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TIME-DEPENDENT NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS DRIVEN BY A FRACTIONAL BROWNIAN MOTION

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Abstract. In this paper we consider a class of time-dependent neutral stochastic functional differential equations with finite delay driven by a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, in a separable real Hilbert space. We prove an existence and uniqueness result of mild solution by means of the Banach fixed point principle. A practical example is provided to illustrate the viability of the abstract result of this work.

1. Introduction

The stochastic functional differential equations have attracted much attention because of their practical applications in many areas such as physics, medicine, biology, finance, population dynamics, electrical engineering, telecommunication networks, and other fields. For more details, one can see Da Prato and Zabczyk [7], and Ren and Sun [14] and the references therein.

In many areas of science, there has been an increasing interest in the investigation of the systems incorporating memory or aftereffect, i.e., there is the effect of delay on state equations. Therefore, there is a real need to discuss stochastic evolution systems with delay. In many mathematical models the claims often display long-range memories, possibly due to extreme weather, natural disasters, in some cases, many stochastic dynamical systems depend not only on present and past states, but also contain the derivatives with delays. Neutral functional differential equations are often used to describe such systems. Very recently, neutral stochastic functional differential equations driven by fractional Brownian motion have attracted the interest of many researchers. One can see [4, 6, 10, 8, 9] and the references therein. The literature concerning the existence and qualitative properties of solutions of time-dependent functional stochastic differential equations is very restricted and limited to a very few articles. This fact is the main motivation of our work. We mention here the recent paper by Ren et al. [13] concerning the existence of mild solutions for a class of stochastic evolution equations driven by fractional Brownian motion in Hilbert space.

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Motivated by the above works, this paper is concerned with the existence and uniqueness of mild solutions for a class of time-dependent neutral functional stochastic differential equations described in the form:

$$\begin{cases} d[x(t)+g(t,x(t-r(t)))] = [A(t)x(t)+f(t,x(t-\rho(t))]dt+\sigma(t)dB^H(t),\ 0\leq t\leq T, \\ x(t)=\varphi(t),\ -\tau\leq t\leq 0, \end{cases} \tag{1.1}$$

in a real Hilbert space X with inner product $< .,. >$ and norm $\|.\|$, where $A(t)$, $t\in[0,T]$ is a family of linear closed operators from a space X into X that generates an evolution system of operators $\{U(t,s),\ 0\leq s\leq t\leq T\}$. B^H is a fractional Brownian motion on a real and separable Hilbert space Y, r, $\rho:[0,+\infty)\rightarrow[0,\tau]$ $(\tau>0)$ are continuous and $f,g:[0,+\infty)\times X\rightarrow X$, $\sigma:[0,+\infty)\rightarrow\mathcal{L}_2^0(Y,X)$, are appropriate functions. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below).

On the other hand, to the best of our knowledge, there is no paper which investigates the study of time-dependent neutral stochastic functional differential equations with delays driven by fractional Brownian motion. Thus we will make the first attempt to study such problem in this paper. Our results are inspired by the one in [4] where the existence and uniqueness of mild solutions to model (1.1) with $A(t)=A$, $\forall t\in[0,T]$, is studied, as well as some results on the asymptotic behavior.

The substance of the paper is organized as follows. Section 2, recapitulate some notations, basic concepts, and basic results about fractional Brownian motion, Wiener integral over Hilbert spaces and we recall some preliminary results about evolution operator. We need to prove a new technical lemma for the $\mathbb{L}^2-$estimate of stochastic convolution integral which is different from that used by [4]. Section 3, gives sufficient conditions to prove the existence and uniqueness for the problem (1.1). In Section 4 we give an example to illustrate the efficiency of the obtained result.

2. Preliminaries

2.1. Evolution families. In this subsection we introduce the notion of evolution family.

Definition 2.1. A set $\{U(t,s):0\leq s\leq t\leq T\}$ of bounded linear operators on a Hilbert space X is called an *evolution family* if

(a) $U(t,s)U(s,r)=U(t,r)$, $U(s,s)=I$ if $r\leq s\leq t$,
(b) $(t,s)\rightarrow U(t,s)x$ is strongly continuous for $t>s$.

Let $\{A(t),\ t\in[0,T]\}$ be a family of closed densely defined linear unbounded operators on the Hilbert space X and with domain $D(A(t))$ independent of t, satisfying the following conditions introduced by [1].

There exist constants $\lambda_0\geq 0$, $\theta\in(\frac{\pi}{2},\pi)$, L, $K\geq 0$, and μ, $\nu\in(0,1]$ with $\mu+\nu>1$ such that

$$\Sigma_\theta\cup\{0\}\subset\rho(A(t)-\lambda_0),\quad \|R(\lambda,A(t)-\lambda_0)\|\leq\frac{K}{1+|\lambda|} \tag{2.1}$$

and

$$\|(A(t)-\lambda_0)R(\lambda,A(t)-\lambda_0)\big[R(\lambda_0,A(t))-R(\lambda_0,A(s))\big]\| \leq L|t-s|^{\mu}|\lambda|^{-\nu}, \quad (2.2)$$

for $t,\, s\in\mathbb{R}$, $\lambda\in\Sigma_\theta$ where $\Sigma_\theta := \big\{\lambda\in\mathbb{C}-\{0\} : |\arg\lambda|\leq\theta\big\}$.

It is well known, that this assumption implies that there exists a unique evolution family $\{U(t,s) : 0\leq s\leq t\leq T\}$ on X such that $(t,s)\to U(t,s)\in\mathcal{L}(X)$ is continuous for $t>s$, $U(\cdot,s)\in\mathcal{C}^1((s,\infty),\mathcal{L}(X))$, $\partial_t U(t,s)=A(t)U(t,s)$, and

$$\|A(t)^k U(t,s)\| \leq C(t-s)^{-k} \quad (2.3)$$

for $0<t-s\leq 1$, $k=0,1$, $0\leq\alpha<\mu$, $x\in D((\lambda_0-A(s))^{\alpha})$, and a constant C depending only on the constants in (2.1)-(2.2). Moreover, $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t>s$ and $x\in D(A(s))$ with $A(s)x\in\overline{D(A(s))}$. We say that $A(\cdot)$ generates $\{U(t,s) : 0\leq s\leq t\leq T\}$. Note that $U(t,s)$ is exponentially bounded by (2.3) with $k=0$.

Remark 2.2. If $\{A(t),\, t\in[0,T]\}$ is a second order differential operator A, that is $A(t)=A$ for each $t\in[0,T]$, then A generates a C_0-semigroup $\{e^{At}, t\in[0,T]\}$.

For additional details on evolution system and their properties, we refer the reader to [12].

2.2. Fractional Brownian Motion. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space. A standard fractional Brownian motion (fBm) $\{\beta^H(t), t\in\mathbb{R}\}$ with Hurst parameter $H\in(0,1)$ is a zero mean Gaussian process with continuous sample paths such that

$$R_H(t,s)=\mathbb{E}[\beta^H(t)\beta^H(s)]=\frac{1}{2}\big(t^{2H}+s^{2H}-|t-s|^{2H}\big), \qquad s,t\in\mathbb{R}. \quad (2.4)$$

Remark 2.3. In the case $H>\frac{1}{2}$, it follows from [11]that the second partial derivative of the covariance function

$$\frac{\partial R_H}{\partial t\partial s}=\alpha_H|t-s|^{2H-2},$$

where $\alpha_H=H(2H-2)$, is integrable, and we can write

$$R_H(t,s)=\alpha_H\int_0^t\int_0^s |u-v|^{2H-2}dudv. \quad (2.5)$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X,Y and $\mathcal{L}(Y,X)$. Let $Q\in\mathcal{L}(Y,Y)$ be an operator defined by $Qe_n=\lambda_n e_n$ with finite trace $trQ=\sum_{n=1}^{\infty}\lambda_n<\infty$. where $\lambda_n\geq 0$ $(n=1,2...)$ are non-negative real numbers and $\{e_n\}$ $(n=1,2...)$ is a complete orthonormal basis in Y.

We define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t)=B_Q^H(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n\beta_n^H(t),$$

where β_n^H are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle B^H(t), x\rangle\langle B^H(s), y\rangle = R(s,t)\langle Q(x), y\rangle, \quad x, y \in Y, \ \ t, s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty}\langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space. Let $\phi(s)$; $s \in [0,T]$ be a function with values in $\mathcal{L}_2^0(Y,X)$, such that

$$\sum_{n=1}^{\infty} \|K^*\phi Q^{\frac{1}{2}} e_n\|^2_{\mathcal{L}_2^0} < \infty.$$

The Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s)dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s)e_n d\beta_n^H(s). \tag{2.6}$$

Now, we end this subsection by stating the following result which is fundamental to prove our result. It can be proved by similar arguments as those used to prove Lemma 2 in [6].

Lemma 2.4. *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$, then the above sum in* (2.6) *is well defined as a X-valued random variable and we have*

$$\mathbb{E}\|\int_0^t \psi(s)dB^H(s)\|^2 \le 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

2.3. The stochastic convolution integral. In this subsection we present a few properties of the stochastic convolution integral of the form

$$Z(t) = \int_0^t U(t,s)\sigma(s)dB^H(s), \qquad t \in [0,T],$$

where $\sigma(s) \in \mathcal{L}_2^0(Y,X)$ and $\{U(t,s),\, 0 \le s \le t \le T\}$ is an evolution system of operators.

The properties of the process Z are crucial when regularity of the mild solution to stochastic evolution equation is studied, see [7] for asystematic account of the theory of mild solutions to infinite-dimensional stochastic equations. Unfortunately, the process Z is not a martingale, and standard tools of the martingale theory, yielding e.g. continuity of the trajectories or $\mathbb{L}^2-$estimates are not available. The following result on the stochastic convolution integral Z holds.

Lemma 2.5. *Suppose that $\sigma : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\sup_{t\in[0,T]} \|\sigma(t)\|^2_{\mathcal{L}_2^0} < \infty$, and suppose that $\{U(t,s),\, 0 \le s \le t \le T\}$ is an evolution system of operators* *satisfying* $\|U(t,s)\| \leq Me^{-\beta(t-s)}$, *for some constants* $\beta > 0$ *and* $M \geq 1$ *for all* $t \geq s$. *Then, we have*

$$\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dB^H(s)\|^2 \leq CM^2t^{2H}(\sup_{t\in[0,T]}\|\sigma(t)\|_{\mathcal{L}_2^0})^2.$$

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be the complete orthonormal basis of Y and $\{\beta_n^H\}_{n\in\mathbb{N}}$ is a sequence of independent, real-valued standard fractional Brownian motion each with the same Hurst parameter $H \in (\frac{1}{2}, 1)$. Thus using fractional Itô isometry one can write

$$\begin{aligned}
\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dB^H(s)\|^2 &= \sum_{n=1}^{\infty}\mathbb{E}\|\int_0^t U(t,s)\sigma(s)e_n d\beta_n^H(s)\|^2 \\
&= \sum_{n=1}^{\infty}\int_0^t\int_0^t < U(t,s)\sigma(s)e_n, U(t,r)\sigma(r)e_n > \\
&\quad \times H(2H-1)|s-r|^{2H-2}dsdr \\
&\leq H(2H-1)\int_0^t\{\|U(t,s)\sigma(s)\| \\
&\quad \times \int_0^t \|U(t,r)\sigma(r)\||s-r|^{2H-2}dr\}ds \\
&\leq H(2H-1)M^2\int_0^t\{e^{-\beta(t-s)}\|\sigma(s)\|_{\mathcal{L}_2^0} \\
&\quad \times \int_0^t e^{-\beta(t-r)}|s-r|^{2H-2}\|\sigma(r)\|_{\mathcal{L}_2^0}dr\}ds.
\end{aligned}$$

Since σ is bounded, one can then conclude that

$$\begin{aligned}
\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dB^H(s)\|^2 &\leq H(2H-1)M^2(\sup_{t\in[0,T]}\|\sigma(t)\|_{\mathcal{L}_2^0})^2\int_0^t\{e^{-\beta(t-s)} \\
&\quad \times \int_0^t e^{-\beta(t-r)}|s-r|^{2H-2}dr\}ds.
\end{aligned}$$

Make the following change of variables, $v = t-s$ for the first integral and $u = t-r$ for the second. One can write

$$\begin{aligned}
\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dB^H(s)\|^2 &\leq H(2H-1)M^2(\sup_{t\in[0,T]}\|\sigma(t)\|_{\mathcal{L}_2^0})^2\int_0^t\{e^{-\beta v} \\
&\quad \times \int_0^t e^{-\beta u}|u-v|^{2H-2}du\}dv \\
&\leq H(2H-1)M^2(\sup_{t\in[0,T]}\|\sigma(t)\|_{\mathcal{L}_2^0})^2 \\
&\quad \times \int_0^t\int_0^t |u-v|^{2H-2}dudv.
\end{aligned}$$

By using (2.5), we get that

$$\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dB^H(s)\|^2 \leq CM^2t^{2H}(\sup_{t\in[0,T]}\|\sigma(t)\|_{\mathcal{L}_2^0})^2.$$

□

Remark 2.6. Thanks to Lemma 2.5, the stochastic integral $Z(t)$ is well-defined.

3. Existence and Uniqueness of Mild Solutions

In this section we study the existence and uniqueness of mild solutions of equation (1.1). Henceforth we will assume that the family $\{A(t),\, t\in[0,T]\}$ of linear operators generates an evolution system of operators $\{U(t,s),\, 0\leq s\leq t\leq T\}$. Before stating and proving the main result, we give the definition of mild solutions for equation (1.1).

Definition 3.1. A X-valued process $\{x(t),\ t\in[-\tau,T]\}$, is called a mild solution of equation (1.1) if

i) $x(.)\in\mathcal{C}([-\tau,T],\mathbb{L}^2(\Omega,X))$,
ii) $x(t)=\varphi(t)$, $-\tau\leq t\leq 0$.
iii) For arbitrary $t\in[0,T]$, $x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) &= U(t,0)(\varphi(0)+g(0,\varphi(-r(0)))) - g(t,x(t-r(t))) \\ &\quad - \int_0^t U(t,s)A(s)g(s,x(s-r(s)))ds + \int_0^t U(t,s)f(s,x(s-\rho(s))ds \\ &\quad + \int_0^t U(t,s)\sigma(s)dB^H(s) \quad \mathbb{P}-a.s \end{aligned}$$

We introduce the following assumptions:

$(\mathcal{H}.1)$ *i)* The evolution family is exponentially stable, that is, there exist two constants $\beta>0$ and $M\geq 1$ such that

$$\|U(t,s)\|\leq Me^{-\beta(t-s)}, \qquad for\ all \quad t\geq s,$$

ii) There exist a constant $M_*>0$ such that

$$\|A^{-1}(t)\|\leq M_* \qquad for\ all \quad t\in[0,T].$$

$(\mathcal{H}.2)$ The maps $f,g:[0,T]\times X\to X$ are continuous functions and there exist two positive constants C_1 and C_2, such that for all $t\in[0,T]$ and $x,y\in X$:
i) $\|f(t,x)-f(t,y)\|\vee\|g(t,x)-g(t,y)\|\leq C_1\|x-y\|$.
ii) $\|f(t,x)\|^2\vee\|A^k(t)g(t,x)\|^2\leq C_2(1+\|x\|^2)$, $\quad k=0,1$.

$(\mathcal{H}.3)$ *i)* There exists a constant $0<L_*<\frac{1}{M_*}$ such that

$$\|A(t)g(t,x)-A(t)g(t,y)\|\leq L_*\|x-y\|,$$

for all $t\in[0,T]$ and $x,y\in X$.

ii) The function g is continuous in the quadratic mean sense: for all $x(.)\in\mathcal{C}([0,T],L^2(\Omega,X))$, we have

$$\lim_{t\longrightarrow s}\mathbb{E}\|g(t,x(t))-g(s,x(s))\|^2=0.$$

($\mathcal{H}.4$) *i)* The map $\sigma : [0,T] \longrightarrow \mathcal{L}_2^0(Y,X)$ is bounded, that is: there exists a positive constant L such that $\|\sigma(t)\|_{\mathcal{L}_2^0(Y,X)} \leq L$ uniformly in $t \in [0,T]$.
ii) Moreover, we assume that the initial data $\varphi = \{\varphi(t) : -\tau \leq t \leq 0\}$ satisfies $\varphi \in \mathcal{C}([-\tau,0],\mathbb{L}^2(\Omega,X))$.

The main result of this paper is given in the next theorem.

Theorem 3.2. *Suppose that ($\mathcal{H}.1$)-($\mathcal{H}.4$) hold. Then, for all $T>0$, the equation (1.1) has a unique mild solution on $[-\tau,T]$.*

Proof. Fix $T>0$ and let $B_T := \mathcal{C}([-\tau,T],\mathbb{L}^2(\Omega,X))$ be the Banach space of all continuous functions from $[-\tau,T]$ into $\mathbb{L}^2(\Omega,X))$, equipped with the supremum norm

$$\|x\|_{B_T}^2 = \sup_{-\tau\leq t\leq T} \mathbb{E}\|x(t,\omega)\|^2.$$

Let us consider the set

$$S_T(\varphi) = \{x \in B_T : x(s) = \varphi(s), \text{ for } s \in [-\tau,0]\}.$$

$S_T(\varphi)$ is a closed subset of B_T provided with the norm $\|.\|_{B_T}$. We transform (1.1) into a fixed-point problem. Consider the operator ψ on $S_T(\varphi)$ defined by $\psi(x)(t) = \varphi(t)$ for $t \in [-\tau,0]$ and for $t \in [0,T]$

$$\begin{aligned}
\psi(x)(t) &= U(t,0)(\varphi(0) + g(0,\varphi(-r(0)))) - g(t,x(t-r(t))) \\
&\quad - \int_0^t U(t,s)A(s)g(s,x(s-r(s)))ds + \int_0^t U(t,s)f(s,x(s-\rho(s))ds \\
&\quad + \int_0^t U(t,s)\sigma(s)dB^H(s) \\
&= \sum_{i=1}^5 I_i(t).
\end{aligned}$$

Clearly, the fixed points of the operator ψ are mild solutions of (1.1). The fact that ψ has a fixed point will be proved in several steps. We will first prove that the function ψ is well defined.

Step 1: ψ is well defined. Let $x \in S_T(\varphi)$ and $t \in [0,T]$, we are going to show that each function $t \to I_i(t)$ is continuous on $[0,T]$ in the $\mathbb{L}^2(\Omega,X)$-sense.

We can easily see that $\mathbb{E}\|I_i(t+h) - I_i(t)\|^2 \longrightarrow 0$, $i = 1,2,3$ as $h \longrightarrow 0$.

For the fourth term $I_4(h)$, we suppose $h>0$ (similar calculus for $h<0$). We have

$$\begin{aligned}
|I_4(t+h) - I_4(t)| &\leq \left|\int_0^t (U(t+h,s) - U(t,s))f(s,x(s-\rho(s)))ds\right| \\
&\quad + \left|\int_t^{t+h} (U(t,s)f(s,x(s-\rho(s)))ds\right| \\
&\leq I_{41}(h) + I_{42}(h).
\end{aligned}$$

By Hölder's inequality, we have

$$\mathbb{E}\|I_{41}(h)\| \leq t\mathbb{E}\int_0^t \|U(t+h,s)-U(t,s))f(s,x(s-\rho(s))\|^2 ds.$$

Again exploiting properties of Definition 2.1, we obtain

$$\lim_{h\longrightarrow 0}(U(t+h,s)-U(t,s))f(s,x(s-\rho(s)))=0,$$

and

$$\|U(t+h,s)-U(t,s))f(s,x(s-\rho(s))\|$$

$$\leq Me^{-\beta(t-s)}(e^{-\beta h}+1)\|f(s,x(s-\rho(s))\| \in L^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_{41}(h)\|^2 = 0.$$

On the other hand, by $(\mathcal{H}.1)$, $(\mathcal{H}.2)$, and the Hölder's inequality, we have

$$\mathbb{E}\|I_{42}(h)\| \leq \frac{M^2C_2(1-e^{-2\beta h})}{2\beta}\int_t^{t+h}(1+\mathbb{E}\|x(s-\rho(s))\|^2)ds.$$

Thus

$$\lim_{h\to 0} I_{42}(h)=0.$$

Now, for the term $I_5(h)$, we have

$$\begin{aligned} I_5(h) &\leq \|\int_0^t (U(t+h,s)-U(t,s)\sigma(s)dB^H(s)\| \\ &\quad +\|\int_t^{t+h} U(t+h,s)\sigma(s)dB^H(s)\| \\ &\leq I_{51}(h)+I_{52}(h). \end{aligned}$$

By Lemma 2.4, we get that

$$E|I_{51}(h)|^2 \leq 2Ht^{2H-1}\int_0^t \|[U(t+h,s)-U(t,s)]\sigma(s)\|^2_{\mathcal{L}_2^0} ds.$$

Since

$$\lim_{h\to 0}\|[U(t+h,s)-U(t,s)]\sigma(s)\|^2_{\mathcal{L}_2^0}=0$$

and

$$\|(U(t+h,s)-U(t,s)\sigma(s)\|_{\mathcal{L}_2^0} \leq MLe^{-\beta(t-s)}e^{-\beta h+1} \in \mathbb{L}^1([0,T],\,ds),$$

we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathbb{E}|I_{51}(h)|^2=0.$$

Again by Lemma 2.4, we get that

$$\mathbb{E}|I_{52}(h)|^2 \leq \frac{2Ht^{2H-1}LM^2(1-e^{-2\beta h})}{2\beta}.$$

Thus

$$\lim_{h\to 0}\mathbb{E}|I_{52}(h)|^2=0.$$

The above arguments show that $\lim_{h\to 0} \mathbb{E}\|\psi(x)(t+h) - \psi(x)(t)\|^2 = 0$. Hence, we conclude that the function $t \to \psi(x)(t)$ is continuous on $[0,T]$ in the $\mathbb{L}^2$-sense.

Step 2: Now, we are going to show that ψ is a contraction mapping in $S_{T_1}(\varphi)$ with some $T_1 \le T$ to be specified later. Let $x, y \in S_T(\varphi)$, by using the inequality $(a+b+c)^2 \le \frac{1}{\nu}a^2 + \frac{2}{1-\nu}b^2 + \frac{2}{1-\nu}c^2$, where $\nu := L_* M_* < 1$, we obtain for any fixed $t \in [0,T]$

$$\begin{aligned}
&\|\psi(x)(t) - \psi(y)(t)\|^2 \\
&\le \frac{1}{\nu}\|g(t, x(t-r(t))) - g(t, y(t-r(t)))\|^2 \\
&\quad + \frac{2}{1-\nu}\|\int_0^t U(t,s)A(s)(g(s, x(s-r(s))) - g(s, y(s-r(s)))ds\|^2 \\
&\quad + \frac{2}{1-\nu}\|\int_0^t U(t,s)(f(s, x(s-\rho(s))) - f(s, y(s-\rho(s)))ds\|^2 \\
&= \sum_{k=1}^{3} J_k(t).
\end{aligned}$$

By using the fact that the operator $\|(A^{-1}(t))\|$ is bounded, combined with the condition $(\mathcal{H}.3)$, we obtain that

$$\begin{aligned}
\mathbb{E}\|J_1(t)\| &\le \frac{1}{\nu}\|A^{-1}(t)\|^2 \mathbb{E}|A(t)g(t, x(t-r(t))) - A(t)g(t, y(t-r(t)))\|^2 \\
&\le \frac{L_*^2 M_*^2}{\nu}\mathbb{E}\|x(t-r(t)) - y(t-r(t))\|^2 \\
&\le \nu \sup_{s\in[-\tau,t]} \mathbb{E}\|x(s) - y(s)\|^2.
\end{aligned}$$

By hypothesis $(\mathcal{H}.3)$ combined with Hölder's inequality, we get that

$$\begin{aligned}
\mathbb{E}\|J_2(t)\| &\le \mathbb{E}\|\int_0^t U(t,s)\left[A(t)g(t, x(t-r(t))) - A(t)g(t, y(t-r(t)))\right]ds\| \\
&\le \frac{2}{1-\nu}\int_0^t M^2 e^{-2\beta(t-s)}ds \int_0^t \mathbb{E}\|x(s-r) - y(s-r)\|^2 ds \\
&\le \frac{2M^2 L_*^2}{1-\nu}\frac{1-e^{-2\beta t}}{2\beta} t \sup_{s\in[-\tau,t]} \mathbb{E}\|x(s) - y(s)\|^2.
\end{aligned}$$

Moreover, by hypothesis $(\mathcal{H}.2)$ combined with the Hölder's inequality, we can conclude that

$$\begin{aligned}
E\|J_3(t)\| &\le E\|\int_0^t U(t,s)\left[f(s, x(s-\rho(s))) - f(s, y(s-\rho(s)))\right]ds\|^2 \\
&\le \frac{2C_1^2}{1-\nu}\int_0^t M^2 e^{-2\beta(t-s)}ds \int_0^t \mathbb{E}\|x(s-r) - y(s-r)\|^2 ds \\
&\le \frac{2M^2 C_1^2}{1-\nu}\frac{1-e^{-2\beta t}}{2\beta} t \sup_{s\in[-\tau,t]} \mathbb{E}\|x(s) - y(s)\|^2.
\end{aligned}$$

Hence

$$\sup_{s\in[-\tau,t]} \mathbb{E}\|\psi(x)(s)-\psi(y)(s)\|^2 \le \gamma(t) \sup_{s\in[-\tau,t]} \mathbb{E}\|x(s)-y(s)\|^2,$$

where

$$\gamma(t) = \nu + [L_*^2 + C_1^2]\frac{2M^2}{1-\nu}\frac{1-e^{-2\beta t}}{2\beta}t$$

By condition $(\mathcal{H}.3)$, we have $\gamma(0)=\nu=L_*M_*<1$. Then there exists $0<T_1\le T$ such that $0<\gamma(T_1)<1$ and ψ is a contraction mapping on $S_{T_1}(\varphi)$ and therefore has a unique fixed point, which is a mild solution of equation (1.1) on $[-\tau,T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-\tau,T]$ in finitely many steps. This completes the proof. □

4. An Example

In recent years, the interest in neutral systems has been growing rapidly due to their successful applications in practical fields such as physics, chemical technology, bioengineering, and electrical networks. We consider the following stochastic partial neutral functional differential equation with finite delays τ_1 and τ_2 $(0\le\tau_i\le\tau<\infty,\ i=1,2)$, driven by a cylindrical fractional Brownian motion:

$$\begin{cases} d\left[u(t,\zeta)+G(t,u(t-\tau_1,\zeta))\right] = \Big[\frac{\partial^2}{\partial^2\zeta}u(t,\zeta)+b(t,\zeta)u(t,\zeta) \\ \quad +F(t,u(t-\tau_2,\zeta))\Big]dt+\sigma(t)dB^H(t),\ 0\le t\le T,\ 0\le\zeta\le\pi, \\ u(t,0)=u(t,\pi)=0,\ \ 0\le t\le T, \\ u(t,\zeta)=\varphi(t,\zeta),\ \ t\in[-\tau,0],\ \ 0\le\zeta\le\pi, \end{cases} \tag{4.1}$$

where B^H is a fractional Brownian motion, $b(t,\zeta)$ is a continuous function and is uniformly Hölder continuous in t, F, $G:\mathbb{R}^+\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions.

To study this system, we consider the space $X=L^2([0,\pi])$ and the operator $A:D(A)\subset X\longrightarrow X$ given by $Ay=y''$ with

$$D(A)=\{y\in X: y''\in X, \quad y(0))=y(\pi)=0\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\ge0}$ on X. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n\in\mathbb{N}$ and the corresponding normalized eigenfunctions given by $e_n:=\sqrt{\frac{2}{\pi}}\sin nx$, $n=1,2,...$, in addition $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal basis in X and

$$T(t)x=\sum_{n=1}^{\infty}e^{-n^2t}<x,e_n>e_n,$$

for $x\in X$ and $t\ge0$.

Now, we define an operator $A(t):D(A)\subset X\longrightarrow X$ by

$$A(t)x(\zeta)=Ax(\zeta)+b(t,\zeta)x(\zeta).$$

By assuming that $b(.,.)$ is continuous and that $b(t,\zeta) \leq -\gamma$ $(\gamma > 0)$ for every $t \in \mathbb{R}$, $\zeta \in [0,\pi]$, it follows that the system

$$\begin{cases} u'(t) & = A(t)u(t), \quad t \geq s, \\ u(s) & = x \in X, \end{cases}$$

has an associated evolution family given by

$$U(t,s)x(\zeta) = \left[T(t-s)\exp\left(\int_s^t b(\tau,\zeta))d\tau\right)x \right](\zeta).$$

From this expression, it follows that $U(t,s)$ is a compact linear operator and that for every $s,t \in [0,T]$ with $t > s$

$$\|U(t,s)\| \leq e^{-(\gamma+1)(t-s)}.$$

In addition, $A(t)$ satisfies the assumption $\mathcal{H}_1$ (see [2]).

To rewrite the initial-boundary value problem (4.1) in the abstract form we assume the following:

i) The substitution operator $f : [0,T] \times X \longrightarrow X$ defined by $f(t,u)(.) = F(t,u(.))$ is continuous and we impose suitable conditions on F to verify assumption $\mathcal{H}_2$.

ii) The substitution operator $g : [0,T] \times X \longrightarrow X$ defined by $g(t,u)(.) = G(t,u(.))$ is continuous and we impose suitable conditions on G to verify assumptions $\mathcal{H}_2$ and $\mathcal{H}_3$.

iii) The function $\sigma : [0,T] \longrightarrow \mathcal{L}_2^0(L^2([0,\pi]), L^2([0,\pi]))$ is bounded, that is, there exists a positive constant L such that $\|\sigma(t)\|_{\mathcal{L}_2^0} \leq L < \infty$, where $L := sup_{t\in[0,T]}e^{-t}$. uniformly in $t \in [0,T]$.

If we put

$$\begin{cases} u(t)(\zeta) = u(t,\zeta), \ t \in [0,T], \ \zeta \in [0,\pi], \\ \\ u(t,\zeta) = \varphi(t,\zeta), \ t \in [-\tau,0], \ \zeta \in [0,\pi], \end{cases} \tag{4.2}$$

then the problem (4.1) can be written in the abstract form

$$\begin{cases} d[x(t) + g(t,x(t-r(t)))] = [A(t)x(t) + f(t,x(t-\rho(t))]dt + \sigma(t)dB^H(t), \\ \\ x(t) = \varphi(t), \ -\tau \leq t \leq 0. \end{cases}$$

Furthermore, if we assume that the initial data $\varphi = \{\varphi(t) : -\tau \leq t \leq 0\}$ satisfies $\varphi \in \mathcal{C}([-\tau,0], \mathbb{L}^2(\Omega, X))$, thus all the assumptions of Theorem 3.2 are fulfilled. Therefore, we conclude that the system (4.1) has a unique mild solution on $[-\tau,T]$.

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