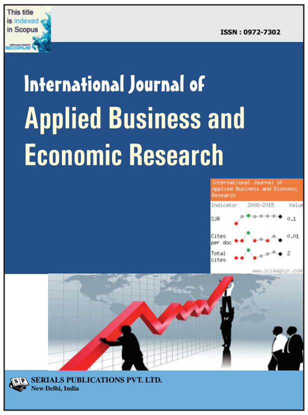


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# Mixed Lagrangian and Multiobjective Fractional Programming Duality with Generalized $\mathfrak{I}$-Convex n-Set Functions

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**ABSTRACT**

***Abstract:*** A vector - valued ratio type mixed Lagrange function is introduced to study mixed saddle point optimality criteria for a class of multiobjective fractional programming problems involving differentiable n-set functions. Further, a mixed Bector type dual is proposed and duality results are established under generalized $(\mathfrak{I},\rho,\theta)$-convexity assumptions on the functions.

***Keywords:*** Fractional programming, n-set functions, Lagrange function, Saddle point, Duality results.

***AMS Classification: Primary***: 90C32; Secondary: 49N15, 26B25.

## 1. INTRODUCTION

Optimization problems involving set functions are extensively studied in the recent years. These problems arise in various areas and have many interesting applications, for example, in fluid flow [4], electric insulator design [5], plasma confinement [5], and many more. First theory of optimizing set functions was developed by Morris [12]. Subsequently several authors [2, 8, 10] have made significant contributions to study duality results for various optimization problems containing n-set functions under different set ups. In [13], Preda introduced $(\mathfrak{I},\rho)$-convexity for n-set functions which was defined by using a sublinear functional. Later, Jo, Kim and Lee [9] extended the concept of $(\mathfrak{I},\rho)$-convexity to generalized $(\mathfrak{I},\rho,\theta)$-convexity for n-set functions and established several sufficient optimality conditions for multiobjective programming problem with inequality and equality constraints. Recently, Preda and Batatorescu [14] established duality results for minmax generalized B-vex program involving n-set functions.

Different types of duals are proposed in the literature. The most significant ones are given by Wolfe [15] and Mond and Weir [11]. Since then several authors have studied Wolfe duality and Mond-Weir duality for different types of optimization problems under different set ups. Recently, it was remarked by Bector and Chandra [3] that while Wolfe dual originated from the Lagrange function of the problem under consideration, the formulation of Mond-Weir dual from the Lagrange function is not fully explored. This led them to introduce the new concept of mixed Lagrange function to study saddle point optimality criteria for scalar- valued nonlinear programming problem under convexity assumptions.

In this paper, the concept of mixed Lagrange function and mixed saddle point for generalized fractional programming problem involving n-set functions is introduced. Duality results are established under generalized $(\Im, \rho, \theta)$-convexity assumptions.

## 2. PRELIMINARIES

Throughout the paper, we assume that $(X, \mathcal{A}, \mu)$ is a finite atomless measure space with $L_1(X, \mathcal{A}, \mu)$ separable. $\boldsymbol{\mathcal{A}^n}$ is n-fold product of a σ-algebra $\mathcal{A}$ of subsets of set X, a pseudometric d on $\boldsymbol{\mathcal{A}^n}$ is defined by

$$d(S, T) = \left(\sum_{i=1}^{n} \mu(S_i \Delta T_i)^2\right)^{1/2}$$

where $S = (S_1, S_2, \ldots, S_n) \in \boldsymbol{\mathcal{A}^n}$, $T = (T_1, T_2, \ldots, T_n) \in \boldsymbol{\mathcal{A}^n}$, and $S_i \Delta T_i$ denotes symmetric difference of sets $S_i$ and $T_i$. For $h \in L_1(X, \boldsymbol{\mathcal{A}}, \mu)$ and $S_i \in \boldsymbol{\mathcal{A}}$, the integral $\int_{S_i} h\, d\mu$ will be denoted by $\langle h, \chi_{S_i} \rangle$, where $\chi_{S_i}$ is the characteristic function of $S_i$.

The following definitions from Jo, Kim and Lee [9] are used in the sequel.

**Definition 2.1** A functional $\Im$ on $\boldsymbol{\mathcal{A}^n} \times \boldsymbol{\mathcal{A}^n} \times L_1^n(X, \boldsymbol{\mathcal{A}^n}, \mu)$ is said to be sublinear in its third argument if for any $S, T \in \boldsymbol{\mathcal{A}^n}$,

$$\Im(S, T; \eta_1 + \eta_2) \leqq \Im(S, T; \eta_1) + \Im(S, T; \eta_2) \qquad \forall\ \eta_1, \eta_2 \in L_1^n(X, \boldsymbol{\mathcal{A}^n}, \mu)$$

$$\Im(S, T; \alpha\eta) = \alpha\, \Im(S, T; \eta) \qquad \forall\ \alpha \geqq 0,\ \alpha \in \mathbf{R}\ \forall\ \eta \in L_1^n(X, \boldsymbol{\mathcal{A}^n}, \mu)$$

**Definition 2.2** Let $\Im$ be a sublinear functional on $\boldsymbol{\mathcal{A}^n} \times \boldsymbol{\mathcal{A}^n} \times L_1^n(X, \boldsymbol{\mathcal{A}^n}, \mu)$. Let the function $F: \boldsymbol{\mathcal{A}^n} \to \mathbf{R}$ be differentiable n-set function, $\theta: \boldsymbol{\mathcal{A}^n} \times \boldsymbol{\mathcal{A}^n} \to \boldsymbol{\mathcal{A}^n} \times \boldsymbol{\mathcal{A}^n}$ with $\theta(S, S^*) \neq 0$, $\forall\ S \neq S^*$ and $\rho \in \mathbf{R}$.

(i) F is said to be $(\Im, \rho, \theta)$-convex at $S^*$ if

$$F(S) - F(S^*) \geqq \Im(S, S^*; DF_{S^*}) + \rho\, d^2(\theta(S, S^*)).$$

(ii) F is said to be $(\Im, \rho, \theta)$-quasiconvex at $S^*$ if for each $S \in \boldsymbol{\mathcal{A}^n}$ such that

$$F(S) \leqq F(S^*), \text{ we have, } \Im(S, S^*; DF_{S^*}) \leqq -\rho\, d^2(\theta(S, S^*)).$$

(iii) F is said to be $(\Im, \rho, \theta)$-pseudoconvex at $S^*$ if for each $S \in \boldsymbol{\mathcal{A}^n}$ such that

$$\Im(S, S^*; DF_{S^*}) \geqq -\rho\, d^2(\theta(S, S^*)) \text{ ,we have, } F(S) \geqq F(S^*).$$

(iv) F is said to be strictly $(\Im, \rho, \theta)$-pseudoconvex at $S^*$ if for each $S \in \boldsymbol{A^n}$,

$$S \neq S^*, \text{ such that } \Im(S, S^*; DF_{S^*}) \geqq -\rho\, d^2(\theta(S, S^*)) \text{ , we have, } F(S) > F(S^*).$$

Following multiobjective fractional programming problem is studied in this paper:

(P) V-minimize $F(S)/G(S) = (F_1(S)/G_1(S), F_2(S)/G_2(S), \ldots, F_p(S)/G_p(S))$

subject to $H_j(S) \leqq 0,\ 1 \leqq j \leqq m,$

where $F = (F_1, F_2, \ldots, F_p) : \boldsymbol{A^n} \to \mathbf{R^p}$, $G = (G_1, G_2, \ldots, G_p) : \boldsymbol{A^n} \to \mathbf{R^p}$, $H = (H_1, H_2, \ldots, H_m) : \boldsymbol{A^n} \to \mathbf{R^m}$ are differentiable n-set functions on $\boldsymbol{A^n}$, and minimization is taken in terms of efficient solutions as defined below. Further, assume that for each i, $1 \leqq i \leqq p$ and for every $S \in \boldsymbol{A^n}$, $F_i(S) \geqq 0$ and $G_i(S) > 0$.

Let $\Gamma = \{S \in \boldsymbol{A^n} \mid H_j(S) \leqq 0, 1 \leqq j \leqq m\}$ denotes the set of feasible solutions of (P).

We now give the following definitions from Bector et al. [2]

**Definition 2.3** A point $S^* \in \Gamma$ is said to be an efficient solution of (P) if $\nexists$ $S \in \Gamma$ such that

$$F_i(S)/G_i(S) \leqq F_i(S^*)/G_i(S^*),\ \forall\, i, 1 \leqq i \leqq p, i \neq k$$

$$F_k(S)/G_k(S) < F_k(S^*)/G_k(S^*) \text{ for some k.}$$

For a vector maximization problem the above definition is modified analogously.

**Definition 2.4** A point $S^* \in \Gamma$ is said to be a regular efficient solution of (P) if there exists $\hat{S} \in \boldsymbol{A^n}$ such that

$$H_j(S^*) + \sum_{r=1}^{n} \left\langle D_r H_{j_{S^*}}, \chi_{\hat{S}_r} - \chi_{S_r^*} \right\rangle < 0\ \ \forall\, j, 1 \leqq j \leqq m.$$

Since for each i, $1 \leqq i \leqq p$ and for every $S \in \boldsymbol{A^n}$, $G_i(S) > 0$, hence, problem (P) can be equivalently written as follows:

(EP) V-minimize $F(S)/G(S) = (F_1(S)/G_1(S), F_2(S)/G_2(S), \ldots, F_p(S)/G_p(S))$

subject to $\quad Q_{ji}(S) = H_j(S)/\ G_i(S) \leqq 0,\ 1 \leqq j \leqq m,\ 1 \leqq i \leqq p.$

Let Q(S) be a matrix of constraint functions of (EP) of order $m \times p$ with $i^{th}$ column given by

$$Q_i(S) = (Q_{1i}(S), Q_{2i}(S), \ldots, Q_{mi}(S))^t, 1 \leqq i \leqq p$$

We associate to the problem (EP) the following set of p-programming problems, $(EP_k^*)$, $1 \leqq k \leqq p$, each with a single objective:

$(EP_k^*)$ Minimize $F_k(S)/G_k(S)$

subject to $F_i(S)/G_i(S) \leqq F_i(S^*)/G_i(S^*)$, $i \neq k, 1 \leqq i \leqq p$,

$$Q_{ji}(S) \leqq 0, \ 1 \leqq j \leqq m, \ 1 \leqq i \leqq p.$$

**Lemma 2.1 [7]** $S^*$ is an efficient solution for (P) (or equivalently to (EP)) if and only if it is an optimal solution for $(EP_k^*)$, for each $1 \leqq k \leqq p$.

**Theorem 2.1 [1] (Necessary Optimality Conditions)** Let $S^*$ be a regular efficient solution for (P) (or equivalently to (EP)) and a regular solution for at least one $(EP_k^*)$, $1 \leqq k \leqq p$. Then there exist $\lambda^* \in \mathbf{R}^p$, $\hat{Y} \in \mathbf{R}^{p\times m}$ such that

$$\langle \sum_{i=1}^{p} \lambda_i^* D_r (F_i / G_i)_{S^*} + \sum_{j=1}^{m}\sum_{i=1}^{p} \hat{y}_{ij} D_r (Q_{ji})_{S^*}, \chi_{S_r} - \chi_{S_r^*} \rangle \geqq 0 \quad \forall\ S_r \in \boldsymbol{A},\ 1 \leqq r \leqq n \tag{2.1}$$

$$\hat{y}_{ij} Q_{ji}(S^*) = 0, 1 \leqq j \leqq m, 1 \leqq i \leqq p, \tag{2.2}$$

$$\hat{y}_{ij} \geqq 0, 1 \leqq j \leqq m, 1 \leqq i \leqq p, \tag{2.3}$$

$$\lambda^* \geq 0,\ \lambda^{*^t} e = 1,\ e = (1, 1, \ldots, 1)^t \in \mathbf{R}^p \tag{2.4}$$

where $$\hat{Y} = \begin{pmatrix} \hat{y}_{11} & \hat{y}_{12} & \cdots & \hat{y}_{1m} \\ \hat{y}_{21} & \hat{y}_{22} & \cdots & \hat{y}_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{y}_{p1} & \hat{y}_{p2} & \cdots & \hat{y}_{pm} \end{pmatrix} = \begin{pmatrix} \hat{Y}_1 \\ \hat{Y}_2 \\ \cdots \\ \hat{Y}_p \end{pmatrix}$$

is a matrix of Lagrange multipliers for the constraints of (P), $\hat{Y}_i$ is the $i^{th}$ row consisting of the Lagrange multipliers for the $i^{th}$ column of constraint matrix Q(S).

$$\text{Set } \lambda_i^* Y_i^* = \hat{Y}_i,\ 1 \leqq i \leqq p.$$

Then (2.1) – (2.3) can be rewritten as

$$\langle D_r \Big(\sum_{i=1}^{p} \lambda_i^* \Big(\frac{F_i + Y_i^* H}{G_i}\Big)\Big)_{s^*}, \chi_{S_r} - \chi_{S_r^*} \rangle \geqq 0 \quad \forall\ S_r \in \boldsymbol{A},\ 1 \leqq r \leqq n$$

$$y_{ij}^* \frac{H_j(S^*)}{G_i(S^*)} = 0,\ 1 \leqq j \leqq m,\ 1 \leqq i \leqq p$$

$$y_{ij}^* \geqq 0,\ 1 \leqq j \leqq m,\ 1 \leqq i \leqq p.$$

## 3. MIXED LAGRANGIAN AND MIXED SADDLE POINT

Let M = {1, 2, …, m}, let $J \subseteq M$ and $K = M \setminus J$ with $|J|$ and $|K|$ denote the cardinality of the sets J and K respectively.

Then the constraint matrix of (EP) can be partitioned as

$$Q(S) = \begin{pmatrix} Q_J(S) \\ Q_K(S) \end{pmatrix},$$

where $Q_J(S)$ is a matrix of order $|J| \times p$ with $i^{th}$ column given by

$$(Q_J(S))_i = \left( H_J(S) \big/ G_i(S) \right)^t, \quad 1 \leqq i \leqq p,$$

and $Q_K(S)$ is a matrix of order $|K| \times p$.

Define a set $\Omega = \{ S \in \boldsymbol{A}^n \mid Q_k(S) = H_k(S)/G_i(S) \leqq 0, \ \forall\, k \in K,\ 1 \leqq i \leqq p\}$.

Note that $\Gamma \subseteq \Omega$.

We now introduce vector-valued Mixed Lagrange function (or Partial Lagrange Function) for the problem (P) (or equivalently for (EP)).

**Definition 3.1** A vector-valued function $L: \Omega \times \mathbf{R}_+^{p\times|J|} \to \mathbf{R}^p$ defined as

$$L(S, Y_J) = (L_1(S, (Y_J)_1), L_2(S, (Y_J)_2), \ldots, L_p(S, (Y_J)_p)),$$

where $L_i(S, (Y_J)_i) = \dfrac{F_i(S) + (Y_J)_i H_J(S)}{G_i(S)}, \quad 1 \leqq i \leqq p,$

$(Y_J)_i$ is the $i^{th}$ row of matrix $Y_J$ of order $p \times |J|$,

is called Mixed Lagrange function for (P).

**Definition 3.2** A point $(S^*, Y_J^*) \in \Omega \times \mathbf{R}_+^{p\times|J|}$ is said to be a mixed saddle point of the Mixed Lagrange function $L$ if the following conditions hold

$$L(S^*, Y_J) \ngeq L(S^*, Y_J^*) \ \forall\ Y_J \in \mathbf{R}_+^{p\times|J|}$$

$$L(S^*, Y_J^*) \ngeq L(S, Y_J^*) \ \forall\ S \in \Omega.$$

Following theorem ensures the existence of a mixed saddle point of L for (P) under $(\Im, \rho, \theta)$-convexity assumptions.

**Theorem 3.1** Let $S^*$ be a regular efficient solution for (P) and regular efficient solution for at least one $(EP_k^*)$, $1 \leqq k \leqq p$. Also, for each i, $1 \leqq i \leqq p$, $j \in J$, $k \in K$, let $(F_i/G_i)$ be $(\Im, \rho_i, \theta)$-convex at $S^*$; $(H_j/G_i)$ be $(\Im, \sigma_{ji}, \theta)$-convex at $S^*$; $(H_k/G_i)$ be $(\Im, \gamma_{ki}, \theta)$-convex at $S^*$. Further, let the sublinear functional $\Im$ satisfy the following assumption

**(C)** For $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in L_1^n(X, \boldsymbol{A^n}, \mu)$,

$$\sum_{r=1}^{n} \langle \eta_r, \chi_{S_r} - \chi_{S_r^*} \rangle \geqq 0, \forall\ S_r \in A, 1 \leqq r \leqq n \Rightarrow \Im(S, S^*; \eta) \geqq 0.$$

Then there exist $\lambda^* \in \mathbf{R}^p$, $Y_J^* \in \mathbf{R}_+^{p \times |J|}$ such that $(S^*, Y_J^*)$ is a mixed saddle point of (P) provided we have $\lambda^* > 0$ and $\sum_{i=1}^{p} \lambda_i^* (\rho_i + \sum_{j \in J} y_{ij}^* \sigma_{ji} + \sum_{k \in K} y_{ik}^* \gamma_{ki}) \geqq 0$.

**Proof.** By Theorem 2.1, there exist $\lambda^* \in \mathbf{R}^p$, $\hat{Y} \in \mathbf{R}^{p \times m}$ such that

$$\langle D_r (\sum_{i=1}^{p} \lambda_i^* \left( \frac{F_i + Y_i^* H}{G_i} \right)_{s^*}, \chi_{S_r} - \chi_{S_r^*} \rangle \geqq 0 \quad \forall\ S_r \in A, 1 \leqq r \leqq n \tag{3.1}$$

$$y_{ij}^* \frac{H_j(S^*)}{G_i(S^*)} = 0, 1 \leqq j \leqq m, 1 \leqq i \leqq p \tag{3.2}$$

$$y_{ij}^* \geqq 0,\ 1 \leqq j \leqq m, 1 \leqq i \leqq p, \lambda^* \geq 0, \lambda^{*t} e = 1.$$

From (3.1), we have

$$\sum_{r=1}^{n} \langle D_r (S^*, \chi_{S_r} - \chi_{S_r^*} \rangle \geqq 0 \quad \forall\ S_r \in A$$

which along with condition (C) on $\Im$ implies

$$\Im(S, S^*; D (\sum_{i=1}^{p} \lambda_i^* \left( \frac{F_i + Y_i^* H}{G_i} \right)_{s^*}) \geqq 0\ \forall\ S \in \boldsymbol{A^n}. \tag{3.3}$$

Now, let $S \in \Omega$. Then by given hypotheses, we have

$$\frac{F_i(S)}{G_i(S)} - \frac{F_i(S^*)}{G_i(S^*)} \geqq \Im(S, S^*; D (\left( \frac{F_i}{G_i} \right)_{s^*}) + \rho_i\, d^2 (\theta(S, S^*)),\ 1 \leqq i \leqq p \tag{3.4}$$

$$\frac{H_j(S)}{G_i(S)} - \frac{H_j(S^*)}{G_i(S^*)} \geqq \Im(S, S^*; D (\left( \frac{H_j}{G_i} \right)_{s^*}) + \sigma_{ji}\, d^2 (\theta(S, S^*)),\ j \in J, 1 \leqq i \leqq p \tag{3.5}$$

$$\frac{H_k(S)}{G_i(S)} - \frac{H_k(S^*)}{G_i(S^*)} \geqq \Im(S, S^*; D (\left( \frac{H_k}{G_i} \right)_{s^*}) + \gamma_{ki}\, d^2 (\theta(S, S^*)), k \in K, 1 \leqq i \leqq p \tag{3.6}$$

Multiply (3.5) by $y_{ij}^* \geqq 0$, $j \in J$, (3.6) by $y_{ik}^* \geqq 0$, $k \in K$ and using (3.2), sublinearity of $\Im$, together with the fact that $S \in \Omega$, we get

$$y_{ij}^* \frac{H_j(S)}{G_i(S)} \geqq \Im(S, S^*; D (\left( y_{ij}^* \frac{H_j}{G_i} \right)_{s^*}) + y_{ij}^* \sigma_{ji}\, d^2 (\theta(S, S^*)),\quad j \in J \tag{3.7}$$

$$0 \geqq \Im(S, S^*; D\left(\left(y^*_{ik}\frac{H_k}{G_i}\right)_{S^*}\right) + y^*_{ik}\gamma_{ki}d^2(\theta(S, S^*)),\ k \in K \tag{3.8}$$

It follows from (3.4), (3.7), and (3.8) together with $\lambda^*_i > 0$, $1 \leqq i \leqq p$, sublinearity of $\Im$, that

$$\sum_{i=1}^{p}\lambda^*_i\left(\frac{F_i(S)}{G_i(S)} - \frac{F_i(S^*)}{G_i(S^*)} + \frac{(Y^*_J)_i H_J(S)}{G_i(S)}\right)$$

$$\geqq \Im(S, S^*; \left(\sum_{i=1}^{p}\lambda^*_i\left(\frac{F_i + (Y^*_J)_i H_J + (Y^*_K)_i H_K}{G_i}\right)\right)_{S^*})$$

$$+ \sum_{i=1}^{p}\lambda^*_i\left(\rho_i + \sum_{j\in J} y^*_{ij}\sigma_{ji} + \sum_{k\in K} y^*_{ik}\gamma_{ki}\right)d^2(\theta(S, S^*)).$$

Using (3.3) and assumption that $\sum_{i=1}^{p}\lambda^*_i\left(\rho_i + \sum_{j\in J} y^*_{ij}\sigma_{ji} + \sum_{k\in K} y^*_{ik}\gamma_{ki}\right) \geqq 0$, we get

$$\sum_{i=1}^{p}\lambda^*_i\left(L(S, Y^*_J) - L(S^*, Y^*_J)\right) \geqq 0 \quad \forall\ S \in \Omega. \tag{3.9}$$

As $\lambda^* > 0$, from (3.9) we get that, for any $S \in \Omega$,

$$L(S^*, Y^*_J) \ngeq L(S, Y^*_J). \tag{3.10}$$

Further, for any $Y_J \in \mathbf{R}_+^{p\times|J|}$,

$$L_i(S^*, Y_J) - L_i(S^*, Y^*_J) = \frac{(Y_J)_i H_J(S^*)}{G_i(S^*)} - \frac{(Y^*_J)_i H_J(S^*)}{G_i(S^*)}$$

which together with (3.2) and the facts that $Y_J \geqq 0$, $S^* \in \Gamma$ and $G(S^*) > 0$, yields

$$L_i(S^*, Y_J) - L_i(S^*, Y^*_J) \leqq 0, \qquad 1 \leqq i \leqq p.$$

This completes the proof of the Theorem.

Next theorem does not require any convexity conditions on the functions and hence its proof follows along the similar lines as the proof of Theorem 6 in [6].

**Theorem 3.2** Let $(S^*, Y^*_J)$ be a mixed saddle point of mixed L. Then $S^*$ is feasible to (P), $Y^*_i H(S^*) = 0$, $1 \leqq i \leqq p$, and $S^*$ is an efficient solution of (P).

## 4. MIXED DUAL AND DUALITY

Necessary optimality conditions for an efficient solution of (P) (Theorem 2.1), developed in Section 2, motivate us to introduce the following mixed dual for (P) (or equivalently for (EP)).

(D) V-maximize $(L_1(S, (Y_J)_1), L_2(S, (Y_J)_2), \ldots, L_p(S, (Y_J)_p))$

subject to $\langle D_r(\sum_{i=1}^{p}\lambda_i\left(\frac{F_i+Y_i H}{G_i}\right))_T, \chi_{S_r}-\chi_{T_r}\rangle \geqq 0 \ \forall \ S_r \in \mathcal{A}, 1 \leqq r \leqq n$ (4.1)

$$\frac{(Y_K)_i H_K(T)}{G_i(T)} \geqq 0, 1 \leqq i \leqq p \quad (4.2)$$

$$Y = (Y_J, Y_K) \geqq 0,\ Y_J \in \mathbf{R}^{p\times|J|},\ Y_K \in \mathbf{R}^{p\times|K|},\ \lambda \geq 0,\ \lambda^t e = 1, T \in \mathcal{A}^n.$$

**Theorem 4.1 (Weak Duality)** Let S be feasible for (P) and let (T, Y,λ) be feasible for (D). Further, let the following conditions be satisfied

$\sum_{i=1}^{p}\lambda_i\left(\frac{F_i+(Y_J)_i H_J}{G_i}\right)$ is $(\mathfrak{I},\rho,\theta)$-pseudoconvex at T, $\frac{(Y_K)_i H_K}{G_i}$ is $(\mathfrak{I},\sigma_i,\theta)$-quasiconvex at T, for each i, $\lambda > 0$ with $\rho + \sum_{i=1}^{p}\lambda_i\sigma_i \geqq 0$.

Also, let the sublinear functional $\mathfrak{I}$ satisfies the assumption

$$\mathfrak{I}(S, T; D(\sum_{i=1}^{p}\lambda_i\left(\frac{F_i+Y_i H}{G_i}\right))_T) > 0. \quad (4.3)$$

Then we have $\frac{F(S)}{G(S)} \nleq L(S, Y_J)$.

**Proof.** On the contrary, let $\frac{F(S)}{G(S)} \leq L(S, Y_J)$. Then there exists an index r such that

$$\frac{F_i(S)}{G_i(S)} \leqq L_i(T,(Y_J)_i), \forall i, 1 \leqq i \leqq p, i \neq r \quad (4.4)$$

$$\frac{F_r(S)}{G_r(S)} < L_r(T,(Y_J)_r) \quad (4.5)$$

Now, as S is feasible for (P), we have $\frac{H_j(S)}{G_i(S)} \leqq 0, \forall j \in J$. Also, $y_{ij} \geqq 0, \forall j \in J, 1 \leqq i \leqq p$, hence, we get

$$\frac{(Y_J)_i H_J(S)}{G_i(S)} \geqq 0, 1 \leqq i \leqq p. \quad (4.6)$$

It follows from (4.4), (4.5) and (4.6) together with $\lambda > 0$ that

$$\sum_{i=1}^{p}\lambda_i\left(\frac{F_i(S)+(Y_J)_i H_J(S)}{G_i(S)}\right) < \sum_{i=1}^{p}\lambda_i\left(\frac{F_i(T)+(Y_J)_i H_J(T)}{G_i(T)}\right)$$

Using $(\Im, \rho, \theta)$-pseudoconvexity of $\sum_{i=1}^{p} \lambda_i \left(\frac{F_i + (Y_J)_i H_J}{G_i}\right)$ at T, we get

$$\Im(S, T; D(\sum_{i=1}^{p} \lambda_i \left(\frac{F_i + (Y_J)_i H_J}{G_i}\right))_T) < -\rho\, d^2(\theta(S, T)). \tag{4.7}$$

Also feasibility of S along with $y_{ij} \geqq 0$, $\forall\, j \in J$, and (4.2) gives

$$\sum_{k \in K} y_{ik} \frac{H_k(S)}{G_i(S)} \leq \sum_{k \in K} y_{ik} \frac{H_k(T)}{G_i(T)}.$$

By $(\Im, \sigma_i, \theta)$-quasiconvexity of $\sum_{k \in K} y_{ik} \frac{H_k}{G_i}$ at T, we get

$$\Im(S, T; D(\sum_{k \in K} y_{ik} \frac{H_k}{G_i})_T) < -\sigma_i\, d^2(\theta(S, T)). \tag{4.8}$$

Multiplying (4.8) by $\lambda_i > 0$ and summing over i, adding it to (4.7) and using sublinearity of $\Im$, gives

$$\Im(S, T; D(\sum_{i=1}^{p} \lambda_i \left(\frac{F_i + Y_i H}{G_i}\right))_T) \leqq -(\rho + \sum_{i=1}^{p} \lambda_i \sigma_i) d^2(\theta(S, T))$$

which in view of hypothesis (iii) implies

$$\Im(S, T; D(\sum_{i=1}^{p} \lambda_i \left(\frac{F_i + Y_i H}{G_i}\right))_T) \leqq 0.$$

This contradicts (4.3). Hence the result.

**Remark 4.1** Weak Duality theorem can also be proved if along with condition (4.3) any one of the following sets of conditions holds

**(I)** $\sum_{i=1}^{p} \lambda_i \left(\frac{F_i + (Y_J)_i H_J}{G_i}\right)$ is strictly $(\Im, \rho, \theta)$-pseudoconvex at T, $\frac{(Y_K)_i H_K}{G_i}$ is $(\Im, \sigma_i, \theta)$-quasiconvex at T, $\forall$ i, $1 \leqq i \leqq p$, $\rho + \sum_{i=1}^{p} \lambda_i \sigma_i \geqq 0$.

**(II)** $\sum_{i=1}^{p} \lambda_i \left(\frac{F_i + (Y_J)_i H_J}{G_i}\right)$ is $(\Im, \rho, \theta)$-quasiconvex at T, $\frac{(Y_K)_r H_K}{G_r}$ is strictly $(\Im, \sigma_r, \theta)$-quasiconvex at T, for atleast one r, with $\lambda_r > 0$, $\frac{(Y_K)_i H_K}{G_i}$ is $(\Im, \sigma_i, \theta)$-quasiconvex at T, $\forall$ i, $i \neq r$, $\rho + \sum_{i=1}^{p} \lambda_i \sigma_i \geqq 0$.

**Theorem 4.2 (Strong Duality)** Let $S^*$ be a regular efficient solution of (P) and regular optimal solution for at least one $(EP_k^*)$, $1 \leqq k \leqq p$. Then there exist $\lambda^* \in \mathbf{R}^p$ and $Y^* \in \mathbf{R}^{p\times m}$ such that $(S^*, \lambda^*, Y^*)$ is feasible for (D). Furthermore, if any of the sets of conditions of Weak Duality theorem (Theorem 4.1 and Remark 4.1) holds, then $(S^*, \lambda^*, Y^*)$ is an efficient solution for (D).

**Proof.** It follows from Theorem 3.1 that there exist $\lambda^* \in \mathbf{R}^p$ and $Y^* \in \mathbf{R}^{p\times m}$ such that

$$\langle D_r (\sum_{i=1}^{p} \lambda_i^* \left( \frac{F_i + Y_i^* H}{G_i} \right))_{s^*}, \chi_{S_r} - \chi_{S_r^*} \rangle \geqq 0 \quad \forall\ S_r \in \mathcal{A}, 1 \leqq r \leqq n$$

$$y_{ij}^* \frac{H_j(S^*)}{G_i(S^*)} = 0, 1 \leqq j \leqq m, 1 \leqq i \leqq p \tag{4.9}$$

$$y_{ij}^* \geqq 0, 1 \leqq j \leqq m, 1 \leqq i \leqq p, \lambda^* \geq 0, \ \lambda^{*t} e = 1.$$

Let M = {1, 2, …, m} be partitioned into two disjoint subsets J and K such that K = M \ J. Then from (4.9), we have

$$\frac{(Y_K)_i H_K(S^*)}{G_i(S^*)} = \sum_{k\in K} y_{ik}^* \frac{H_k(S^*)}{G_i(S^*)} = 0 \quad \forall\ i, 1 \leqq i \leqq p.$$

Hence $(S^*, \lambda^*, Y^*)$ is feasible for (D). Moreover, in view of (4.9), we have

$$\frac{F_i(S^*)}{G_i(S^*)} = \frac{F_i(S^*) + (Y_J^*)_i H_J(S^*)}{G_i(S^*)} = L_i(S^*, Y_J^*), \forall\ i, 1 \leqq i \leqq p.$$

Thus, $\dfrac{F(S^*)}{G(S^*)} = L(S^*, Y_J^*)$.

That is, the value of (P)-objective at $S^*$ is equal to the value of (D)-objective at $(S^*, Y^*, \lambda^*)$. Hence, it follows from the Weak Duality theorem that $(S^*, Y^*, \lambda^*)$ is an efficient solution for (D).

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