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A FIXED POINT THEOREM IN DISLOCATED S_b-METRIC SPACES

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Abstract: In this paper we introduced dislocated S_b-metric spaces and some of its properties. Also we have discussed a fixed point theorem for a self mapping on a complete dislocated S_b-metric space.

Keywords: dS_b-metric space, dS_b-convergent, dS_b-Cauchy sequence, dS_b continuous.

Subject Classification: 54E35, 47H10.

1. INTRODUCTION

The concept of metric spaces has been generalized in many ways. The generalization of metric spaces were proposed by several mathematicians like Gahler[5,6] (2-metric space), Dhage[3,4] (D-metric space), Bakhtin [1] (b-metric space), K.S.Ha. *et al.* [7], Mustafa and Sims[11,12] (G-metric space), Souayah, Mlaiki [13](S_b-metric space), Sedghi *et al.* [14,15] (S_b-metric space using the concept of S-metric space).

The dislocation in metric space was introduced by P. Hitzler [8] in which the self distance of points need not be necessarily zero. Dislocated metric spaces play vital role in topology, logical programming and electronic engineering. Fixed point theory is an important area of study in pure and applied mathematics and it is flourishing area of research. The study of fixed point theory has been extensively developed in the past decades. In the present paper we introduced a new concept dislocated S_b - metric space and some of its properties. Also we have proved a fixed point theorem for a self mapping on a complete dislocated S_b - metric space.

Throughout this paper $R = (-\infty, \infty)$, $R^+ = (0, \infty)$, $R_0^+ = [0, \infty)$, N denotes the set of all natural numbers.

2. PRELIMINARIES

Definition 2.1

Let X be a non empty set. A distance function on a set X is a function $d: X \, x \, X \to R_0^+$ satisfying the following properties.

(1) $d(x, y) \geq 0$, for all $x, y \in X$.

(2) $d(x, y) = d(y, x)$, for all $x, y \in X$.

(3) $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$.

(4) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called the dislocated metric (simply d-metric) in X and the pair (X, d) is called dislocated metric space (or simply d-metric space).

Definition 2.2

A sequence $\{x_n\}$ in d-metric space (X, d) with respect to d is said to be d-converge to $x \in X$ provided that for all $\epsilon > 0$ there exists $n_0 \in N$ such that $d(x_n, x) < \epsilon$, for all $n \geq n_0$(or) $d - lim_{n\to\infty}\, d(x_n, x) = d - lim_{n\to\infty}\, d(x, x_n) = 0$.

Definition 2.3

We call a sequence in d-metric space is a d-Cauchy sequence provided that for all $\epsilon > 0$ there exists $n_0 \in N$ such that $d(x_n, x_m) < \epsilon$, for all $m, n \geq n_0$

Definition 2.4

A d-metric space (X, d) is called d-complete if every d-Cauchy sequence in X converges with respect to d in X.

Definition 2.5

Let X be a non empty set and $b \geq 1$ be a given real number. Suppose that a mapping S_b: $X^3 \to [0, \infty)$ is a function satisfying the following properties:

(a) $S_b\,(x, y, z) > 0$, for all $x, y, z \in X$ with $x \neq y \neq z$.

(b) $S_b\,(x, y, z) = 0$ if and only if $x = y = z$.

(c) $S_b\,(x, y, z) \leq b[S_b\,(x, x, a) + S_b\,(y, y, a) + S_b\,(z, z, a)]$ for all $x, y, z, a \in X$.

Then the function S_b is called S_b -metric on X and the pair (X, S_b) is called S_b-metric space.

Definition 2.6

Let X be a non empty set. An S-metric on X is a function S: $X^3 \to [0, \infty)$ that satisfies the condition for all $x, y, z, t \in X$:

(i) $S(x, y, z) = 0$ if and only if $x = y = z$

(ii) $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$.

The pair (X, S) is called a S-metric space.

3. DISLOCATED S_b-METRIC SPACE

Definition 3.1

Let X is a non empty set and $b \geq 1$ be a given real number. Suppose that a mapping $\rho : X^3 \to R_0^+$ is a function satisfying the following properties:

(a) $\rho(x, y, z) > 0$, for all $x, y, z \in X$ with $x \neq y \neq z$.

(b) $\rho(x, y, z) = 0$ implies $x = y = z$.

(c) $\rho(x, y, z) \leq b[\rho(x, x, a) + \rho(y, y, a) + \rho(z, z, a)]$ for all $x, y, z, a \in X$.

(d) $\rho(x, y, z) = \rho(y, z, x) = \rho(z, x, y) = \rho(y, x, z) = \rho(z, y, x) = \rho(x, z, y)$,

(e) $\rho(x, x, y) = \rho(y, x, x)$ for all $x, y, z \in X$.

Then the function $\rho$is called dislocated S_b-metric on X(simply dS_b-metric) and the pair (X, ρ) is called dislocated S_b-metric space (simply $d\, S_b$-metric space).

Example 3.2

Let $X = [0, 1]$. The distance function d: $XxXxX \to R_0^+$ is defined as $d(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$. Then d is dS_b-metric and the pair (X, d) is dS_b-metric space.

Example 3.3

Let $X = [0, 1]$. Define d: $XxXxX \to R_0^+$ by $d(x, y, z) = \begin{cases} z - y - x, \, if \; x \leq y \leq z \\ 2(x - y - z), otherwise \end{cases}$.

Then (X, d) is not dS_b-metric but S_b-metric with $b \geq 4$.

Definition 3.4

Let (X, ρ) be a dS_b-metric space. Then for each $x \in X$, $r > 0$ we define the dS_b- open ball $B_{dS_b}(x, r)$ and the dS_b- closed ball $B_{dS_b}[x, r]$ with center x and radius r as follows:

$B_{dS_b}(x, r) = \{y \in X \mid \rho(y, y, x) < r\}$

$$B_{dS_b}[x, r] = \{y \in X \mid \rho(y, y, x) \leq r\}$$

Definition 3.5

A sequence $\{x_n\}$ in dS_b - metric space (X, ρ) is said to be dS_b-converge to $x \in X$ provided that for $\epsilon > 0$, there exists a $n_0 \in N$ such that $\rho(x_n, x_n, x) < \epsilon$ (or) $\rho(x, x_n, x_n) < \epsilon$, for all $n \geq n_0$ and we denote as $dS_b - lim_{n\to\infty}\, x_n = x$.

Definition 3.6

A sequence $\{x_n\}$ in dS_b- metric space (X, ρ) is said to dS_b-Cauchy sequence provided that for $\epsilon > 0$, there exists a $n_0 \in N$ such that $\rho(x_n, x_m, x_l) \leq \epsilon$ for all $l, m, n \geq n_0$.

Definition 3.7

A dS_b-metric space (X, ρ) is called complete if every dS_b- cauchy sequence is dS_b-convergent in X.

Lemma 3.8

Every converging sequence in a dS_b-metric space is a dS_b - Cauchy sequence.

Proof: Let $\{x_n\}$ be a sequence which converges to some $x \in X$. Let $\epsilon > 0$ be chosen. Then there exists $n_0 \in N$ such that $\rho\,(x_n, x_n, x) < \epsilon/3b$, for all $n \geq n_0$. For $m,n \geq n_0$, we have $\rho(x_n, x_m, x_l) \leq b[\rho\,(x_n, x_n, x) + \rho\,(x_m, x_m, x) + \rho\,(x_l, x_l, x)] < \epsilon$. Hence $\{x_n\}$ is a dS_b-Cauchy sequence.

Remark 3.9

If a sequence $\{x_n\}$ in dS_b- metric space (X, ρ) is dS_b-Cauchy sequence then for for $\epsilon > 0$, there exists a $n_0 \in N$ such that (i) $\rho\,(x_n, x_m, x_m) \leq \epsilon$ for all $m, n \geq n_0$, (ii) $\rho(x_m, x_m, x_n) \leq \epsilon$ for all $m, n \geq n_0$ are equivalent.

Lemma 3.10

Limits in dS_b- metric space are unique.

Proof: Let $\{x_n\}$ be the sequence in dS_b-metric space having the limits x, y and z. Then as $n \to \infty$, $\rho(x_n, x_n, x) \to 0$, $\rho(x_n, x_n, y) \to 0$ and $\rho(x_n, x_n, z) \to 0$. Now $\rho(x, y, z) \leq 0$. But $\rho(x, y, z) \geq 0$. Therefore $\rho(x, y, z) = 0$ which implies $x = y = z$. Hence the limits in dS_b - metric space are unique.

Definition 3.10

Any set containing a ball with center a is sometimes called dS_b neighbourhood in dS_b-metric space (i.e) $N_{dSb}(x) = \{y \in X \mid \rho\,(y, x, x) < \epsilon\}$ where $\epsilon > 0$.

Theorem 3.11

Let (X, ρ) be a dS_b-metric space. Then for all $x \in X$, $\rho(x, x, x) = 0$ iff $N_{dSb}(x) \neq \emptyset$, for all $x \in X$ and $\epsilon > 0$.

Proof: Since $N_{dSb}(x) = \{y \in X \mid \rho(y, x, x) < \epsilon\}$ where $\epsilon > 0$ is an open ball in dS_b-metric space (X, ρ) with center x. Suppose that $\rho(x, x, x) = 0$, for all $x \in X$ and for $\epsilon > 0$, we have $\rho(x, x, x) < \epsilon$ implies $x \in N_{dSb}(x)$. Hence $N_{dSb}(x) \neq \emptyset$, for all $x \in X$.

Conversely, suppose that $N_{dS_b}(x) \neq \emptyset$ for $\epsilon>0$ we have $N_{dSb}(x) = \{y \in X \mid \rho(y, x, x) < \epsilon\}$. Then there exists $a \in X$ such that $\rho(a, x, x) = r/b < \epsilon$. By rectangular property and symmetry property it follows that $\rho(x, x, x) \leq b[\rho(x, x, a) + \rho(x, x, a) + \rho(x, x, a)] = 3r < \epsilon$. Hence for all $\epsilon > 0$, $\rho(x, x, x) < \epsilon$ this shows that $\rho(x, x, x) = 0$.

Theorem 3.12

Let (X, ρ) be a dS_b-metric space. Then ρ is S_b-metric iff $N_{dSb}(x) \neq \emptyset$, for all $x \in X$ and $\epsilon > 0$.

Proof: Let ρ be S_b-metric. Then we have $\rho(x, x, x) = 0$, for all $x \in X$. By Theorem 3.11, it follows that $N_{dSb}(x) \neq \emptyset$, for all $x \in X$ and $\epsilon > 0$. Conversely, let $N_{dSb}(x) \neq \emptyset$ for all $x \in X$ and $\epsilon > 0$. Since $N_{dSb}(x) = \{y \in X \mid \rho(y, x, x) < \epsilon\}$, then $\rho(x, x, x) < \epsilon$. Hence $\rho(x, x, x) = 0$ for all $x \in X$, $\rho(x, x, x) = 0$ and then $\rho(y, x, x) = 0$ iff $x = y$. Also the rectangle inequality holds. Therefore ρ is a S_b-metric.

Definition 3.13

Let (X, ρ) and (Y, ρ') be dS_b-metric spaces and let $f: X \to Y$ be a function. Then f is said to be dS_b-continuous at a point $p \in X$ if for every $\in > 0$ there is a $\delta > 0$ such that $\rho' (f(x), f(x), f(p)) < \epsilon$ whenever $\rho(x, x, p) < \delta$.

If f is a dS_b- continuous at every point of a subset A of X, we say that f is dS_b-continuous on A.

Theorem 3.14

A mapping $f: X \to Y$ of a dS_b metric spaces (X, ρ) and (Y, ρ') is dS_b-continuous at a point $x_0 \in X$ if and only if $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$.

Proof: Assume that f is dS_b-continuous at x_0. Then for given $\in > 0$ there is a $\delta > 0$ such that $\rho' (f(x), f(x), f(p)) < \epsilon$ whenever $\rho(x, x, p) < \delta$. Let $\{x_n\}$ be a sequence in X such that $x_n \to x_0$. Then there is $n_0 \in N$ such that $\rho(x_n, x_n, x_0) < \delta$ for all $n \geq n_0$. Hence for all $n \geq n_0$, $\rho' (f(x_n), f(x_n), f(x_0)) < \epsilon$. This shows that $f(x_n) \to f(x_0)$. Conversely, assume that f is not dS_b-continuous at x_0. Then for given $\delta > 0$ such that $\rho' (f(x), f(x), f(x_0)) \geq \epsilon$. Choosing $\delta = \dfrac{1}{n}$, there is an x_n satisfying $\rho(x_n, x_n, x_0) < \dfrac{1}{n}$. But $\rho' (f(x_n), f(x_n), f(x_0)) \geq \epsilon$. This contradicts that $f(x_n) \to f(x_0)$. Thus f dS_b is continuous at x_0.

Theorem 3.15

Let $f : R_0^+ \to [0,3^{-1})$ be non increasing function. Let (X, ρ) be complete dS_b metric space and let $T: X \to X$ be a ds_b continuous mapping such that

$\rho(Tx, Ty, Tz) \leq f(\rho(x, y, z))\,[\rho(x, x, Tx) + \rho(y, y, Ty) + \rho(z, z, Tz)]\ldots$ (3.1) for all $x, y, z \in X$. Moreover if f is a constant function then T has a unique fixed point.

Proof: Let $x_{n+1} = Tx_n$ for each $n \in N$. Assume that $\rho(x_n, x_n, x_{n+1}) \neq 0$ for each $n \in N$. Then by (1) we get,

$$\begin{aligned}\rho(x_{n+1}, x_{n+2}, x_{n+3}) &= \rho(Tx_n, Tx_{n+1}, Tx_{n+2}) \\ &\leq f\big(\rho(x_n, x_{n+1}, x_{n+2})\big)[\rho(x_n, x_n, Tx_n) + \rho(x_{n+1}, x_{n+1}, Tx_{n+1}) + \rho(x_{n+2}, x_{n+2}, Tx_{n+2})] \\ &\leq \frac{1}{3}[\rho(x_n, x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+1}, x_{n+2}) + \rho(x_{n+2}, x_{n+2}, x_{n+3})]\end{aligned}$$

This implies that $\rho(x_{n+1}, x_{n+2}, x_{n+3}) \leq \rho(x_n, x_n, x_{n+1})$.

Hence $\{\rho(x_n, x_n, x_{n+1})\}$ is a non negative non increasing sequence.

Thus there exists $\alpha \geq 0$ such that $\rho(x_n, x_n, x_{n+1}) \to \alpha$ as $n \to \infty$.

Now we claim that $\alpha = 0$. Suppose that $\alpha > 0$. Then,

$$\begin{aligned}\rho(x_{n+1}, x_{n+2}, x_{n+3}) &\leq f(\alpha)[\rho(x_n, x_n, Tx_n) + \rho(x_{n+1}, x_{n+1}, Tx_{n+1}) + \rho(x_{n+2}, x_{n+2}, Tx_{n+2})] \\ &\leq 3\alpha f(\alpha)\end{aligned}$$

This is impossible since $f(\alpha) < \frac{1}{3}$.

Thus $\alpha = 0$ (i.e) $dS_b - lim_{n\to\infty} x_n = x_{n+1} = Tx_n$.

Now we have to prove that $\{x_n\}$ is a dS_b Cauchy sequence.

From (1) we have,

$$\begin{aligned}\rho(x_n, x_m, x_l) &= \rho(Tx_n, Tx_m, Tx_l) \\ &\leq f(\rho(x_n, x_n, x_m))\,[\rho(x_n, x_n, Tx_n) + \rho(x_m, x_m, Tx_m) + \rho(x_l, x_l, Tx_l)] \\ &\leq \frac{1}{3}[\rho(x_n, x_n, Tx_n) + \rho(x_m, x_m, Tx_m) + \rho(x_l, x_l, x_{l+1})]\end{aligned}$$

There exists $n_0 \in N$ such that for $l, n, m \geq n_0$, $\rho(x_n, x_n, x_{n+1}) < \epsilon$, $\rho(x_m, x_m, x_{m+1}) < \epsilon$ and $\rho(x_l, x_l, x_{l+1}) < \epsilon$

Hence $\rho(x_n, x_m, x_l) \leq \frac{1}{3}(2\epsilon + \epsilon) = \epsilon$

This proves that $\{x_n\}$ is a dS_b Cauchy sequence.

Since X is complete, there exists $a \in X$ such that $x_n \to a$ as $n \to \infty$.

Now lets prove that a is a fixed point. (i.e) $Ta = a$.

(i) If T is dS_b continuous then, $Ta = \lim_{n\to\infty} Tx_n = x_{n+1} = a$, then a is a fixed point of T.

(ii) If f is a constant function then we have,

$$\rho(a, a, Ta) \le b[\rho(a, a, x_{n+1}) + \rho(a, a, x_{n+1}) + \rho(Ta, Ta, x_{n+1})]$$

$$= b[2\rho(a, a, x_{n+1}) + \rho(Ta, Ta, Tx_n)]$$

$$\le b[2\rho(a, a, x_{n+1}) + f(\rho(a, a, x_{n+1}))\,[\rho(a, a, Ta) + \rho(a, a, Ta) + \rho(x_n, x_n, Tx_n)]]$$

Letting $n \to \infty$ we get,

$$\rho(a, a, Ta) \le b[f\,[2\rho(a, a, Ta)]].$$

Similarly, $\rho(Ta, a, a) \le b[f\,[2\rho(a, a, Ta)]]$ as $n \to \infty$.

Hence $|\rho(a, a, Ta) - \rho(Ta, a, a)| \le 0$.

Thus $Ta = a$ and a is the fixed point of T.

Finally we have to prove that the uniqueness of the point.

Assume that $Ta = a$ and $Tb = b$ with $a \ne b$, $a, b \in X$.

Now, from (3.1) we have,

$$\rho(Ta, Ta, Ta) \le f(\rho(a, a, a))\,[\rho(a, a, Ta) + \rho(a, a, Ta) + \rho(a, a, Ta)]$$

$$= f(\rho(a, a, a))\,[3\rho(a, a, a)] \tag{3.2}$$

Since f: $R_0^+ \to [0, 3^{-1})$ be non increasing function, so the inequality (3.2) is possible if $\rho(a, a, a) = 0$.

Similarly we can get $\rho(b, b, b) = 0$.

Consider,

$$\rho(a, a, b) = \rho(Ta, Ta, Tb)$$

$$\le f(\rho(a, a, b))\,[2\rho(a, a, Ta) + \rho(b, b, Tb)]$$

$$= f\,(\rho(a, a, b))\,[2\rho(a, a, a) + \rho(b, b, b)]$$

This shows that $\rho(a, a, b) = 0$ and so $a = b$ which proves the uniqueness of the fixed point of T.

Example 3.16

Let $X = [0,1]$. Define $\rho : X^3 \to R_0^+$ by $\rho(x, y, z) = x + y + z$. Then ρ is a dS_b-metric and (X, ρ) is a complete dS_b-metric space. Define f: $R_0^+ \to [0, 3^{-1})$ by $f(u) = (u + 2)^{-1}$ and T: $X \to X$ by $T(x) = x/2$. Here f is non increasing function and T is continuous in X. For all $x, y, z \in X$ we get,

$$\rho(Tx, Ty, Tz) \leq f(\rho(x, y, z))\, [\rho(x, x, Tx) + \rho(y, y, Ty) + \rho(z, z, Tz)]$$

$$\rho\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right) \leq f(x+y+z)\left[\rho\left(x,x,\frac{x}{2}\right)+\rho\left(y,y,\frac{y}{2}\right)+\rho\left(z,z,\frac{z}{2}\right)\right]$$

$$\frac{x}{2}+\frac{y}{2}+\frac{z}{2} \leq \frac{1}{x+y+z+2}\left[\frac{5(x+y+z)}{2}\right]$$

$x + y + z \leq 3$, for all $x, y, z \in X$

T satisfies the condition (3.1) and satisfies the hypothesis of Theorem 3.15 and $x = 0$ is a unique fixed point.

Conflict of Interest

Authors declare that there is no conflict of interest regarding this article.

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