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GENERALIZED HAMILTON VARIATIONAL PRINCIPLES FOR NON-LINEAR ANALYSES OF BEAM-COLUMN STRUCTURES*

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ABSTRACT

In this paper, the generalized Hamilton variational principles for non-linear analyses of Timoshenko-type and Euler-type beam-column structures are established in general case, and consequently, the corresponding mathematical models can be presented correctly, including coupling non-linear partial differential equations of motion with variable coefficients, the boundary conditions and the initial conditions. In the present work, the geometrical and material property of structures as well as the loads may be all arbitrary, except assumed that the cross-section is symmetrical about the inertia axes and the loads acting on the structure are conservation. In addition, the inertia of the neutral layer and rotation is considered as well. The generalized variational principles and mathematical models presented in the paper are the foundations for the correct theoretical analyses and numerical calculations of beam-column structures.

***Keywords*:** beam-column structures, non-linearity and non-uniformity of material, finite deformation, generalized Hamilton variational principle

1. INTRODUCTION

It is well known that the beam-column structures are widely applied to different fields of science and technology. Whether in the civil engineering, mechanical engineering, aerospace engineering or in the emerging fields of science and technology, one can find the application background of the beam-column structures[1~7]. With the development of science and technology, the length of the beam-column structures has become longer, the thickness has become thinner, and the weight has become lighter in the practical application. So in order to provide more suitable design theory, the large deformation of the structures, non-linearity of materials and so on should be considered for the analysis of the non-linear characteristics of the structures.

Compared with linear systems, non-linear systems may possess the more complex properties, for example, bifurcation, chaos, saltation, instability, and so on. To explore the non-linear phenomena of structures and to explain their mechanism, it is essential to establish correct and rational mathematical models. Up to now, there exist many theories for the non-linear analysis of beam-column structures. Bisshopp, *et al.*[8], Jenkins, *et al.* [9], Kerr [10], and Chen [11] presented theories for the non-linear analysis of beam-column structures by using the arc-coordinate, and studied the large deformations of beams and frames. Antman [12] set up the large deformation theory of elastic slender rods, and discussed the non-uniqueness of equilibrium states in tension, shear and necking instability of elastic rods. The advantages of these theories are very delicacy and concinnity, and they can be applied to explain the mechanism of the non-linear phenomena of structures from the analytic solutions of simple problems. However, it is very difficult to apply these theories to more complex problems, such as, complex structures, complex loads, complex solution domain etc. On the other hand, generally speaking, for non-linear analyses of structures in science and technology, the axial displacement and deformation of structures are relatively smaller, but the transverse

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displacements and the rotations caused by them are relatively larger, and the effect of the rotation on the axial deformation must be considered. Hence, it is necessary to develop the more relevance non-linear theory of the beam-column structures mentioned above to analyze more complex problems conveniently.

In the present paper, the Hamilton variational principle is extensively applied to the non-linear, non-uniform beam-column structures with variable cross-sections, and the corresponding Hamilton variational principles are established, in which, the non-linearity and non-uniformity of materials, the finite deformation of structures as well as the inertia of the neutral layer and rotation are all considered, and all the geometrical property of structures and the loads acting on the structures may be arbitrary, except assumed that the cross-section is symmetrical about the inertia axes and the loads are conservative. Further, from the different applications of beam-column structures, one divides them into two kinds, the one is slender, its length is much bigger than the size of the cross-section, called as Euler-type beam-column structures, another is short and thick, its length is close to the size of the cross-section, called as Timoshenko-type beam-column structures. According to this classification, the Euler-type and Timoshenko-type displacement modes are adopted, respectively, and the generalized Hamilton variational principles for non-linear mechanical analyses of two kinds of structures are presented, hence, the corresponding mathematical models can be obtained. The variational principles and mathematical models are the foundations for the correct theoretical analyses and numerical calculations of structures as well.

2. HAMILTON VARIATIONAL PRINCIPLE (I) FOR NON-LINEAR ANALYSES OF TIMOSHENKO-TYPE BEAM-COLUMN STRUCTURES

Consider a beam-column structure resting on an elastic foundation and subjected to arbitrarily distributed tangent and transverse loads (Fig.1). Let ox be the neutral axis of beam, oy, oz be the inertia axes of cross-section. It is assumed that the cross-section is symmetrical about the inertia axes, and A is the area of cross-section, l is the length, ρ is the mass density (a given constant). It is also assumed that $q_u(x, t)$, $q_v(x, t)$, $q_w(x, t)$ are the components of load in x-, y- and z-directions, respectively, namely, the tangent and transverse components. In addition, at the end of beam, for example, at the end $x = l$, the structure may be subjected to the given axial force and shear forces as well as moments.

If the length of the structure is not much larger than the size of cross-section, namely, we consider Timoshenko-type beam-column structures here, the displacements u_1, v_1, w_1 of the neutral layer in x-, y- and z-directions may be given as [13]

$$\begin{cases} u_1 = u(x,t) + y\varphi(x,t) + z\psi(x,t) \\ v_1 = v(x,t), \qquad w_1 = w(x,t) \end{cases} \tag{1}$$

in which, u, v, w are the displacements of the neutral axis, φ and ψ are the rotational angles of the cross-section about z- and y- directions, respectively. y and z are the distances to the neutral axis.

Geometry relation: In the case of small deformations, the non-zero strain components can be given as

$$\varepsilon_x = \frac{\partial u}{\partial x} + y\frac{\partial \varphi}{\partial x} + z\frac{\partial \psi}{\partial x},\ \gamma_{xy} = \frac{\partial v}{\partial x} + \varphi, \gamma_{xz} = \frac{\partial w}{\partial x} + \psi \tag{2}$$

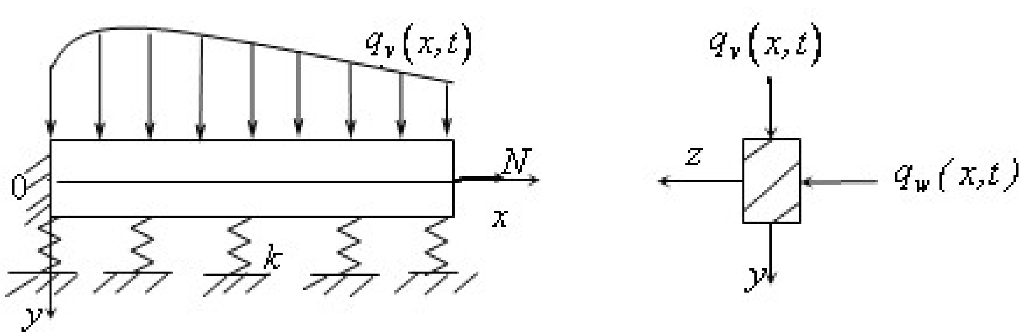

Figure 1: Physical Model of Beam-column

Constitutive relation: From the theory of beams, the effect of the stress components σ_y, σ_z and τ_{yz} on the deformation may be neglected. If the material of beam is a sort of cubic nonlinear elastic materials, when the effect of transverse shear deformations is considered, we have the constitutive relations as follows:

$$\begin{cases} \sigma_x = E_1\varepsilon_x + E_2\varepsilon_x{}^2 + E_3\varepsilon_x{}^3 \\ \tau_{xz} = G\gamma_{xz}, \qquad \tau_{xy} = G\gamma_{xy} \end{cases} \tag{3}$$

in which, E_1 is the linear elastic modulus of material, and E_2 and E_3 are the generalized moduli, which are used to describe the non-linear effect of material. For a non-uniform material, the moduli are the given functions of x. The constitutive relation (3) may be used to describe the mechanical characteristic of the concrete to a certain extent [14-15].

It would be also assumed that the foundation is a sort of cubic nonlinear materials, and a generalized Winkler model is adopted to simulate the resistance of foundation. Hence, we have

$$p_s = k_{s1}s + k_{s2}s^2 + k_{s3}s^3, s = u, v, w \tag{4}$$

in which, p_u, p_v, p_w are the tangent and transverse components of foundation resistances, respectively. k_{s1}, k_{s2} and k_{s3} are the corresponding linear and non-linear rigidity coefficients.

Generalized Hamilton variational principle (I): In all possible displacements satisfying the geometry relations and the displacement boundary conditions as well as having the designated motion at the initial and terminate time, the actual displacements u, v, w, φ, ψ make the functional Π in (5) arrive at the stationary value, that is, $\delta\Pi = 0$, in which

$$\Pi = \int_0^T H dt = \int_0^T -(U - W - T)dt \tag{5}$$

in which, $H = -(U - W - T)$ is the Hamilton function, T is the kinetic energy of structure, $U = U_1 + U_2$ is the strain energy, U_1 and U_2 are the strain energies caused by the normal and shear strains, respectively. W is the work done by the given distributed external loads, the end forces and moments, as well as the resistance of foundation. The representations of T, U and W are given as follows:

$$T = \int_0^l \frac{1}{2}\rho\left\{A\left((\frac{\partial u}{\partial t})^2 + (\frac{\partial v}{\partial t})^2 + (\frac{\partial w}{\partial t})^2\right) + I_z(\frac{\partial \varphi}{\partial t})^2 + I_y(\frac{\partial \psi}{\partial t})^2\right\}dx \tag{6a}$$

where, the last two terms are the kinetic energies caused by the rotation, $I_y = \iint_A z^2 dA,\ I_z = \iint_A y^2 dA$ are the inertia moments of cross-section, respectively.

Observing the non-linearity of material, the corresponding strain energy densities caused by the normal and shear strains are

$$\begin{cases} W_1 = \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x = \frac{1}{2}E_1\varepsilon_x^2 + \frac{1}{3}E_2\varepsilon_x^3 + \frac{1}{4}E_3\varepsilon_x^4 \\ W_2 = \kappa_1\int_0^{\gamma_{xy}} \tau_{xy} d\gamma_{xy} + \kappa_2\int_0^{\gamma_{xz}} \tau_{xz} d\gamma_{xz} = \frac{1}{2}\kappa_1 G\gamma_{xy}^2 + \frac{1}{2}\kappa_1 G\gamma_{xz}^2 \end{cases}$$

in which, κ_1 and κ_2 are the shear modified coefficients of the cross-section in y- and z- directions, which denote a modifiability of “assumption for the shear strain on the cross-section is constant” [13]. Hence,

$$U_1 = \iint_A \int_0^l W_1 dx dA = U_{11} + U_{12} + U_{13} \tag{6b}$$

in which

$$U_{11} = \frac{1}{2}\int_0^l \left[E_1 A\left(\frac{\partial u}{\partial x}\right)^2 + I_z\left(\frac{\partial \varphi}{\partial x}\right)^2 + I_y\left(\frac{\partial \psi}{\partial x}\right)^2 \right] dx$$

$$U_{12} = \frac{1}{3}\int_0^l E_2 \left\{ A\left(\frac{\partial u}{\partial x}\right)^3 + 3I_z \frac{\partial u}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right)^2 + 3I_y \frac{\partial u}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)^2 \right\} dx$$

$$U_{13} = \frac{1}{4}\int_0^l E_3 [A\left(\frac{\partial u}{\partial x}\right)^4 + 6I_z\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial \varphi}{\partial x}\right)^2 + 6I_y\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial \psi}{\partial x}\right)^2$$
$$+ 6\bar{I}_{yz}\left(\frac{\partial \varphi}{\partial x}\right)^2\left(\frac{\partial \psi}{\partial x}\right)^2 + 6\bar{I}_z\left(\frac{\partial \varphi}{\partial x}\right)^4 + 6\bar{I}_y\left(\frac{\partial \psi}{\partial x}\right)^4]dx$$

and $I_y = \iint_A z^2 dA, I_z = \iint_A y^2 dA$, $\bar{I}_{yz} = \iint_A y^2 z^2 dA, \bar{I}_z = \iint_A y^4 dA, \bar{I}_y = \iint_A z^4 dA$ are the inertia moments and the high order inertia moments of cross-section about the inertia axes, respectively. Substituting it into (6b) yields the expression of U_1 in terms of the generalized displacements u, v, w, φ, ψ. And also, we have the expression of U_2 as

$$U_2 = \frac{1}{2}\iint_A \int_0^l W_2 dx dA = \frac{1}{2}\int_0^l GA\left[\kappa_1\left(\varphi + \frac{\partial v}{\partial x}\right)^2 + \kappa_2\left(\frac{\partial w}{\partial x} + \psi\right)^2 \right] dx \tag{6c}$$

Suppose that q_u, q_v, q_w are the designated distributed loads in x–, y–, z–directions, respectively, and $N, \bar{T}_v, \bar{T}_w, \bar{M}_\varphi, \bar{M}_\psi$ are the designated axial and tangent forces as well as moments at the end $x = l$, respectively, and assume that these loads are all conservative. Then, the work done by these forces is given as

$$W = \int_0^l \sum_{s=u,v,w} q_s s dx - \frac{N}{2}\int_0^l \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \right] dx - \int_0^l \sum_{s=u,v,w} \left(\frac{1}{2}k_{s1}s + \frac{1}{3}k_{s2}s^2 + \frac{1}{4}k_{s3}s^3 \right) s dx$$
$$+ Nu(l) + \bar{T}_v v(l) + \bar{T}_w w(l) + \bar{M}_\varphi \varphi(l) + \bar{M}_\psi \psi(l) \tag{6d}$$

For a non-uniform material, the moduli in (6) are the given functions of x, so they can not be moved out from the integral and differential.

Operating the variation calculation on the expressions (6a)-(6d), substituting the resultant equations into the variation equation $\delta\Pi = 0$ for (5), and observing that the motion of the beam-column at the initial and terminate time is given, hence, $\delta u = \delta v = \delta w = \delta\varphi = \delta\psi = 0$ at $t = 0$ and $t = T$. At the same time, observing arbitrariness of the variables δu, δv, δw, $\delta\varphi$, $\delta\psi$ on $[0, l]$ and at the end $x = l$, we can obtain the differential equations of motion, the boundary conditions at $x = l$, as well as the initial conditions of Timoshenko-type beam-column structures, which are composed of a non-linear and non-uniform elastic material. One can see that the differential equations of motion are a set of coupling non-linear partial differential equations with respect to u, v, w, φ, ψ, and the boundary conditions are non-linear as well. Generally speaking, it is very difficult to obtain the analytic solutions of problem. We can only obtain the numerical solutions by numerical methods.

3. HAMILTON VARIATIONAL PRINCIPLE (II) FOR NON-LINEAR ANALYSES OF TIMOSHENKO-TYPE BEAM-COLUMN STRUCTURES

As also, consider a beam-column structure resting on an elastic foundation shown in Fig. 1, and adopt the displacement fields in Eq.(1), but here think that the rotation of the cross-section for y–,z–directions is bigger, and the relation between the displacement and the deformation should be non-linear. Hence, we have the equations as follows.

The displacement field is

$$\begin{cases} u_1 = u(x,t) + y\varphi(x,t) + z\psi(x,t) \\ v_1 = v(x,t), \qquad w_1 = w(x,t) \end{cases} \tag{7}$$

Geometry relation: In the case of finite deformations, the non-zero strain components may be given as

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} + y\dfrac{\partial \varphi}{\partial x} + z\dfrac{\partial \psi}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}\right)^2 + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \gamma_{xy} = \varphi + \dfrac{\partial v}{\partial x}, \gamma_{xz} = \psi + \dfrac{\partial w}{\partial x} \end{cases} \tag{8}$$

Constitutive relation: From the theory of beams, if the material of structure is a sort of linear elastic materials, when the effect of transverse shear deformations is considered, then the constitutive relations can be expressed as

$$\sigma_x = E\varepsilon_x, \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}, \tau_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz} \tag{9}$$

in which, E, ν are the linear elastic modulus and Poisson ratio of material, and they may be the given functions of x, generally.

Generalized Hamilton variational principle (II): In all possible displacements satisfying the geometry relations and the displacement boundary conditions as well as having the designated motion at the initial and terminate time, the actual displacements u, v, w, φ, ψ make the functional Π in (10) arrive at the stationary value, that is, $\delta\Pi = 0$, in which

$$\Pi = \int_0^T H dt = \int_0^T -(U - W - T) dt \tag{10}$$

and $H = -(U - W - T)$ is the Hamilton function, T is the kinetic energy of structure, $U = U_1 + U_2$ is the strain energy, U_1 and U_2 are the strain energies caused by the normal and shear strains, respectively. W is the work done by the given external loads and the foundation counterforce. The representations of T and U as well as W are given as

$$T = \int_0^l \frac{1}{2}\rho\left\{A\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right) + I_z\left(\frac{\partial \varphi}{\partial t}\right)^2 + I_y\left(\frac{\partial \psi}{\partial t}\right)^2\right\}dx \tag{11a}$$

in which, the last two terms are the kinetic energy caused by the rotation.

For a linear elastic material, the corresponding strain energy densities caused by the normal and shear strains are

$$\begin{cases} W_1 = \dfrac{1}{2}\sigma_x\varepsilon_x = \dfrac{1}{2}E\left(\dfrac{\partial u}{\partial x} + y\dfrac{\partial \varphi}{\partial x} + z\dfrac{\partial \psi}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}\right)^2 + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2\right)^2 \\ W_2 = \dfrac{G}{2}\left(\kappa_1\gamma_{xy}^2 + \kappa_2\gamma_{xz}^2\right) = \dfrac{1}{2}G\left[\kappa_1\left(\varphi + \dfrac{\partial v}{\partial x}\right)^2 + \kappa_2\left(\psi + \dfrac{\partial w}{\partial x}\right)^2\right] \end{cases}$$

where, $G = \dfrac{E}{2(1+\nu)}$. So, the total strain energy is given as

$$U = U_1 + U_2 = U_{11} + U_{12} + U_2 \tag{11b}$$

in which

$$U_1 = \iint_A \int_0^l W_1 dxdA$$
$$= \frac{1}{2}\int_0^l EA\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^4 + \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4 + \frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2\left(\frac{\partial w}{\partial x}\right)^2\right]dx$$
$$+\frac{1}{2}\int_0^l E\left[I_z\left(\frac{\partial \varphi}{\partial x}\right)^2 + I_y\left(\frac{\partial \psi}{\partial x}\right)^2\right]dx = U_{11} + U_{12}$$

$$U_2 = \iint_A \int_0^l W_2 dxdA = \frac{1}{2}\int_0^l GA\left[\kappa_1\left(\varphi + \frac{\partial v}{\partial x}\right)^2 + \kappa_2\left(\psi + \frac{\partial w}{\partial x}\right)^2\right]dx$$

Also assume that the material of foundation is linear elastic, and k_u, k_v, k_w are the corresponding linear rigidity moduli in the tangent and transverse direction, respectively. Thus, the work done by the external forces is

$$W = \int_0^l \sum_{s=u,v,w} q_s s dx - \frac{N}{2}\int_0^l\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]dx - \frac{1}{2}\int_0^l \sum_{s=u,v,w} k_s s^2 dx$$
$$+ Nu(l) + \bar{T}_v v(l) + \bar{T}_w w(l) + \bar{M}_\varphi \varphi(l) + \bar{M}_\psi \psi(l) \tag{11c}$$

Operating the variation calculation on the expressions (11a)-(11c), and substituting the resultant equations into $\delta\Pi = 0$ for (10), we can obtain the differential equations of motion, the boundary conditions of forces, as well as the initial conditions of Timoshenko-type beam-column structures with finite deformations, which are composed of a non-uniform linear elastic material.

For a non-uniform linear elastic beam-column structure with variable cross-sections, all the material and geometrical parameters are the given functions of x, which can not be moved out from the integral and differential in (11). One can see that the differential equations of motion are a set of coupling non-linear partial differential equations with variable coefficients with respect to displacements u, v, w, φ, ψ. And also the boundary conditions of forces are also coupling and non-linear.

4. HAMILTON VARIATIONAL PRINCIPLE (III) FOR NON-LINEAR ANALYSES OF EULER-TYPE BEAM-COLUMN STRUCTURES

Again consider a beam-column structure resting on an elastic foundation shown in Fig. 1. But here, we will study the Euler-type beam-column structures, that is, the length of structures is much larger than the size of cross-section, and the effect of transverse shear deformations may be neglected. Assume that rotation of the cross-section about y–, z–directions is larger, so we have the following displacement field:

$$u_1 = u(x,t) - y\frac{\partial v}{\partial x} - z\frac{\partial w}{\partial x}, v_1 = v(x,t), w_1 = w(x,t) \tag{12}$$

From the theory of finite deformations, the geometry equations are given as

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - y\frac{\partial^2 v}{\partial x^2} - z\frac{\partial^2 w}{\partial x^2} = A_1 - A_2 \tag{13}$$

Here, for convenience, we have introduced the symbols

$$\begin{cases} A_1(u,v,w) = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}\right)^2 + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2, A_2(v,w) = y\dfrac{\partial^2 v}{\partial x^2} + z\dfrac{\partial^2 w}{\partial x^2} \\ B(v,w) = I_z\left(\dfrac{\partial^2 v}{\partial x^2}\right)^2 + I_y\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 \end{cases} \tag{14}$$

in which, $I_y, I_z, \overline{I}_y, \overline{I}_z, \overline{I}_{xy}$ are the inertia moments and high order moments of the cross-section about *y*–,*z*–axes, respectively.

If assume that the material of structure is a sort of thrice nonlinear materials, we have the constitutive equation

$$\sigma_x = E_1\varepsilon_x + E_2\varepsilon_x{}^2 + E_3\varepsilon_x{}^3 \tag{15}$$

in which, E_1 is the linear elastic modulus of material, E_2 and E_3 are the generalized moduli describing the non-linear effect of material. If the material is non-uniform, these moduli are given functions of *x*.

Generalized Hamilton variational principle (III): In all possible displacements satisfying the geometry relations and the displacement boundary conditions as well as having the designated motion at the initial and terminate time, the actual displacements *u*, *v*, *w* make the functional Π in (16) arrive at the stationary value, that is, $\delta\Pi = 0$, in which

$$\Pi = \int_0^T H dt = \int_0^T -(U - W - T)dt \tag{16}$$

and $H = -(U - W - T)$ is the Hamilton function, *T* is the kinetic energy of structure, $U = U_1 + U_2 + U_3$ is the strain energy, U_1, U_2 and U_3 are the strain energies caused by the linear, quadric and thrice elasticity, respectively. *W* is the work done by the given external forces.

In the present case, the kinetic energy of structure is given as:

$$T = \int_0^l \frac{1}{2}\rho\left\{A(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2) + I_z\left(\frac{\partial^2 v}{\partial t\partial x}\right)^2 + I_y\left(\frac{\partial^2 w}{\partial t\partial x}\right)^2\right\}dx \tag{17a}$$

in which, the last two terms are the kinetic energy caused by the rotation.

The strain energy density is given as

$$W_1 = \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x = \frac{1}{2}E_1\varepsilon_x^2 + \frac{1}{3}E_2\varepsilon_x^3 + \frac{1}{4}E_3\varepsilon_x^4$$

Observing the geometry relation (13), and setting the strain energy $U = U_1 + U_2 + U_3$, then we have the expressions as follows:

$$\begin{cases} U_1 = \dfrac{1}{2}\iint\limits_A \int_0^l E_1\varepsilon_x^2 dxdydz = \dfrac{1}{2}\int_0^l E_1\left[A(A_1)^2 + B\right]dx \\ U_2 = \dfrac{1}{3}\iint\limits_A \int_0^l E_2\varepsilon_x^3 dxdydz = \dfrac{1}{3}\int_0^l E_2\left[A(A_1)^3 + 3A_1B\right]dx \\ U_3 = \dfrac{1}{4}\iint\limits_A \int_0^l E_3\varepsilon_x^4 dxdydz = U_{31} + U_{32} + U_{33} \\ \quad = \dfrac{1}{4}\int_0^l E_3A(A_1)^4 dx + \dfrac{3}{2}\int_0^l E_3(A_1)^2 Bdx \\ \quad + \dfrac{1}{4}\int_0^l E_3\left\{\overline{I}_z\left(\dfrac{\partial^2 v}{\partial x^2}\right)^4 + 6\overline{I}_{yz}\left(\dfrac{\partial^2 v}{\partial x^2}\right)^2\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 + \overline{I}_y\left(\dfrac{\partial^2 w}{\partial x^2}\right)^4\right\}dx \end{cases} \tag{17b}$$

At the same time, we have the expression of the work as

$$W = \int_0^l \sum_{s=u,v,w} q_s s dx - \frac{N}{2}\int_0^l \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] dx - \int_0^l \sum_{s=u,v,w} \left(\frac{1}{2}k_{s1}s + \frac{1}{3}k_{s2}s^2 + \frac{1}{4}k_{s3}s^3\right) s dx$$
$$+ Nu(l) + \bar{T}_v v(l) + \bar{M}_y \frac{\partial v}{\partial x}(l) + \bar{T}_w w(l) + \bar{M}_z \frac{\partial w}{\partial x}(l) \tag{17c}$$

here, $\bar{M}_Y, \bar{M}_Y$ are the moments corresponding to the rotations $\frac{\partial v}{\partial x}(l), \frac{\partial w}{\partial x}(l)$ at the end $x = l$. And the foundation counterforce is given by (4).

Operating the variation calculation on the expressions (17a)-(17c) and substituting the obtained results into the variation equation $\delta\Pi = 0$ for (16), observing $\delta u = \delta v = \delta w = 0$ at $t = 0$ and $t = T$ due to the beam-column has the given motion at the initial and terminate time, at the same time, observing arbitrariness of the variables δu, δv, δw on [0, l] and arbitrariness of the variables δu, δv, δw, $\delta\frac{\partial v}{\partial x}, \delta\frac{\partial w}{\partial x}$ at the end $x = l$, we can obtain the differential equations of motion, the boundary conditions of forces, as well as the initial conditions of Euler-type beam-column structures, which are composed of a non-linear and non-uniform elastic material in the case of finite deformations.

For a non-linear and non-uniform elastic beam-column structure with variable cross-sections, all the material moduli and geometrical parameters are the given functions of x, they can not be moved out from the integral and differential in (17), hence the differential equations of motion are a set of coupling non-linear partial differential equations with variable coefficients with respect to the displacements u, v, w.

It can be seen that if let the moduli describing the non-linear effect of material be zeros, that is, $E_2 = E_3 = 0$, we can obtain the Hamilton variational principle of Euler-type beam-column structures, which are composed of a non-uniform linear elastic material in the case of finite deformations.

Generalized Hamilton variational principle (IV): In all possible displacements satisfying the geometry relations and the displacement boundary conditions as well as having the designated motion at the initial and terminate time, the actual displacements u, v, w make the functional Π in (18) arrive at the stationary value, that is, $\delta\Pi = 0$, in which

$$\Pi = \int_0^T H dt = \int_0^T -(U - W - T) dt \tag{18}$$

and $H = -(U - W - T)$ is the Hamilton function. T is the kinetic energy of structure given by (17a), U is the strain energy given by (17b) with $E_2 = E_3 = 0$, W is the work done by the given external forces and the foundation counterforce. For non-linear elastic foundations, W is given by (17c), and for linear elastic foundations, it is given by (11c).

4. CONCLUSIONS

In this paper, the Hamilton variational principle is applied to non-linear mechanical analyses of beam-column structures, and the corresponding Hamilton variational principles are establish in general case. From these variational principles, the correct mathematical models for non-linear analyses of beam-column structures can be obtained exactly.

According to different applications to engineering and science, the beam-column structures are divided into slender and short beams. The former is called as Euler-type beam-column structures, and the latter is called as Timoshenko-type beam-column structures. Hence, we present the Hamilton variational principles of non-linear analyses and the corresponding mathematical models for the two kinds of structures by using the Euler-type and Timoshenko-type displacement mode, respectively.

In the variational principles and mathematical models, the material of structures and foundations is a sort of non-linear and non-uniform elastic materials or a sort of linear and non-uniform elastic materials: the deformation of structures is finite or small; the loads subjected to structures are arbitrary but they must be conservative; the geometric characteristics of structures are also arbitrary except assumed that the cross-section is symmetrical about the inertia axes; further, the inertia of the neutral layer and rotation is also considered. Consequently, the theories presented in the present work are very general and complete, and large numbers of problems of beam-column structures may be described by these theories or their simplified forms, for example, Liyanapathirana, *et al.* [16], Mei, *et al.*[17], Lee, *et al.*[18].

Some conclusions are listed as follows:

(1) From the generalized Hamilton variational principles (I) and (II), we obtain the differential equations of motion, the boundary conditions at the given end forces, as well as the initial conditions of Timoshenko-type beam-column structures, which are composed of a non-linear and non-uniform elastic material in the case of small or finite deformations, respectively. They are a set of non-linear partial differential equations with variable coefficients in terms of five unknown displacements u, v, w, φ, ψ generally.

For Euler-type beam-column structures, from the generalized Hamilton variational principles (III) and (IV), we obtain the corresponding mathematical models, but they are a set of non-linear differential equations with variable coefficients with respect to three unknown displacements u, v, w. If the inertia forces of structures may be ignored, the non-linear mathematical models of the corresponding static problems can be yielded.

(2) If let the parameters describing the non-linear effect of materials be zeros, we may obtain the non-linear theories of Timoshenko- and Euler- type beam-column structures, which are composed of a linear and non-uniform elastic material in the case of finite deformations, respectively.

(3) As degeneration or simplification of the theories, the mathematical models corresponding to the axial motion, plane motion of Timoshenko- and Euler-type beam-column structures under small or finite deformations can be obtained. As expect, the differential equations of motion and the boundary conditions at the given end forces in these models are non-linear as well.

(4) If all non-linear terms in these theories may be ignored, we will obtain the mathematical theories of linear elastic Timoshenko- and Euler-type beam-column structures under small deformations. Evidently, all the governing equations and boundary conditions are uncoupled and linear with respect to unknown displacements, but the coefficients of governing equations are variable.

(5) For appropriate modification or generalization to the theories above, the non-linear theories of beam-column structures with another constitutive relation can also be yielded, for example, thermoelastic beam-column structures with voids, visco-elastic beam-column structures with voids etc.[19]. We can also consider the aerodynamics behavior of beam-column structures in supersonic airflow, in this case, the structures considered will be subjected to an aerodynamic pressure, the governing differential equations of motion and the boundary conditions can not be obtained directly from the Hamilton variational principle.

Hence, the non-linear theories presented by this paper are very general. But it is need to point out that the finite deformation of structures is limited, because the geometrical non-linearity considered here is the finite rotation caused by the transverse displacement, and the effect of the rotation on the axial deformation should be considered. When the axial displacement and rotation are all large, these non-linear theories may be inapplicable, one has to establish another models to satisfy the demand of engineering and science, for example, Hu, *et al.* [20-21].

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