^{n-2}}{n(n-1)(1-(\varepsilon^2/2))^{(n-1)}}\frac{\partial^n(g_1(x,\epsilon)p(x,\epsilon))}{\partial x^n} \tag{3.10}$$

Following similar logic for ϵ we have the coefficient:

$$\frac{((\varepsilon^2/2)\epsilon+\varepsilon x)^{n-2}}{n(n-1)(1-(\varepsilon^2/2))^{(n-1)}}\frac{\partial^n(g_2(x,\epsilon)p(x,\epsilon))}{\partial \epsilon^n} \tag{3.11}$$

These coefficients can now be used to calculate a 2nd order inhomogeneous differential equation for the moments of $H(y_x, y_\epsilon|x,\epsilon)$. We start by expanding the $\partial^2/\partial x^2$, and $\partial^2/\partial\epsilon^2$ in equation (3.4).

Since, we assume from section 2.2, that x, ϵ are uncorrelated, equation (3.4) can be written:

$$\begin{aligned}\frac{\partial p(x,\epsilon,t)}{\partial t} = \frac{1}{2}\sum_{i=0}^{\infty}\frac{(-1)^{(i)}}{i!}\Bigg(&\frac{\partial^i(g_1(x,\epsilon)p(x,\epsilon,t))}{\partial x^i}\frac{\partial^2 a_x^i}{\partial x^2} + \frac{\partial^{i+2}(g_1(x,\epsilon)p(x,\epsilon,t))}{\partial x^{i+2}}a_x^i \\ &+2\frac{\partial^{i+1}(g_1(x,\epsilon)p(x,\epsilon,t))}{\partial x^{i+1}}\frac{\partial a_x^i}{\partial x}\Bigg) \\ +\frac{1}{2}\sum_{j=0}^{\infty}\frac{(-1)^{(j)}}{j!}\Bigg(&\frac{\partial^j(g_2(x,\epsilon)p(x,\epsilon,t))}{\partial \epsilon^j}\frac{\partial^2 a_\epsilon^j}{\partial \epsilon^2} + \frac{\partial^{j+2}(g_1(x,\epsilon)p(x,\epsilon,t))}{\partial \epsilon^j}a_\epsilon^j \\ &+2\frac{\partial^{j+1}(g_1(x,\epsilon)p(x,\epsilon,t))}{\partial \epsilon^{j+1}}\frac{\partial a_\epsilon^j}{\partial \epsilon}\Bigg)\end{aligned} \tag{3.12}$$

The coefficients for $\frac{\partial^n(g_1(x,\epsilon)p(x,\epsilon,t))}{\partial x^n}$ from equation (3.12) are now given by:

$\frac{\partial^2 a^0}{\partial x^2}(g_1(x,\epsilon)p(x,\epsilon,t))$ for $n=0$,

$(\frac{\partial^2 a_x^1}{\partial x^2} + 2\frac{\partial a^0}{\partial x})\frac{\partial(g_1(x,\epsilon)p(x,\epsilon,t))}{\partial x}$ for $n=1$, and:

$$\frac{(-1)^n a_x^{n-2} + 2n\frac{\partial a_x^{n-1}}{\partial x} + n(n-1)\frac{\partial^2 a_x^n}{\partial x^2}}{n!}\frac{\partial^n(g_1(x,\epsilon)p(x,\epsilon,t))}{\partial x^n} \tag{3.13}$$

for $n \geq 2$. Similarly, for ϵ we have:

$$\frac{(-1)^n a_\epsilon^{n-2} + 2n\frac{\partial a_\epsilon^{n-1}}{\partial \epsilon} + n(n-1)\frac{\partial^2 a_\epsilon^n}{\partial \epsilon^2}}{n!}\frac{\partial^n(g_2(x,\epsilon)p(x,\epsilon,t))}{\partial \epsilon^n} \tag{3.14}$$

We now make the assumption that H is a normalised probability distribution with expectation zero for x and ϵ. Ie, $\frac{\partial a_0}{\partial x} = 0$, $a_x^1 = 0$, and $a_\epsilon^1 = 0$. These assumptions ensure the coefficients with $n = 0, 1$ equate to zero on both sides of equation (3.7). The proposition follows by equating equations (3.10)/(3.13) and (3.11)/(3.14). $\square$

4. Monte-Carlo Methods & Numerical Simulations

In this section, we give a brief overview of McKean stochastic differential equations, before introducing how the particle method, discussed in the book by Guyon & Henry-Labordère: [8], can be used in their simulation. We then go on to present numerical results from the bid-offer model discussed above, placing particular emphasis on understanding how quantum effects become apparent through small transformations applied to a classical Black-Scholes system.

4.1. McKean Stochastic Differential Equations. McKean nonlinear stochastic differential equations were introduced in [18], and refer to SDEs, where the drift & volatility coefficients depend on the underlying probability law for the stochastic process. Following notation from [8] we have:
$dX_t = b(t, X_t, \mathbb{P}_t)dt + \sigma(t, X_t, \mathbb{P}_t)dW_t$

These are then related to the nonlinear Fokker Planck equation:

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sum_{i,j}\frac{\partial^2(\sigma_i(t,x,\mathbb{P}_t)\sigma_j(t,x,\mathbb{P}_t)p(t,x))}{\partial x_i \partial x_j} - \sum_i \frac{\partial(b^i(t,x,\mathbb{P}_t))}{\partial x_i} \tag{4.1}$$

In this case, we can write equation (3.1) in this form. We have for $r = 0$, $b^1(t, x, \epsilon, \mathbb{P}_t) = b^2(t, x, \epsilon, \mathbb{P}_t) = 0$ and

$\sigma_1(t, x, \epsilon, \mathbb{P}_t) = \sqrt{g_1(x,\epsilon)\mathbb{E}^p\left[\frac{H(x-y_x,\epsilon-y_\epsilon|x,\epsilon)}{p(x,\epsilon,t)}\right]}$

$\sigma_2(t, x, \epsilon, \mathbb{P}_t) = \sqrt{g_2(x,\epsilon)\mathbb{E}^p\left[\frac{H(x-y_x,\epsilon-y_\epsilon|x,\epsilon)}{p(x,\epsilon,t)}\right]}.$

Therefore, we can simulate the solution to equation (3.1) by first calculating the function $H(x-y_x, \epsilon-y_\epsilon)$ using a moment matching algorithm, and then simulating the following McKean SDE, with uncorrelated Wiener processes dW^1, dW^2:

$$\begin{aligned} dx &= \sqrt{\frac{g_1(x,\epsilon)}{p(x,\epsilon,t)}\mathbb{E}^{p(y)}\left[H(x-y_x,\epsilon-y_\epsilon|x,\epsilon)\right]}dW^1 \\ d\epsilon &= \sqrt{\frac{g_2(x,\epsilon)}{p(x,\epsilon,t)}\mathbb{E}^{p(y)}\left[H(x-y_\epsilon,\epsilon-y_\epsilon|x,\epsilon)\right]}dW^2 \end{aligned} \tag{4.2}$$

The simulation of the above SDE relies on the *particle method* outlined in Guyon & Henry-Labordère's book *Nonlinear Option Pricing* chapters 10, 11 (cf: [8]).

Each path (x^i, ϵ^i) now interacts with the other paths: $(x^j, \epsilon^j), j \neq i$ during the simulation process, and the convergence of the method relies on the so called *propagation of the chaos* property. This states:

Definition 4.1. For all functions $\phi(x,\epsilon,t) \in C_0(\mathbb{R}^2)$:

$$\frac{1}{N}\sum_{j=1}^{N}\phi(x^j,\epsilon^j) \xrightarrow{N\to\infty} \int_{\mathbb{R}^2}\phi(x,\epsilon,t)p(x,\epsilon,t)dxd\epsilon \tag{4.3}$$

In our case, the SDE (4.2), is a McKean-Vlasov process, and we have from Guyon, Henry-Labordère (cf: [8] Theorem 10.3), and originally Sznitman (cf: [22]), that the propagation of the chaos property holds.

4.2. Particle Method. The first step is to discretize the SDE: (4.2), as follows:

$$\begin{aligned} dx^i &= \left(\sum_{j=1}^{N} H(x^j - x^i, \epsilon^j - \epsilon^i)\frac{P(x^j,\epsilon^j)}{P(x^i,\epsilon^i)}g_1(x^i,\epsilon^i)\right)^{0.5} dW^{1,i} \\ d\epsilon^i &= \left(\sum_{j=1}^{N} H(x^j - x^i, \epsilon^j - \epsilon^i)\frac{P(x^j,\epsilon^j)}{P(x^i,\epsilon^i)}g_2(x^i,\epsilon^i)\right)^{0.5} dW^{2,i} \end{aligned} \tag{4.4}$$

where $P(x^j,\epsilon^j)$ represents a suitably discretized probability function. The algorithm then proceeds as follows:

(1) Solve for the moments of the "blurring" function $H(x - y_x, \epsilon - y_\epsilon | x, \epsilon)$ using propositions 3.2, and 3.3.
(2) Choose a parameterised distribution to approximate $H(x-y_x,\epsilon-y_\epsilon|x,\epsilon)$, and fit the parameters using the calculated moments. For example, approximate $H(x-y_x,\epsilon-y_\epsilon|x,\epsilon)$ as a univariate/bivariate normal distribution.
(3) Simulate the 1st time step, t_1, using the value of $H(0,0|x_0,\epsilon_0)$, for starting positions x_0,ϵ_0.
(4) After each simulation, allocate the simulated paths into discrete probability buckets: $P(x^j,\epsilon^j)$, for paths $j = 1$ to N.
(5) Proceed from the t_{k-1} to t_k time-step, using (4.4), the value of $H(x - y_x, \epsilon - y_\epsilon|x,\epsilon)$, and the discrete buckets at t_{k-1}.
(6) Iterate steps 4 & 5 until the final maturity: t_F.

4.3. Modelling the Market Fear Factor. We can see from (4.4), that small translations, will lead to a variance scaling factor:
$\sum_{j=1}^{N} H(x^j - x^i, \epsilon^j - \epsilon^i)\frac{P(x^j,\epsilon^j)}{P(x^i,\epsilon^i)}$

This will have the impact of reducing the volatility of those paths which lie in the middle of the "bell curve", owing to the negative curvature of the probability law at these points - probability mass is spread by the "blurring" function to lower probability points.

Similarly, at the extremes of the probability density curve where the curvature is positive, probability mass is spread to areas with net higher probability. In essence the market memory of a recent extreme event, will lead to a higher market volatility at the next time step.

This effect differs from the negative skew observed in local volatility models (for example the work by Dupire: cf [6]), and from stochastic volatility models (for example Heston: [11]), in the sense that the increase in volatility is linked to recent random moves in the tail of the probability distribution, rather than to the level of the stochastic volatility or a static function of the price, and time.

To highlight the difference, in the process given by equation (4.4), one could allow for periodic rebalancing of the process. For example, one could replace the unconditional probability, with the probability conditional on the previous step. In this way, the level of the volatility would depend purely on a "memory" of recent price history, rather than on the absolute level of the market price, or an additional random variable. The market responds to large moves with a heightened fear factor. The study of modelling such processes with rebalancing, will involve advanced techniques for calculating the conditional probabilities, and we defer detailed study to a future work.

4.4. Numerical Results. In this section, we simulate the one-factor process described in section 2.1, and 3.1. In this case, we approximate $H(y)$ using a normal distribution using the moments from proposition 3.2: $N(\frac{\varepsilon}{3}, \frac{\varepsilon^2}{18})$.

The non-zero 1st moment, will lead to an upside/downside bias to the "market fear factor" effect. Essentially, by introducing a translation in the negative x direction, one introduces downside 'fear' into the model.

Figures 1 & 2 below, illustrate the results from a 2 step Monte-Carlo process, with $g(x) = 0.01x^2$, starting value: $x_0 = 1$, 100K Monte-Carlo paths, and 500 discrete probability buckets. The scatter plot shows the magnitude of the proportional return on the 1st time-step on the horizontal axis, and the second time-step on the vertical axis:

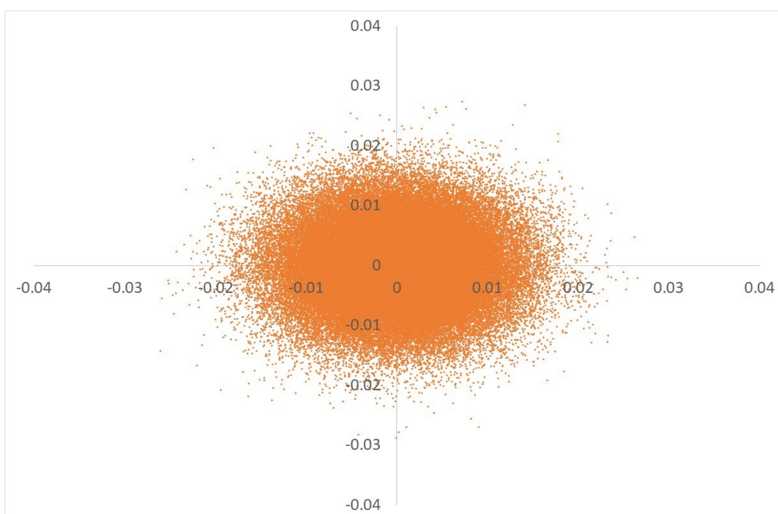

FIGURE 1. Figure 1: $\varepsilon = 0$, horizontal axis represents the proportional return for the first time-step, vertical axis represents the second time-step.

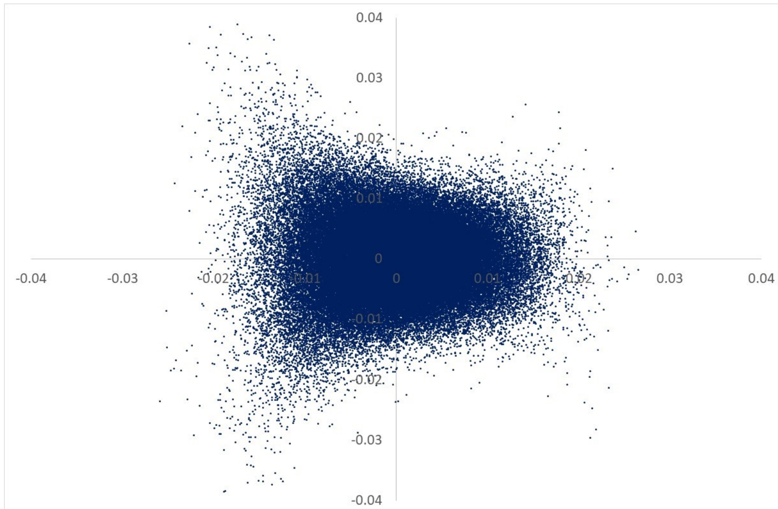

FIGURE 2. Figure 2: The results for $\varepsilon = 0.02$, horizontal axis represents the proportional return for the first time-step, vertical axis represents the second time-step.

Figure 1 shows the results for $\varepsilon = 0$. This is a classical Black-Scholes system, and there is no correlation between the magnitude & direction of the 1st and 2nd time-steps.

Figure 2 shows the proportional returns for $\varepsilon = 0.02$. The volatility of the second step is reduced on those paths where the first time-step has been small. There is a slight increased second step volatility for those paths with large positive first steps, and significant second step volatility for those paths with a large negative first step. In effect, the drop in market prices has introduced "fear" into these paths.

The final chart (next page) shows the probability distributions for the natural logarithm of the simulated value after 50 one day time-steps. The non-zero translation results in a natural skewness in the distribution.

5. Conclusions

In this paper, we demonstrate how unitary transformations can be used to model novel quantum effects in the Quantum Black-Scholes system of Accardi & Boukas (cf [1]).

We show how these quantum stochastic processes can also be modelled using nonlocal diffusions, and simulated using the particle method outlined by Guyon & Henry-Labordère in [8].

By introducing a bid-offer spread parameter, and extending the Accardi-Boukas framework to 2 variables, we show how rotations, in addition to translations, can be applied. Thus, a richer representation of the information contained in the current market leads to a wider variety of unitary transformations that can be used.

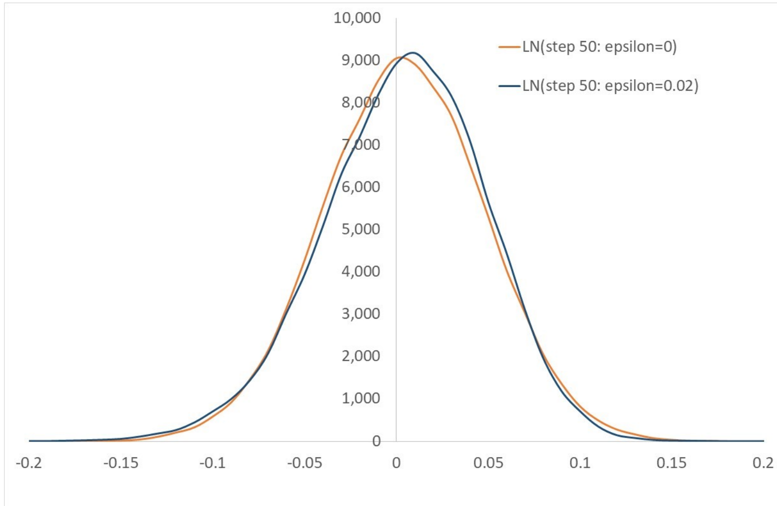

Figure 3. Figure 3: Distribution for the natural log of the final price after 50 one day time-steps. 100K Monte-Carlo paths, and 500 discrete probability buckets.

In section 4, using a Monte-Carlo simulation, we illustrate how introducing a translation to the one dimensional model leads to a skewed distribution, whereby recent market down moves leads to increased volatility going forward. In effect, the market retains memory of recent significant moves.

In [6], Dupire shows how to calibrate a local volatility to the current vanilla option smile. This enables a Monte-Carlo simulation that is fully consistent with current market option prices. Carrying out the same analysis, using the new Quantum Fokker-Planck equations, is another important next step to consider as a future development of the current work.

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