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Approximation by Fejér Operator

S. N. Dubey

Abstract: The present paper embodies the notion of the approximation of a 2π-periodic continuous function satisfying a Lipschitz condition of order a with constant M, i.e. $\text{Lip}_M\alpha$ for $0 < \alpha < 1$ by Fejér operator. As a matter of course, our result may be deemed to be a quantitative version of Fejér's Theorem. Constants of approximation have inter alia been taken into notice quite meticulously.

1. Introduction and Main Result

Let $S_n(f)$ designate the nth partial sum of the Fourier series of f. The Fejér sums of f are defined as follows:

$$\sigma_n(f) = \frac{S_0(f) + S_1(f) + \ldots + S_n(f)}{(n+1)};$$

for this concept the reference may be made to DeVore [4].

The well-known theorem of Fejér states that $\sigma_n(f)$ converges to f at every point of continuity of f, for more detailed study of these ideas, the reference may be made to Zygmund {[8], p. 89}.

It is worth-mentioning and contextual also that the vital theme of the theory of approximation of functions is to connect the smoothness of the function f with the rate of convergence of $\sigma_n(f)$ to 0. As a matter of rule, the smoother the function, the faster the error tends to zero for $n \to \infty$. The smoothness properties upon f are quite often expressed in terms of Lipschitz classes, modulus of continuity, modulus of smoothness, bounded variation and differentiability; for these notions, the reference may be made to Butzer and Nessel [2].

So far as the approximation of f by Fejér's operator $(\sigma_n(f))$ is concerned, we have the following result:

$$\| f - \sigma_n(f) \| \le (1 + \pi)\, w(f, n^{-1/2}). \tag{1.1}$$

For more details in this direction, the reference may be made to DeVore {[4], p. 35}.

The main object of this paper is to ponder over the approximation of the function f by the Fejér operator $\sigma_n(f)$ in terms of Lipschitz condition of order α $(0 < 1 < 1)$, i.e. $\text{Lip}_M\alpha$ where M is some constant, by taking utmost care of all constants.

More precisely, we prove the following:

Theorem: Let f be a 2π-periodic continuous function satisfying $\mathrm{Lip}_M\alpha$ for $0 < \alpha < 1$. Then the approximation of f by Fejér operator $\sigma_n(f)$ is given by:

$$\left|\sigma_n(f) - f\right| \le \left[M\pi^{\alpha}\left\{\frac{2}{(1+\alpha)} + \frac{1}{2(1-\alpha)}\right\} n^{-\alpha}\right]. \tag{1.2}$$

2. Proof of the Theorem

It is well known that {Bary [1], p. 140 – (49.4)}

$$\sigma_n(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} g_x(t) K_n(t)\, dt,$$

where

$$g_x(t) = f(x+t) + f(x-t) - 2f(x),$$

and

$$K_n(t) = \frac{1}{(n+1)}\sum_{k=0}^{n} D_k(t)\,.$$

In order to evaluate the above integral we break it up into two intervals as follows:

$$\sigma_n(x) - f(x) = \frac{1}{\pi}\left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right) g_x(t)\, K_n(t)\, dt$$

$$= \alpha + \beta, \text{ say.}$$

By virtue of the facts that $g_x(t)$ belongs to $\mathrm{Lip}_M\alpha$, we have $|g_x(t)| \le Mt^{\alpha}$, where M is some constant and $0 < \alpha < 1$, and $|K_n(t)| \le 2n$ (for $n \ge 1$), for this inequality, the reference may be made to Bary {[1], p. 140 – (49.5)}; we ascertain

$$|\alpha| = \left|\frac{1}{\pi}\int_0^{\pi/n} g_x(t)\, K_n(t)\, dt\right| \le \frac{2Mn}{\pi}\int_0^{\pi/n} \left|g_x(t)\right| dt$$

$$\le \frac{2Mn}{\pi}\int_0^{\pi/n} t^{\alpha} \le \frac{2Mn}{\pi}\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\pi/n} \le \frac{2Mn}{\pi(\alpha+1)}\cdot\left(\frac{\pi}{n}\right)^{\alpha+1}$$

$$\le \frac{2M\pi^{\alpha}}{(1+\alpha)}\cdot n^{-\alpha} \tag{2.1}$$

Secondly, we estimate β. In view of the fact that $|K_n(t)| \leq \frac{\pi^2}{2(n+1)t^2}$ for $0 < |t| < \pi$, for this fact the references may be made to Bary {[1], p. 134 – (47.6)} and Powell and Shah {[5], p. 120 – (5.27)}. Then we have the following

$$|\beta| = \left| \frac{1}{\pi} \int_{\pi/n}^{\pi} g_x(t) K_n(t) dt \right|$$

$$\leq \frac{1}{\pi} \int_{\pi/n}^{\pi} Mt^{\alpha} \frac{\pi^2}{2(n+1)t^2} dt \leq \frac{M\pi}{2(n+1)} \int_{\pi/n}^{\pi} t^{\alpha-2} dt .$$

Now substituting (π/s) for t, we obtain

$$|\beta| = \frac{M\pi^{\alpha}}{2(n+1)} \int_1^n \frac{1}{s^{\alpha}} ds \leq \frac{M\pi^{\alpha}}{2(n+1)(1-\alpha)} (n^{-\alpha+1} - 1)$$

$$\leq \frac{M\pi^{\alpha}}{2(1-\alpha)} \cdot n^{-\alpha} . \tag{2.2}$$

On collecting the estimates (2.1) and (2.2), we eventually obtain

$$|\sigma_n(f) - f| \leq \left\{ \frac{2M\pi^{\alpha}}{(1+\alpha)} \cdot n^{-\alpha} + \frac{M\pi^{\alpha}}{2(1-\alpha)} \cdot n^{-\alpha} \right\}$$

$$\leq \left[M\pi^{\alpha} \left\{ \frac{2}{(1+\alpha)} + \frac{1}{2(1-\alpha)} \right\} n^{-\alpha} \right].$$

This completes the proof of the Theorem.

Remark: The similar estimates may be ascertained by assuming f to be a 2π-periodic continuous function of bounded variation on [–π, π].

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S. N. Dubey

Professor and Head,

Department of Engineering Mathematics,

Gyan Ganga Institute of Technology & Sciences,

Jabalpur, India