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BASIC RESULTS ON NONLINEAR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATION VIA SUCCESSIVE APPROXIMATIONS AND INTEGRAL INEQUALITY

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Abstarct: In this paper, we study an initial value problem for an implicit nonlinear fractional differential equation with the Riemann-Liouville derivative. The existence, uniqueness, continuous dependence of solutions on initial conditions, parameters and functions involved in the equation are discussed via successive approximations approach. The same results are derived by the integral inequality established by Haiping Ye *et al.* Further existence and uniqueness of solution also proved by the Banach contraction principle.

Key words: Implicit fractional equation, Riemann-Liouville derivative, Successive approximations, Integral inequality, Fixed point theorem, Fractional calculus; Existence and uniqueness of solutions, Continuous dependence.

Mathematics Subject Classification: 26A33,26D15, 34A08, 34A09, 34A12, 34A34, 39B12, 34G20.

1. INTRODUCTION

Consider the the following nonlinear implicit fractional differential equation of the form:

$$D^{\alpha} x(t) = f(t, x(t), D^{\alpha} x(t)), \quad 0 < t \leq b, \tag{1.1}$$

$$D^{\alpha-1} x(t) \big|_{t=0} = x_0, \tag{1.2}$$

where $0 < \alpha < 1$, the unknown $x(\cdot)$ takes values in the real space $\mathbb{R}$; f: $\in C((0, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and and x_0 is a given element of $\mathbb{R}$. The operator D^{α} denotes the Riemann-Liouville fractional derivative operator.

Fractional differential equations arise in variety of applications and their study is of great interest. Several authors have studied the equations (1.1) in different forms of non-integer differential equations and their special versions with different view point, see [3, 7, 11, 12, 15, 17, 18, 26, 27, 28] and the reference are given therein. Theory of fractional differential equations has been extensively studied in recent years, refer the monographs of Abbas *et al.* [1, 2], Das [6], Kai Diethelm [8], Kilbas *et al.* [16], Miller and Ross [19], Podlubny I. [23], and Sabatier *et al.* [24].

The purpose of the present paper is to study the existence, uniqueness and continuous dependence of solutions to the initial value problem (1.1)-(1.2). The main tools employed in the analysis are based on the applications of the successive approximations approach, the integral inequality and the Banach fixed point theorem.

Furthermore, our aim is to generalize the some results presented by [4, 5, 22, 23, 25]. Also we extends the results studied for fractional differential equations with Liouville-Caputo derivative by [13, 14, 20] to an implicit nonlinear equations with Riemann-Liouville derivative.

The paper is organized as follows. In Section 2, we present the preliminaries. Section 3 deal with the our main result and the qualitative properties of solutions via successive approximations approach. Section 4 concern uniqueness of solution and the properties of solutions via integral inequality. In section 5, we will prove the existence and uniqueness of solution by fixe point theorem.

2. PRELIMINARIES AND HYPOTHESES

We shall setforth some preliminaries that will be used in our subsequent discussion.

Definition 2.1 *A real function $f(t), t>0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^p g(t)$, where $g(t) \in C[0, \infty)$, and it is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu$, $n \in \mathbb{N}$.*

Definition 2.2 *A function $f \in C_\mu$ $\mu \geq -1$ is said to be Riemann-Liouville fractional integrable of order $\alpha \in \mathbb{R}^+$ if*

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds < \infty,$$

where Γ is the Euler gamma function and if $\alpha = 0$, then $I^0 f(t) = f(t)$.

Definition 2.3 *The fractional derivative in the Caputo sense is defined as*

$$\frac{d^\alpha f(t)}{dt^\alpha} = I^{n-\alpha}\left(\frac{d^n f(t)}{dt^n}\right) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$ and $f \in C_{-1}^n$.

Definition 2.4 *A two-parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\lambda,\mu}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\lambda k+\mu)}, (\lambda > 0, \mu > 0).$$

The following generalized singular integral inequality established by Hiping Ye, Jianming Gao and Yongsheng Ding [9] is crucial in the proof of our results.

Lemma 2.5 *Suppose $\beta > 0$, $a(t)$ is be continuous function locally integrable on $0 \le t \le T$ (some $T \le \infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \le t \le T$, $g(t) \le M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \le t \le T$ with*

$$u(t) \le a(t) + g(t)\int_0^t (t-s)^{\beta-1}u(s)ds, \quad t \in [0,T].$$

Then

$$u(t) \le a(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\Big]ds, 0 \le t \le T.$$

Lemma 2.6 Let $\alpha, \beta \in [0, \infty)$. Then

$$\int_0^t s^{\alpha-1}(t-s)^{\beta-1} = t^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

3. RESULTS VIA SUCCESSIVE APPROXIMATIONS

Let us suppose that $f(t,x,y)$ is defined in a domain D of a space (t, x, y), and define a region $R(h, K) \subset D$ as a set of points $(t, x, y) \in D$, which satisfy the following inequalities:

$$R(h,K): \{0 < t < h, \quad |t^{1-\alpha}x(t) - \frac{x_0}{\Gamma(\alpha)}| \le K, \quad y \in \mathbb{R}\}, \tag{3.1}$$

where h and K are constants.

3.1. Existence and Uniqueness of Solution:

The following main result deals with existence and uniqueness of the solution for the problem (1.1)-(1.2) by the ideas of successive approximations.

Theorem 3.1 *Let $f(t, x, y)$ be a real-valued continuous function, defined in the domain D, satisfying in D the condition:*

(H_1) There exists $p \in C([0, b], \mathbb{R}_+)$ and $L \in (0,1)$ such that

$$|f(t,x,y) - f(t,\bar{x},\bar{x})| \le p(t)|x-\bar{x}| + L|y-\bar{y}|.$$

(H2) $\max_{0\le t\le h} f(t, x, y) \le M < \infty$, for all $(t, x, y) \in D$.

Let also

$$K \ge \frac{Mh}{\Gamma(1+\alpha)}.$$

Then the initial value problem (1.1)-(1.2) has a unique and continuous solution on $(0, h]$.

Proof. Following the fractional calculus and results proved in, one can say that the equation

$$x(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), D^\alpha x(s)) ds \tag{3.2}$$

is equivalent to the initial value problem (1.1)-(1.2).

The method of proof of this theorem is based on the ideas due to M. A. Al-Bassam [4], E. Pitcher and W. E. Sewell [22] and I. Podlubny [23].

Now we are set to define certain approximation to a solution of (1.1)-(1.2). First of all we stat with an approximation to a solution and improve it by iteration. It is expected that these iterations converge to a solution of (1.1)-(1.2) in the limit. The importance of equation (3.2) now springs up. In this connection, we mention that the estimates can be handled easily with integrals rather that derivatives.

A rough approximation to a solution of (1.1)-(1.2) is just given condition:

$$x_0(t) = \frac{x_0 t^{\alpha-1}}{\Gamma(\alpha)}. \tag{3.3}$$

We may get a better approximation by substituting x_0 (t) in the right of sides of (3.2), thus obtaining a new approximation x_1 (t) give by

$$x_1(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_0(s), D^\alpha x_0(s)) ds. \tag{3.4}$$

To get a still better approximation we repeat the process thereby defining

$$x_2(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_1(s), D^\alpha x_1(s)) ds. \tag{3.5}$$

In general,

$$x_n(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_{n-1}(s), D^\alpha x_{n-1}(s)) ds, \quad n = ,1,2, \dots \tag{3.6}$$

We will show that $\lim_{n\to\infty})$ x_n (t) exists and gives the desired solution $x(t)$ of the equation (3.2) on $(0, h]$.

First, it can be shown by induction that for $0 < t \le h$, we have x_n $(t) \in R(h, K)$ for all n. Indeed,

$$|t^{1-\alpha} x_n(t) - \frac{x_0}{\Gamma(\alpha)}| \le \frac{Mh}{\Gamma(1+\alpha)} \le K, \quad foralln = 1,2,\cdots. \tag{3.7}$$

Then for $n = 1$ and using hypothesis (H_2), we have

$$
\begin{aligned}
|t^{1-\alpha}x_1(t) - \tfrac{x_0}{\Gamma(\alpha)}| &\le \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, x_0(s), D^\alpha x_0(s))|ds \\
&\le \frac{t^{1-\alpha}}{\Gamma(\alpha)} M \int_0^t (t-s)^{\alpha-1}ds \\
&= \frac{t^{1-\alpha}}{\Gamma(\alpha)} M \frac{t^\alpha}{\alpha} \\
&= \frac{Mt}{\Gamma(1+\alpha)} \\
&\le \frac{Mh}{\Gamma(1+\alpha)} \\
&\le K.
\end{aligned}
\tag{3.8}
$$

This proves (3.7) is true for $n = 1$. Now we assume that for $n = m$, $x_m(t)$ is defined on $0 < t \le h$ satisfies the (3.7). Therefore x_{m+1} is defined on $0 < t \le h$ and from (3.6), we obtain

$$
\begin{aligned}
|t^{1-\alpha}x_{m+1}(t) - \tfrac{x_0}{\Gamma(\alpha)}| &\le \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, x_m(s), D^\alpha x_m(s))|ds \\
&\le \frac{t^{1-\alpha}}{\Gamma(\alpha)} M \int_0^t (t-s)^{\alpha-1}ds \\
&= \frac{t^{1-\alpha}}{\Gamma(\alpha)} M \frac{t^\alpha}{\alpha} \\
&= \frac{Mt}{\Gamma(1+\alpha)} \\
&\le \frac{Mh}{\Gamma(1+\alpha)} \\
&\le K.
\end{aligned}
\tag{3.9}
$$

Thus x_{m+1} satisfies (3.7).

Further, it can be shown by induction that for all n

$$
|x_n(t) - x_{n-1}| \le M\left(\frac{P}{1-L}\right)^{n-1}\frac{(t^\alpha)^n}{\Gamma(1+n\alpha)}, \quad (0 \le t \le h) \tag{3.10}
$$

In fact, using (3.6) and (H_2), for $n = 1$, we have

$$
\begin{aligned}
|x_1(t) - x_0| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, x_0(s), D^\alpha x_0(s))|ds \\
&\le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&= \frac{M}{\Gamma(\alpha)}\frac{t^\alpha}{\alpha} \\
&= \frac{Mt^\alpha}{\Gamma(1+\alpha)}, \quad (0 < t \le h).
\end{aligned}
\tag{3.11}
$$

Also for $n = 2$, using (3.6) and (H_1), we observe that

$$|x_2(t) - x_1(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_1(s), D^\alpha x_1(s)) - f(s, x_0(s), D^\alpha x_0(s))| ds$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s)|x_1(s) - x_0(s)| + L|D^\alpha (x_1(s) - x_0(s))|] ds. \tag{3.12}$$

But by hypothesis (H_1), for any $t \in [0, h]$ and $x_1(t), x_0(t) \in R(h, K)$, we have

$$|D^\alpha (x_1(t) - x_0(t))| = |f(t, x_1(t), D^\alpha x_1(t)) - f(t, x_0(t), D^\alpha x_0(t))|$$
$$\leq [p(t)|x_1(t) - x_0(t)| + L|D^\alpha (x_1(t) - x_0(t))|].$$

This implies

$$|D^\alpha (x_1(t) - x_0(t))| \leq \frac{p(t)}{1-L} |x_1(t) - x_0(t)|. \tag{3.13}$$

Using (3.13) in (3.12) and recall (3.11), we get

$$|x_2(t) - x_1(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s)|x_1(s) - x_0(s)| + L \frac{p(s)}{1-L} |x_1(s) - x_0(s)|] ds$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\frac{p(s)}{1-L}] |x_1(s) - x_0(s)| ds$$
$$\leq \frac{1}{\Gamma(\alpha)} (\frac{P}{1-L}) \int_0^t (t-s)^{\alpha-1} |x_1(s) - x_0(s)| ds$$
$$\leq \frac{1}{\Gamma(\alpha)} (\frac{P}{1-L}) \int_0^t (t-s)^{\alpha-1} \frac{Ms^\alpha}{\Gamma(1+\alpha)} ds$$
$$= M \frac{1}{\Gamma(1+\alpha)} \frac{1}{\Gamma(\alpha)} (\frac{P}{1-L}) \int_0^t (t-s)^{\alpha-1} s^\alpha ds$$
$$= M \frac{1}{\Gamma(1+\alpha)} \frac{1}{\Gamma(\alpha)} (\frac{P}{1-L}) \frac{\Gamma(\alpha)\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} t^{2\alpha}$$
$$= M (\frac{P}{1-L}) \frac{(t^\alpha)^2}{\Gamma(1+2\alpha)}, \tag{3.14}$$

where $P = \max\{p(t) : t \in [0, b]\}$. This proves the result in (3.10) is true for $n = 2$. Let us assume that this result holds for $n = n-1$, i.e.

$$|x_{n-1}(t) - x_{n-2}| \leq M (\frac{P}{1-L})^{n-2} \frac{(t^\alpha)^{n-1}}{\Gamma(1+(n-1)\alpha)}, \quad (0 \leq t \leq h) \tag{3.15}$$

Then using (3.6), (H_1) and (3.15), we have

$$|x_n(t) - x_{n-1}(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_{n-1}(s), D^\alpha x_{n-1}(s)) - f(s, x_{n-2}(s), D^\alpha x_{n-2}(s))| ds$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s)|x_{n-1}(s) - x_{n-2}(s)| + L|D^\alpha (x_{n-1}(s) - x_{n-2}(s))|] ds$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s)|x_{n-1}(s) - x_{n-2}(s)| + L \frac{p(s)}{1-L} |x_{n-1}(s) - x_{n-2}(s)|] ds$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\frac{p(s)}{1-L}] |x_{n-1}(s) - x_{n-2}(s)| ds$$
$$\leq \frac{1}{\Gamma(\alpha)} (\frac{P}{1-L}) \int_0^t (t-s)^{\alpha-1} |x_{n-1}(s) - x_{n-2}(s)| ds$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})\int_0^t (t-s)^{\alpha-1}M(\frac{P}{1-L})^{n-2}\frac{(s^\alpha)^{n-1}}{\Gamma(1+(n-1)\alpha)}ds\\
&= M\frac{1}{\Gamma(1+(n-1)\alpha)}\frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})^{n-1}\int_0^t (t-s)^{\alpha-1}s^{(n-1)\alpha}ds\\
&= M\frac{1}{\Gamma(1+(n-1)\alpha)}\frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})^{n-1}\frac{\Gamma(\alpha)\Gamma(1+(n-1)\alpha)}{\Gamma(1+n\alpha)}t^{n\alpha}\\
&= M(\frac{P}{1-L})^{n-1}\frac{(t^\alpha)^n}{\Gamma(1+n\alpha)}.
\end{aligned} \tag{3.16}$$

This means that the result in (3.10) is true for all n.

Now let us consider the series

$$\lim_{n\to\infty}[x_n(t) - x_0(t)] = \sum_{j=1}^{\infty} [x_j(t) - x_{j-1}(t)]. \tag{3.17}$$

According to the estimate (3.10), for $0 \leq t \leq$ h, the absolute value of terms of the series (3.10) are less than the corresponding terms of the convergent numeric series

$$M\sum_{j=1}^{\infty}\frac{(\frac{P}{1-L})^{j-1}h^{j\alpha}}{\Gamma(1+j\alpha)} = \frac{M}{\frac{P}{1-L}}\sum_{j=1}^{\infty}\frac{(\frac{P}{1-L})^{j}h^{j\alpha}}{\Gamma(1+j\alpha)} = \frac{M(1-L)}{P}[E_{\alpha,1}(\frac{P}{1-L}h^\alpha) - 1], \tag{3.18}$$

where $E_{\lambda,\mu}$ is the Mittag-Leffler function. This shows that the series converges uniformly and absolutely on the interval $0 \leq t \leq h$. Obviously, each term $[x_j(t) - x_{j-1}(t)]$ of the series is a continuous function of t for $0 \leq t \leq h$. Therefore, the sum of the series $x^*(t)$, is a continuous function for $0 \leq t \leq h$, and hence we have

$$x(t) = \lim_{n\to\infty} x_n(t) = x_0(t) + x^*(t),$$

is a continuous function for $0 < t \leq h$

We now show that the limit function $x(t)$ satisfies the fractional integral (3.2). By the definition of successive approximations

$$x_n(t) = x_0(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, x_{n-1}(s), D^\alpha x_{n-1}(s))ds. \tag{3.19}$$

In view of (3.19), we have

$$\begin{aligned}
&|x(t) - x_0(t) - \int_0^t (t-s)^{\alpha-1}f(s, x(s), D^\alpha x(s))ds|\\
&= |x(t) - x_n(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, x_{n-1}(s), D^\alpha x_{n-1}(s))ds\\
&\quad - \int_0^t (t-s)^{\alpha-1}f(s, x(s), D^\alpha x(s))ds|\\
&\leq |x(t) - x_n(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, x_{n-1}(s), D^\alpha x_{n-1}(s)) - f(s, x(s), D^\alpha x(s))|ds\\
&\leq |x(t) - x_n(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\frac{P}{1-L})|x_{n-1}(s) - x(s)|ds
\end{aligned}$$

$$= |x(t) - x_n(t)| + (\frac{P}{1-L})\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x_{n-1}(s) - x(s)|ds$$

$$\leq |x(t) - x_n(t)| + (\frac{P}{1-L})\frac{1}{\Gamma(\alpha)}\max_{0<t\leq h}|x_{n-1}(t) - x(t)|\int_0^t (t-s)^{\alpha-1}ds$$

$$= |x(t) - x_n(t)| + (\frac{P}{1-L})\frac{1}{\Gamma(\alpha)}\max_{0<t\leq h}|x_{n-1}(t) - x(t)|\frac{t^\alpha}{\alpha}$$

$$\leq |x(t) - x_n(t)| + (\frac{P}{1-L})\frac{1}{\Gamma(1+\alpha)}\max_{0<t\leq h}|x_{n-1}(t) - x(t)|h^\alpha. \tag{3.20}$$

The uniform convergence of x_n to $x(t)$ now implies that the right hand side of (3.20) tend to zero as $n \to \infty$. But the left side of (3.20) is independent of n. Thus $x(t)$ satisfies the fractional integral equation (3.2). This proves the existence of solution in $0 < t \leq h$.

Let us now prove that if $x(t)$ and $\overline{x}(t)$ are any two solutions of the problem (1.1)-(1.2), then they coincide on $0 < t \leq h$. Therefore, x(t) and $\overline{x}(t)$ as solution we have

$$x(t) = \frac{x_0 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s), D^\alpha x(s))ds, \tag{3.21}$$

and

$$\overline{x}(t) = \frac{x_0 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, \overline{x}(s), D^\alpha \overline{x}(s))ds, (0 < t \leq h). \tag{3.22}$$

We shall now prove by mathematical induction that

$$|\overline{x}(t) - x_n(t)| \leq (\frac{P}{1-L})^n \frac{K}{\Gamma(1+n\alpha)} t^{n\alpha}, n = 1,2,\cdots. \tag{3.23}$$

Since $\overline{x}(t) \in R(h, K)$, we have $|\overline{x}(t) - x_0(t)| \leq K$ for $0 < t \leq h$. Then for $n = 1$, we have

$$\begin{aligned}
|\overline{x}(t) - x_1(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, \overline{x}(s), D^\alpha \overline{x}(s)) - f(s, x_0(s), D^\alpha x_0(s))|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\frac{P}{1-L})|\overline{x}(s) - x_0(s)|ds \\
&\leq \frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})\int_0^t (t-s)^{\alpha-1}Kds \\
&= K\frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})\int_0^t (t-s)^{\alpha-1}ds \\
&= K\frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})\frac{t^\alpha}{\alpha} \\
&= \frac{Kt^\alpha}{\Gamma(1+\alpha)}(\frac{P}{1-L}),
\end{aligned} \tag{3.24}$$

which is (3.23) for $n = 1$. We assume it is true for $n = n–1$ and prove it for exactly n. Then for $n = n–1$, we have

$$|\overline{x}(t) - x_{n-1}(t)| \leq (\frac{P}{1-L})^{n-1} \frac{K}{\Gamma(1+(n-1)\alpha)} t^{(n-1)\alpha}. \tag{3.25}$$

Hence we have for exactly n

$$\begin{aligned} |\overline{x}(t) - x_n(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,\overline{x}(s),D^\alpha \overline{x}(s)) - f(s,x_{n-1}(s),D^\alpha x_{n-1}(s))| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\frac{P}{1-L}) |\overline{x}(s) - x_{n-1}(s)| ds \\ &\leq \frac{1}{\Gamma(\alpha)} (\frac{P}{1-L}) \int_0^t (t-s)^{\alpha-1} (\frac{P}{1-L})^{n-1} \frac{K}{\Gamma(1+(n-1)\alpha)} s^{(n-1)\alpha} ds \\ &= \frac{1}{\Gamma(\alpha)} \frac{K}{\Gamma(1+(n-1)\alpha)} (\frac{P}{1-L})^n \int_0^t (t-s)^{\alpha-1} s^{(n-1)\alpha} ds \\ &= K \frac{(\frac{P}{1-L} t^\alpha)^n}{\Gamma(1+n\alpha)}. \end{aligned} \tag{3.26}$$

Thus by induction the inequality (3.23) holds for all n on $(0, h]$. In the right hand side, we recognize-up to the constant multiplier K-the general term of the series for the Mittag-Leffler function $E_{\alpha,1}(\frac{P}{1-L} t^\alpha)^n$, and therefore for all t

$$\lim_{n\to\infty} (\frac{P}{1-L} t^\alpha)^n = 0. \tag{3.27}$$

Taking the limit of (3.23) or (3.26) as $n \to \infty$, we have

$$\overline{x}(t) = \lim_{n\to\infty} x_n(t), \quad (0 < t \leq h). \tag{3.28}$$

But recall that $\lim_{n\to\infty} x_n(t) = x(t), \quad (0 < t \leq h)$. Thus $\overline{x}(t) = x(t), \quad (0 < t \leq h)$. This proves the uniqueness of the solution of the problem (1.1)-(1.2), which completing the proof of the theorem.

3.2. Dependence on Initial Conditions

In this section, we study the continuous dependence of solutions of the initial value problem (1.1)-(1.2). For this we consider the changes in the solutions which are caused by small changes in the conditions (1.2):

$$D^{\alpha-1} x(t)|_{t=0} = x_0 + \delta, \tag{3.29}$$

where δ is arbitrary constant.

The following theorem is a generalization of M. A. Al-Bassam and I. Podlubny results and deals with the continuous dependence of solutions of equation (1.1) on given initial conditions.

Theorem 3.2 *Let the assumptions of Theorem 3.1 hold. If x(t) is a solution of the equation (1.1) satisfying the initial condition (1.2), and $\overline{x}(t)$ is a solution of the same equation satisfying the initial condition (3.29), then for $0 < t \leq h$ the following holds:*

$$|x(t) - \overline{x}(t)| \leq |\delta| t^{\alpha-1} E_{\alpha,\alpha}\left(\frac{P}{1-L} t^{\alpha}\right). \tag{3.30}$$

Proof. In accordance with Theorem 3.1, we have

$$x(t) = \lim_{n\to\infty} x_n(t), \tag{3.31}$$

$$x_0(t) = \frac{x_0 t^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.32}$$

$$x_n(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_{n-1}(s), D^{\alpha} x_{n-1}(s)) ds, \quad n = ,1,2, \ldots \tag{3.33}$$

and

$$\overline{x}(t) = \lim_{n\to\infty} \overline{x}_n(t), \tag{3.34}$$

$$\overline{x}_0(t) = \frac{(x_0+\delta) t^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.35}$$

$$\overline{x}_n(t) = \overline{x}_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{x}_{n-1}(s), D^{\alpha} \overline{x}_{n-1}(s)) ds, \quad n = ,1,2, \ldots \tag{3.36}$$

From (3.32) and (3.35), it directly follows that

$$\begin{aligned} |x_0(t) - \overline{x}_0(t)| &\leq \left| \frac{x_0 t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(x_0+\delta) t^{\alpha-1}}{\Gamma(\alpha)} \right| \\ &\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} |x_0 - x_0 - \delta| \\ &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned} \tag{3.37}$$

Using subsequently the relationships (3.33) and (3.36), hypothesis (H_1), the inequality (3.37), and the rule for the Riemann-Liouville fractional differentiation of the power function, we obtain

$$\begin{aligned} |x_1(t) - \overline{x}_1(t)| &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_0(s), D^{\alpha} x_0(s)) - f(s, \overline{x}_0(s), D^{\alpha} \overline{x}_0(s))| ds \\ &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{P}{1-L}\right) |x_0(s) - \overline{x}_0(s)| ds \\ &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{P}{1-L}\right) \frac{|\delta| s^{\alpha-1}}{\Gamma(\alpha)} ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{P}{1-L}\right)\frac{|\delta|}{\Gamma(\alpha)^2}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds \\
&= \frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{P}{1-L}\right)\frac{|\delta|t^{2\alpha-1}}{\Gamma(2\alpha)} \\
&= |\delta|t^{\alpha-1}\sum_{j=0}^{1}\frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma((j+1)\alpha)}.
\end{aligned} \tag{3.38}$$

Similarly, we have

$$\begin{aligned}
|x_2(t)-\overline{x}_2(t)| &\leq \frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\frac{P}{1-L}\right)|x_1(s)-\overline{x}_1(s)|ds \\
&\leq \frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\frac{P}{1-L}\right)\left[|\delta|s^{\alpha-1}\sum_{j=0}^{1}\frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma((j+1)\alpha)}\right]ds \\
&\leq \frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{P}{1-L}\right)\frac{|\delta|}{\Gamma(\alpha)^2}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds \\
&\quad + \left(\frac{P}{1-L}\right)^2\frac{|\delta|}{\Gamma(\alpha)\Gamma(2\alpha)}\int_0^t (t-s)^{\alpha-1}s^{2\alpha-1}ds \\
&= \frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{P}{1-L}\right)\frac{|\delta|t^{2\alpha-1}}{\Gamma(2\alpha)} + \left(\frac{P}{1-L}\right)^2\frac{|\delta|t^{3\alpha-1}}{\Gamma(3\alpha)} \\
&= |\delta|t^{\alpha-1}\sum_{j=0}^{2}\frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma((j+1)\alpha)},
\end{aligned} \tag{3.39}$$

and by induction

$$|x_n(t)-\overline{x}_n(t)| \leq |\delta|t^{\alpha-1}\sum_{j=0}^{n}\frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma((j+1)\alpha)}, \quad n = 1,2,\cdots. \tag{3.40}$$

Taking the limit of (3.40) as $n \to \infty$, we obtain

$$\begin{aligned}
|x(t)-\overline{x}(t)| &\leq |\delta|t^{\alpha-1}\sum_{j=0}^{\infty}\frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma((j+1)\alpha)} \\
&= |\delta|t^{\alpha-1}E_{\alpha,\alpha}\left(\frac{P}{1-L}t^{\alpha}\right)
\end{aligned} \tag{3.41}$$

which ends the proof of Theorem 3.2.

Remark 3.3 *It follows from this theorem that for every ? between 0 and h small changes in initial conditions (1.2) cause only small changes of the solution in the closed interval [ε, h] (which does not contain zero).*

On other hand, the solution may change significantly in $[0, \varepsilon]$. Indeed, if the non-disturbed initial conditions (1.2) are zero (i.e. . $x_0 = 0$, then the non-disturbed solution $y(t)$ is continuous in $[0, \varepsilon]$ and therefore bounded. However, the solution $x(t)$ corresponding to the disturbed initial conditions, may contain terms of the form $\frac{x_0 t^{\alpha-1}}{\Gamma(\alpha)}$, which for $\alpha < 1$ will make the disturbed solution unbounded at $t = 0$.

Remark 3.4 *The inequality (3.30) shows continuous dependence of solutions of the problem (1.1)-(1.2) on initial values. This also provides the uniqueness of the solution by putting the arbitrary constant δ is equal zero.*

3.3. Dependence on Functions Involved Therein:

In this section, we study the continuous dependence of solutions of (1.1) on the given initial data, and function involved therein.

Now, we consider the initial value problem (1.1)-(1.2) with

$$D^{\alpha}y(t) = F(t, y(t), D^{\alpha}y(t)), \quad 0 < t \leq b, \tag{3.42}$$

$$D^{\alpha-1}y(t)|_{t=0} = x_0 + \delta, \tag{3.43}$$

where δ is an arbitrary constant and $F \in C((0, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

The next theorem deals with the closeness of solutions of initial value problem (1.1)-(1.2) and initial value problem (3.42)-(3.43).

Theorem 3.5 *Suppose that f in (1.1)satisfy the hypotheses (H_1)-(H_2). Further suppose, for arbitrarily small constant $\varepsilon > 0$, that*

$$|f(t, x(t), D^{\alpha} x(t)) - F(t, x(t), D^{\alpha} (t))| \leq \varepsilon. \tag{3.44}$$

Then the solution $x(t)$ of the initial value problem (1.1)-(1.2) depends continuously on the functions involved in the right hand side of the equation(1.1).

Proof. Let us define the approximations for the problems (1.1)-(1.2) and (3.42)-(3.43) respectively:

$$x(t) = \lim_{n\to\infty} x_n(t), \tag{3.45}$$

$$x_0(t) = \frac{x_0 t^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.46}$$

$$x_n(t) = x_0(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x_{n-1}(s), D^{\alpha}x_{n-1}(s))ds, \quad n = ,1,2, \ldots \tag{3.47}$$

and

$$y(t) = \lim_{n\to\infty} y_n(t), \tag{3.48}$$

$$y_0(t) = \frac{(x_0+\delta)t^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.49}$$

$$y_n(t) = y_0(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F(s, y_{n-1}(s), D^{\alpha}y_{n-1}(s))ds, \quad n = ,1,2, \ldots \tag{3.50}$$

By following a similar arguments as in the proof of Theorem 3.2 and from the hypotheses, it follows that

$$\begin{aligned} |x_0(t) - y_0(t)| &\leq \left| \frac{x_0 t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(x_0+\delta)t^{\alpha-1}}{\Gamma(\alpha)} \right| \\ &\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} |x_0 - x_0 - \delta| \\ &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned} \tag{3.51}$$

Using subsequently the relationships (3.47) and (3.50), assumption (3.44) , the inequality (3.51), and the rule for the Riemann-Liouville fractional differentiation of the power function, we obtain

$$\begin{aligned} |x_1(t) - y_1(t)| &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_0(s), D^\alpha x_0(s)) - F(s, y_0(s), D^\alpha y_0(s))| ds \\ &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_0(s), D^\alpha x_0(s)) - f(s, y_0(s), D^\alpha y_0(s))| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y_0(s), D^\alpha y_0(s)) - F(s, y_0(s), D^\alpha y_0(s))| ds \\ &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{P}{1-L}\right) |x_0(s) - y_0(s)| ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varepsilon ds \\ &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{P}{1-L}\right) \frac{|\delta| s^{\alpha-1}}{\Gamma(\alpha)} ds \\ &\leq \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)} + \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{P}{1-L}\right) \frac{|\delta|}{\Gamma(\alpha)^2} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} ds \\ &= \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)} + \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{P}{1-L}\right) \frac{|\delta| t^{2\alpha-1}}{\Gamma(2\alpha)} \\ &= \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)} + |\delta| t^{\alpha-1} \sum_{j=0}^{1} \frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma((j+1)\alpha)}. \end{aligned} \tag{3.52}$$

Similarly, we have

$$\begin{aligned} |x_2(t) - y_2(t)| &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{P}{1-L}\right) |x_1(s) - y_1(s)| ds \\ &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{P}{1-L}\right) \frac{\varepsilon s^\alpha}{\Gamma(1+\alpha)} ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{P}{1-L}\right) \left[|\delta| s^{\alpha-1} \sum_{j=0}^{1} \frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma((j+1)\alpha)} \right] ds \\ &\leq \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)} + \left(\frac{P}{1-L}\right) \frac{\varepsilon t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{P}{1-L}\right) \frac{|\delta|}{\Gamma(\alpha)^2} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} ds \\ &\quad + \left(\frac{P}{1-L}\right)^2 \frac{|\delta|}{\Gamma(\alpha)\Gamma(2\alpha)} \int_0^t (t-s)^{\alpha-1} s^{2\alpha-1} ds \\ &= \varepsilon t^\alpha \sum_{j=0}^{1} \frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma(1+(j+1)\alpha)} + \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{P}{1-L}\right) \frac{|\delta| t^{2\alpha-1}}{\Gamma(2\alpha)} + \left(\frac{P}{1-L}\right)^2 \frac{|\delta| t^{3\alpha-1}}{\Gamma(3\alpha)} \\ &= \varepsilon t^\alpha \sum_{j=0}^{1} \frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma(1+(j+1)\alpha)} + |\delta| t^{\alpha-1} \sum_{j=0}^{2} \frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma((j+1)\alpha)}, \end{aligned} \tag{3.53}$$

and by induction for all n, we have

$$|x_n(t) - y_n(t)| \leq \varepsilon t^{\alpha} \sum_{j=0}^{n-1} \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma(1+(j+1)\alpha)} + |\delta| t^{\alpha-1} \sum_{j=0}^{n} \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma((j+1)\alpha)}. \tag{3.54}$$

Taking the limit of (3.54) as $n \to \infty$, we obtain

$$|x(t) - y(t)| \leq \varepsilon t^{\alpha} \sum_{j=0}^{\infty} \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma(1+(j+1)\alpha)} + |\delta| t^{\alpha-1} \sum_{j=0}^{\infty} \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma((j+1)\alpha)}$$

$$= \varepsilon t^{\alpha} E_{\alpha,\alpha+1}(\frac{P}{1-L} t^{\alpha}) + |\delta| t^{\alpha-1} E_{\alpha,\alpha}(\frac{P}{1-L} t^{\alpha}). \tag{3.55}$$

From (3.55) it follows that the solutions of the initial value problem (1.1)-(1.2) depends continuously on the functions involved in the right hand side of (1.1). This completes the proof of Theorem 3.5.

Remark 3.6 *The result given in Theorem 4.3 relates the solutions of the initial value problem (1.1)-(1.2) and (3.42)-(3.43) in the sense that if f is close to F, x_0 is close to $x_0 + \delta$ (i.e. depends upon δ and as δ is smaller and smaller), then not only the solutions of initial value problem (1.1)-(1.2) and (3.42)-(3.43) are close to each other, but also depends continuously on the functions involved the right hand side of (1.1).*

Next, consider the initial value problem (1.1)-(1.2) together with system of equations:

$$D^{\alpha} y(t) = f_k(t, y(t), D^{\alpha} y(t)), \quad t \in [0, b] \tag{3.56}$$

$$D^{\alpha-1} y(t)|_{t=0} = x_0 + \delta_k, \tag{3.57}$$

where $f_k \in C((0, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and δ_k is sequence of elements in $\mathbb{R}$ for $k = 1,2,?$.

The next corollary deals with the convergence of solutions of initial value problem (1.1)-(1.2) and initial value problem (3.56)-(3.57).

As an immediate consequence of Theorem 3.5, we have the following corollary.

Corollary 3.7 *Suppose that f in (1.1)satisfy the hypotheses (H_1)-(H_2). Further suppose, for arbitrarily small constants $\varepsilon_k > 0$, that*

$$|f(t, x(t), D^{\alpha} x(t)) - f_k(t, x(t), D^{\alpha}(t))| \leq \varepsilon_k, \tag{3.58}$$

where $\varepsilon_k \to 0$, and $\delta_k \to 0$ as $k \to \infty$. If $x(t)$ and $y_k(t)$ $(k = 1, 2, ...)$ are respectively the solutions of initial value problem (1.1)-(1.2) and (3.56)-(3.57) on J. Then as $k \to \infty$, $y_k(t) \to x(t)$ on $0 < t \leq h$.

Proof. For $k = 1,2, ...,$ the conditions of Theorem 3.5 hold. As an application of Theorem 3.5 yields

$$|x_n(t) - y_{kn}(t)| \le \varepsilon_k t^\alpha \sum_{j=0}^{n-1} \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma(1+(j+1)\alpha)} + |\delta_k| t^{\alpha-1} \sum_{j=0}^{n} \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma((j+1)\alpha)}, \quad (3.59)$$

where $x_n(t)$ and $y_{kn}(t)$ are iterations as defined in Theorem 3.5 for $n = 0, 1, 2, \ldots$. Now we allow $n \to \infty$, we have

$$|x(t) - y_k(t)| \le \varepsilon_k t^\alpha \sum_{j=0}^{\infty} \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma(1+(j+1)\alpha)} + |\delta_k| t^{\alpha-1} \sum_{j=0}^{\infty} \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma((j+1)\alpha)}$$

$$= \varepsilon_k t^\alpha E_{\alpha,\alpha+1}(\frac{P}{1-L} t^\alpha) + |\delta_k| t^{\alpha-1} E_{\alpha,\alpha}(\frac{P}{1-L} t^\alpha). \quad (3.60)$$

As $k \to \infty$, the required result follows from (3.60).

Remark 3.8 *The result obtained in Corollary provide sufficient conditions that ensures solutions of initial value problem (3.56)-(3.57) will converge to the solutions of initial value problem (1.1)-(1.2).*

3.4. Dependence on Parameters

In this section, we study the continuous dependence of solutions on certain parameters.

We consider the differential equations of fractional order:

$$D^\alpha x(t) = G(t, x(t), D^\alpha x(t), \mu_1), \quad (3.61)$$

$$D^\alpha y(t) = G(t, y(t), D^\alpha y(t), \mu_2), \quad (3.62)$$

for $t \in (0, b]$, where $G \in C((0, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and μ_1, μ_2 are real parameters and with initial condition given by (1.2).

The following theorem states the continuous dependency of solutions to (3.61)-(1.2) and (3.62)-(1.2) on parameters.

Theorem 3.9 *Let $G(t, x, y, \mu)$ be a real-valued continuous function satisfy:*

(H3) There exists $q \in C([0, b], \mathbb{R}_+)$ and $\overline{L} \in (0, 1)$ such that

$$|G(t, x, y, \mu) - G(t, \overline{x}, \overline{y}, \mu)| \le q(t)|x - \overline{x}| + \overline{L}|y - \overline{y}|.$$

(H4) There exists $n \in C([0, b], \mathbb{R}_+)$ such that

$$|G(t, x, y, \mu_1) - G(t, x, y, \mu_2)| \le n(t)|\mu_1 - \mu \ \ 2|.$$

If $x(t)$ and $y(t)$ are respectively solutions of (3.61) and (3.62), then

$$|x(t) - y(t)| \le N|\mu_1 - \mu_2| t^\alpha E_{\alpha,\alpha+1}(\frac{Q}{1-\overline{L}} t^\alpha), \quad (3.63)$$

where $Q = \max\{q(t)\colon t \in [0, b]\}$ and $N = \max\{n(t)\colon t \in [0, b]\}$.

Proof. By following a similar arguments as in the proof of Theorem 3.5 and from the hypotheses, it follows that

$$|x_n(t) - y_n(t)| \leq N|\mu_1 - \mu_2|t^\alpha \sum_{j=0}^{n-1} \frac{(\frac{Q}{1-L})^j t^{j\alpha}}{\Gamma(1+(j+1)\alpha)}. \tag{3.64}$$

where $x_n(t)$ and $y_n(t)$ are iterations as defined in Theorem 3.5 for $n = 1, 2, \ldots$. Now we allow $n \to \infty$, we have

$$|x(t) - y(t)| \leq N|\mu_1 - \mu_2|t^\alpha \sum_{j=0}^{\infty} \frac{(\frac{Q}{1-L})^j t^{j\alpha}}{\Gamma(1+(j+1)\alpha)}$$

$$= N|\mu_1 - \mu_2|t^\alpha E_{\alpha,\alpha+1}(\frac{Q}{1-L} t^\alpha). \tag{3.65}$$

From (3.65), it follows that the solutions (3.61) and (3.62) with (1.2) depend continuously on the parameters μ_1, μ_2.

Remark 3.10 *The result dealing with the property of a solution called "dependence of solutions on parameters". Here the parameters are scalars. Notice that the initial conditions do not involve parameters. The dependence on parameters are an important aspect in various physical problems.*

4. RESULTS VIA INTEGRAL INEQUALITY

Integral inequalities play an important role in the qualitative analysis of the solutions to differential and integral equations.The method of inequalities which provides explicit estimates on unknown functions has been a significant source in the study of many qualitative properties of solutions of various differential, integral and finite difference equations. It enable us to obtain valuable information about solutions without explicit knowledge of the solution process. In many cases while studying the behavior of solutions, the method which works very effectively to establish existence does not yield other properties of the solutions in ready fashion and one often needs some new ideas and methods in the analysis. The theory of integral inequalities with explicit estimates has emerged as an interesting and fascinating topic of applicable analysis with a wide range of applications.

Therefore, in this, section we will prove uniqueness of solution and continuous dependence on the orders of equations and initial conditions; on parameters and function involved therein via integral inequality. According to N. Heymans and I. Podlubny [10], it is possible to discuss the physical interpretation about initial conditions expressed in terms of Riemann-Liouville fractional derivatives and hence, our discussion here is meaningful.

4.1. Uniqueness of Solution

The existence and uniqueness of the initial value problem (1.1)-(1.2) have been discussed in the previous section. The following theorem proves the uniqueness of solutions to (1.1)-(1.2) without the existence part.

Theorem 4.1 *If the hypothesis* (H_1) *is holds, then the initial value problem (1.1)-(1.2) has at most one mild solution on* $(0, b], (b \leq \infty)$.

Proof. Let $x(t)$ and $y(t)$ be any two solutions of the initial value problem (1.1)-(1.2) and $u(t) = |x(t) - y(t)|$. Then by hypothesis, we have

$$\begin{aligned} u(t) &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s), D^\alpha x(s)) - f(s, y(s), D^\alpha y(s))| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\frac{P}{1-L}) |x(s) - y(s)| ds \\ &= \frac{1}{\Gamma(\alpha)} (\frac{P}{1-L}) \int_0^t (t-s)^{\alpha-1} |x(s) - y(s)| ds. \end{aligned} \tag{4.1}$$

Now a suitable application of Lemma 2.5 (with $g(t) = \frac{1}{\Gamma(\alpha)} (\frac{P}{1-L})$ and $\alpha(t) = 0$) to (4.1) yields

$$u(t) = |x(t) - y(t)| \leq 0,$$

which implies $x(t) = y(t)$ for $t \in (0, b]$. Thus there is at most one solution to the initial value problem (1.1)-(1.2) on $(0, b]$.

4.2. Dependence on Order and Initial Conditions

In this section, we shall consider the solutions of two initial value problems with neighboring orders and neighboring initial values. It is important to note that here we are considering a question which does not arise in the solution of differential equations of integer order.

The following theorem dealt with the dependence of the solution on the order and the initial condition to a certain implicit fractional differential equation with Riemann-Liouville fractional derivatives via integral inequality.

Theorem 4.2 *Let* $\alpha > 0$ *and* $\beta > 0$ *such that* $0 < \alpha - \beta < \alpha \leq 1$. *Let the function* f *in (1.1) satisfy the hypotheses* (H_1) *and* (H_2). *For* $0 < t \leq b$, *assume that* $x(t)$ *and* $y(t)$ *are respectively solutions of the initial value problems (1.1)-(1.2) and*

$$D^{\alpha-\beta} y(t) = f(t, y(t), D^\alpha y(t)), \quad 0 < t \leq b, \tag{4.2}$$

$$D^{\alpha-\beta-1} x(t)|_{t=0} = y_0, \tag{4.3}$$

Then, for $0 < t \leq b$, the following holds:

$$|y(t) - x(t)| \leq w(t) + \int_0^t [\sum_{j=1}^{\infty} \frac{(\frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})\Gamma(\alpha-\beta))^j}{\Gamma(j(\alpha-\beta))} (t-s)^{j(\alpha-\beta)-1} w(s)] ds, \tag{4.4}$$

where

$$w(t) = |\frac{y_0}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} - \frac{x_0}{\Gamma(\alpha)}t^{\alpha-1}| + |\frac{t^{\alpha-\beta}}{(\alpha-\beta)\Gamma(\alpha-\beta)} - \frac{t^{\alpha}}{\Gamma(1+\alpha)}|M + |\frac{t^{\alpha-\beta}}{\alpha-\beta}[\frac{1}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha)}]|M. \tag{4.5}$$

Proof. The solutions of the initial value problems (1.1)-(1.2) and (4.2)-(4.3) are respectively given by

$$x(t) = \frac{x_0}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, x(s), D^{\alpha}x(s))ds, \tag{4.6}$$

and

$$x(t) = \frac{y_0}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}f(s, y(s), D^{\alpha}y(s))ds. \tag{4.7}$$

Let $u(t) = |y(t) - x(t)|$. Then by hypothesis, we have

$$\begin{aligned} u(t) \le\ & |\frac{y_0}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} - \frac{x_0}{\Gamma(\alpha)}t^{\alpha-1}| \\ & +|\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}f(s,y(s),D^{\alpha}y(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-\beta-1}f(s,y(s),D^{\alpha}y(s))ds| \\ & +|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-\beta-1}f(s,y(s),D^{\alpha}y(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-\beta-1}f(s,x(s),D^{\alpha}x(s))ds| \\ & +|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-\beta-1}f(s,x(s),D^{\alpha}x(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s),D^{\alpha}x(s))ds| \\ \le\ & |\frac{y_0}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} - \frac{x_0}{\Gamma(\alpha)}t^{\alpha-1}| + |\frac{t^{\alpha-\beta}}{(\alpha-\beta)\Gamma(\alpha-\beta)} - \frac{t^{\alpha}}{\Gamma(1+\alpha)}|M \\ & +|\frac{t^{\alpha-\beta}}{\alpha-\beta}[\frac{1}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha)}]|M + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-\beta-1}(\frac{P}{1-L})|y(s)-x(s)|ds \\ =\ & w(t) + \frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})\int_0^t (t-s)^{\alpha-\beta-1}u(s)ds. \end{aligned} \tag{4.8}$$

Now a suitable application of Lemma 2.5 (with $g(t) = \frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})$ and $a(t) = w(t)$, $w(t)$ as given in (4.5)), to (4.8) which gives

$$u(t) \le w(t) + \int_0^t [\sum_{j=1}^{\infty} \frac{(\frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})\Gamma(\alpha-\beta))^j}{\Gamma(j(\alpha-\beta))}(t-s)^{j(\alpha-\beta)-1}w(s)]ds. \tag{4.9}$$

This concludes the theorem.

Remark 4.3 *It follows from Theorem 4.2 that for every ε between 0 and b small changes in order of differential equations and initial conditions cause only small changes of the solution in the closed interval $[\varepsilon, b]$ (which does not contain zero).*

Corollary 4.4 *Under the hypotheses of Theorem 4.2, if $\beta = 0$ and $y_0 = x_0 + \delta$, where δ is an arbitrary constant, then*

$$|y(t)-x(t)| \leq |\delta| t^{\alpha-1} E_{\alpha,\alpha}(\frac{P}{1-L} t^{\alpha}), \quad (0 < t \leq b). \tag{4.10}$$

Proof. If $\beta = 0$ and $y_0 = x_0 + \delta$, then from (4.5) we have $w(t) = \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)}$. By recalling the discussion of Theorem 4.2, we obtain

$$\begin{aligned}
u(t) &\leq \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^t [\sum_{j=1}^{\infty} \frac{(\frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})\Gamma(\alpha))^j}{\Gamma(j\alpha)} (t-s)^{j\alpha-1} \frac{|\delta| s^{\alpha-1}}{\Gamma(\alpha)}] ds \\
&= \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\delta|}{\Gamma(\alpha)} \int_0^t [\sum_{j=1}^{\infty} \frac{(\frac{P}{1-L})^j}{\Gamma(j\alpha)} (t-s)^{j\alpha-1} s^{\alpha-1}] ds \\
&= \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\delta|}{\Gamma(\alpha)} [\sum_{j=1}^{\infty} \frac{(\frac{P}{1-L})^j}{\Gamma(j\alpha)} \int_0^t (t-s)^{j\alpha-1} s^{\alpha-1} ds] \\
&= \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + |\delta| [\sum_{j=1}^{\infty} \frac{(\frac{P}{1-L})^j}{\Gamma(\alpha)\Gamma(j\alpha)} \frac{\Gamma(j\alpha)\Gamma(\alpha) t^{(j+1)\alpha-1}}{\Gamma((j+1)\alpha)}] \\
&= \frac{|\delta| t^{\alpha-1}}{\Gamma(\alpha)} + |\delta| t^{\alpha-1} [\sum_{j=1}^{\infty} \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma((j+1)\alpha)}] \\
&= |\delta| t^{\alpha-1} \sum_{j=0}^{\infty} \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma((j+1)\alpha)} \\
&= |\delta| t^{\alpha-1} E_{\alpha,\alpha}(\frac{P}{1-L} t^{\alpha}).
\end{aligned}$$

Remark 4.5 *It follows from Corollary 4.4 that for every ε between 0 and b small changes initial conditions cause only small changes of the solution in the closed interval [ε, b] (which does not contain zero). Also we observer that the estimates on solutions in case of successive approximations approach and integral inequality are identical. Further, the Corollary 4.4 generalize the results studied in [4, 9, 22, 23].*

4.3. Dependence on Functions Involved Therein

In this section, we study the continuous dependence of solutions of (1.1) on the given initial data, and function involved therein.

The next theorem deals with the closeness of solutions of initial value problem (1.1)-(1.2) and initial value problem (3.42)-(3.43).

Theorem 4.6 *Suppose all assumptions of Theorem 3.5 are satisfied. Then the solution x(t) of the initial value problem (1.1)-(1.2) depends continuously on the functions involved in the right hand side of the equation(1.1).*

Proof. Suppose $x(t)$ and $y(t)$ are respectively solutions of the problems (1.1)-(1.2) and (3.42)-(3.43). Let $u(t) = |x(t) - y(t)|$. Then by hypotheses, we obtain

$$\begin{aligned}|x(t)-y(t)| &\le \frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),D^\alpha x(s))-F(s,y(s),D^\alpha y(s))|ds\\
&\le \frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),D^\alpha x(s))-f(s,y(s),D^\alpha y(s))|ds\\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s),D^\alpha y(s))-F(s,y(s),D^\alpha y(s))|ds\\
&\le \frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\frac{P}{1-L})|x(s)-y(s)|ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varepsilon ds\\
&\le \frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)}+\frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\frac{P}{1-L})|x(s)-y(s)|ds\\
&\le \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}+\frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)}+(\frac{P}{1-L})\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds.\end{aligned} \tag{4.12}$$

Hence by a suitable application of Lemma 2.5 (with $g(t)=\frac{1}{\Gamma(\alpha)}(\frac{P}{1-L})$ and $a(t)=\frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}+\frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)}$ to (4.12) yields

$$\begin{aligned}u(t) &\le \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}+\frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)}+\int_0^t [\sum_{j=1}^\infty \frac{(\frac{P}{1-L})^j}{\Gamma(j\alpha)}(t-s)^{j\alpha-1}(\frac{\varepsilon s^\alpha}{\Gamma(1+\alpha)}+\frac{|\delta|s^{\alpha-1}}{\Gamma(\alpha)})]ds\\
&= \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}+\frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)}+\sum_{j=1}^\infty \frac{(\frac{P}{1-L})^j}{\Gamma(j\alpha)}[\frac{\varepsilon}{\Gamma(1+\alpha)}\int_0^t (t-s)^{j\alpha-1}s^\alpha ds+\frac{|\delta|}{\Gamma(\alpha)}\int_0^t (t-s)^{j\alpha-1}s^{\alpha-1}ds]\\
&= \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}+\frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)}+\sum_{j=1}^\infty \frac{(\frac{P}{1-L})^j}{\Gamma(j\alpha)}\Gamma(j\alpha)[\frac{\varepsilon}{\Gamma(1+\alpha)}\frac{\Gamma(1+\alpha)t^{(j+1)\alpha}}{\Gamma((j+1)\alpha+1)}+\frac{|\delta|}{\Gamma(\alpha)}\frac{\Gamma(\alpha)t^{(j+1)\alpha-1}}{\Gamma((j+1)\alpha)}]\\
&= \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}+\frac{|\delta|t^{\alpha-1}}{\Gamma(\alpha)}+\sum_{j=1}^\infty \frac{(\frac{P}{1-L})^j}{\Gamma((j+1)\alpha+1)}\varepsilon t^{(j+1)\alpha}+\sum_{j=1}^\infty \frac{(\frac{P}{1-L})^j}{\Gamma((j+1)\alpha)}|\delta|t^{(j+1)\alpha-1}\\
&= \varepsilon t^\alpha\sum_{j=0}^\infty \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma((j+1)\alpha+1)}+|\delta|t^{\alpha-1}\sum_{j=0}^\infty \frac{(\frac{P}{1-L})^j t^{j\alpha}}{\Gamma((j+1)\alpha)}\\
&= \varepsilon t^\alpha E_{\alpha,\alpha+1}(\frac{P}{1-L}t^\alpha)+|\delta|t^{\alpha-1}E_{\alpha,\alpha}(\frac{P}{1-L}t^\alpha).\end{aligned} \tag{4.13}$$

This completes the proof of Theorem 4.6.

Remark 4.7 *The result given in Theorem 4.6 relates the solutions of the initial value problem (1.1)-(1.2) and (3.42)-(3.43) in the sense that if f is close to F, x_0 is close to $x_0+\delta$ (i.e. depends upon δ and as δ is smaller and smaller), then not only the solutions of initial value problem (1.1)-(1.2) and (3.42)-(3.43) are close to each other, but also depends continuously on the functions involved the right hand side of (1.1). Further we have seen that the estimates on solutions in case of successive approximations approach and integral inequality are coincides.*

The next corollary deals with the convergence of solutions of initial value problem (1.1)-(1.2) and initial value problem (3.56)-(3.57).

As an immediate consequence of Theorem 4.6, we have the following corollary.

Corollary 4.8 *Suppose all assumptions of Corollary 3.7 hold and if x(t) and y_k (t) (k = 1,2,...) are respectively the solutions of initial value problem (1.1)-(1.2) and (3.56)-(3.57) on J, then as $k \to \infty$, $y_k(t) \to x(t)$ on $0 < t \leq b$.*

Proof. By following a similar arguments as in the proof of Theorem 4.6 and from the hypotheses, it follows that

$$|x(t) - y_k(t)| \leq \frac{\varepsilon_k t^\alpha}{\Gamma(1+\alpha)} + \frac{|\delta_k| t^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{P}{1-L}\right) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s) - y_k(s)| ds. \tag{4.14}$$

Hence by a suitable application of Lemma 2.5 (with $g(t) = \frac{1}{\Gamma(\alpha)} \left(\frac{P}{1-L}\right)$ and $a(t) = \frac{\varepsilon_k t^\alpha}{\Gamma(1+\alpha)} + \frac{|\delta_k| t^{\alpha-1}}{\Gamma(\alpha)}$ to (4.14) and then integrate term by term, we have

$$\begin{aligned} |x(t) - y_k(t)| &\leq \varepsilon_k t^\alpha \sum_{j=0}^{\infty} \frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma(1+(j+1)\alpha)} + |\delta_k| t^{\alpha-1} \sum_{j=0}^{\infty} \frac{\left(\frac{P}{1-L}\right)^j t^{j\alpha}}{\Gamma((j+1)\alpha)} \\ &= \varepsilon_k t^\alpha E_{\alpha,\alpha+1}\left(\frac{P}{1-L} t^\alpha\right) + |\delta_k| t^{\alpha-1} E_{\alpha,\alpha}\left(\frac{P}{1-L} t^\alpha\right). \end{aligned} \tag{4.15}$$

As $k \to \infty$, the required result follows from (4.15).

Remark 4.9 *The result obtained in Corollary provide sufficient conditions that ensures solutions of initial value problem (3.56)-(3.57) will converge to the solutions of initial value problem (1.1)-(1.2). Moreover, under the approach of successive and approximations and integral inequality, we get same results as proved in Corollary 3.7 and Corollary 4.8.*

4.4. Dependence on Parameters

In this section, we study the continuous dependence of solutions on certain parameters.

The following theorem states the continuous dependency of solutions to (3.61)-(1.2) and (3.62)-(1.2) on parameters.

Theorem 4.10 *Suppose all assumptions of Theorem 3.9 hold and if x(t) and y(t) are respectively solutions of (3.61) and (3.62), then*

$$|x(t) - y(t)| \leq N|\mu_1 - \mu_2| t^\alpha E_{\alpha,\alpha+1}\left(\frac{Q}{1-L} t^\alpha\right). \tag{4.16}$$

Proof. *Suppose x(t) and y(t) are respectively solutions of the problems (3.61)-(1.2) and (3.62)-(1.2). Let u(t)=|x(t)-y(t)|. Then by hypotheses and following a similar arguments as in the proof of Theorem 4.6, we have*

$$
\begin{aligned}
u(t) &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|G(s,x(s),D^\alpha x(s),\mu_1) - G(s,y(s),D^\alpha y(s),\mu_2)|ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|G(s,x(s),D^\alpha x(s),\mu_1) - G(s,y(s),D^\alpha y(s),\mu_1)|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|G(s,y(s),D^\alpha y(s),\mu_1) - G(s,y(s),D^\alpha y(s),\mu_2)|ds \\
&\le \frac{N|\mu_1-\mu_2|t^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\frac{Q}{1-\overline{L}})|x(s)-y(s)|ds \\
&\le \frac{N|\mu_1-\mu_2|t^\alpha}{\Gamma(1+\alpha)} + (\frac{Q}{1-\overline{L}})\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds.
\end{aligned}
\tag{4.17}
$$

Hence by a suitable application of Lemma 2.5 (with $g(t) = \frac{1}{\Gamma(\alpha)}(\frac{Q}{1-\overline{L}})$ and $a(t) = \frac{N|\mu_1-\mu_2|t^\alpha}{\Gamma(1+\alpha)}$ to (4.17) and then integrate term by term, which implies

$$
\begin{aligned}
u(t) &\le N|\mu_1 - \mu_2|t^\alpha \sum_{j=0}^{\infty} \frac{(\frac{Q}{1-\overline{L}})^j t^{j\alpha}}{\Gamma(1+(j+1)\alpha)} \\
&= N|\mu_1 - \mu_2|t^\alpha E_{\alpha,\alpha+1}(\frac{Q}{1-\overline{L}}t^\alpha).
\end{aligned}
\tag{4.18}
$$

From (4.18), it follows that the solutions (3.61) and (3.62) with (1.2) depend continuously on the parameters μ_1, μ_2. This completes the proof of Theorem 4.10.

Remark 4.11 *The result dealing with the property of a solution called "dependence of solutions on parameters". Here the parameters are scalars. Notice that the initial conditions do not involve parameters. The dependence on parameters are an important aspect in various physical problems. Moreover, under the approach of successive and approximations and integral inequality, we get same results as proved in Theorem 3.9 and Theorem 4.10.*

5. RESULT VIA FIXED POINT THEOREM

Let us recall some definitions and notation related to our result.

Let $C([0, b], \mathbb{R})$ denotes the Banach space of all continuous real valued functions defined on $[0, b]$ with norm

$$\| x \| = \sup\{|x(t)| : t \in [0, b]\}.$$

For $t \in [0, b]$, we define $x_r = t^r x(t)$, $r \ge 0$. Let $C_r([0, b], \mathbb{R})$ be the space of all functions such that $x_r \in C([0, b], \mathbb{R})$ which turn out to be a Banach space when endowed with the norm

$$\| x \| = \sup\{t^r \mid x(t) \mid : t \in [0, b]\}.$$

Setting $r = 1 - \alpha$, $0 < \alpha < 1$, and define a set $C_0 = \{x \in C_r([0, b], \mathbb{R})$: $\lim_{t\to 0^+} t^{1-\alpha}x(t) = \frac{x_0}{\Gamma(\alpha)}\}$. It is clear that C_0 is a Banach space with the norm $\|\cdot\|_{1-\alpha}$ and $C_0 \subseteq C_{1-\alpha}([0,b],\mathbb{R})$.

We prove the existence and uniqueness of solution for the problem (1.1) -((1.2) by using Banach contraction principle.

Theorem 5.1 *Assume f satisfies the hypothesis (H_1). Then the initial value problem (1.1)-(1.2) has a unique solution in* $C_0 \subseteq C_{1-\alpha}([0, b], \mathbb{R})$ *if* $d = [\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\frac{P}{1-L})b^\alpha] < 1$.

Proof. Let us define an operator $T: C_0 \to C_0$ by

$$Tx(t) = \frac{x_0}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s),D^\alpha x(s))ds. \tag{5.1}$$

Observe that the problem (1.1)-(1.2) has solutions if and only if the operator T has fixed points. We prove that T is a contraction map. In view of (H_1) and for each $t \in [0, b]$ and $x, y \in C_0$, we have

$$\begin{aligned} t^{1-\alpha}|Tx(t) - Ty(t)| &\le \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),D^\alpha x(s)) - f(s,y(s),D^\alpha y(s))|ds \\ &\le \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\frac{P}{1-L})|x(s)-y(s)|ds \\ &= (\frac{P}{1-L})\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}|x(s)-y(s)|ds \\ &\le (\frac{P}{1-L})\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\parallel x-y \parallel_{1=\alpha} ds \\ &= [(\frac{P}{1-L})\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds]\parallel x-y \parallel_{1=\alpha}. \end{aligned} \tag{5.2}$$

Note that

$$\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}t^{2\alpha-1}. \tag{5.3}$$

Hence, using (5.3) in (5.2), we obtain

$$\begin{aligned} t^{1-\alpha}|Tx(t) - Ty(t)| &\le [(\frac{P}{1-L})\frac{t^{1-\alpha}}{\Gamma(\alpha)}\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}t^{2\alpha-1}]\parallel x-y \parallel_{1=\alpha} \\ &= [\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\frac{P}{1-L})t^\alpha]\parallel x-y \parallel_{1=\alpha} \\ &\le [\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\frac{P}{1-L})b^\alpha]\parallel x-y \parallel_{1=\alpha} \\ &= d\parallel x-y \parallel_{1=\alpha}, \end{aligned} \tag{5.4}$$

which implies

$$\| T_x - Ty|_{1=\alpha} \leq d \| x - y \|_{1=\alpha}. \tag{5.5}$$

From the equation (5.5), it follows that by the definition of d that the operator T is a contraction. Therefore, by the Banach fixed point theorem,we claim that T has a unique fixed point which the required solution in $C_0 \subseteq C_{1-\alpha}$ ([0, b], $\mathbb{R}$) of the problem (1.1)-(1.2). This completes the proof.

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