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THE FORCING RESTRAINED EDGE MONOPHONIC NUMBER OF A GRAPH

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Abstract: For the connected graph $G = (V, E)$ of order at least two, a chord of a path P is an edge joining two non-adjacent vertices of P is called a monophonic Path if it is a chordless path. Let G be a connected graph and M is a minimum restrained edge monophonic set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum restrained edge monophonic set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing restrained edge monophonic number of M, denoted by $f_{em_r}(M)$, is the cardinality of a minimum forcing subset of M. The forcing restrained edge monophonic number of G, denoted by $f_{em_r}(G)$, is $f_{em_r}(G) = \min\{f_{em_r}(M)\}$, Where the minimum is taken over all minimum restrained edge monophonic sets M in G. We determine bounds for if and find the forcing restrained edge monophonic number of certain classes of graphs. For every pair a, b of integers with $0 \le a \le b$ and $b \ge 2$, there exists a connected graph G such that $f_{em_r}(G) = f_{em}(G) = 0$, $em(G) = a$ and $em_r(G) = b$. For every integers a, b and c with $0 \le a < b \le c$ and $b > a + 1$, there exists a connected graph G such that $f_{em_r}(G) = 0$, $f_{em}(G) = a$, $em(G) = b$ and $em_r(G) = c$. For every integers a, b and c with $0 \le a < b < c$ and $c > a + b$, there exists a connected graph G such that $f_{em_r}(G) = f_{em}(G) = a$, $em(G) = b$ and $em_r(G) = c$.

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1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The closed neighborhood of a vertex v is the set $N[v] = N(v) \cup \{v\}$. A vertex v is an extreme vertex if the subgraph induced by its neighbors is complete. A vertex v is a semi-extreme vertex of G if the subgraph induced by its neighbors has a full degree vertex in $N(v)$. In particular, every extreme vertex is a semi-extreme vertex and a semi-extreme vertex need not be an extreme vertex. A chord of a path $u_1, u_2, \ldots, u_k$ in G is an edge $u_i u_j$ with $j \ge i + 2$. A $u - v$ path P is called a monophonic path if it is a chordless path. A set M of vertices is an edge monophonic set if every vertex of G lies on an edge monophonic path joining some pair of vertices in M, and the minimum cardinality of an edge monophonic set is the edge monophonic number $em(G)$ of G. An edge monophonic set of cardinality $em(G)$ is called an em-set of G. The monophonic domination number of a graph G was studied in [9]. A set M of vertices of a connected graph G is a restrained edge monophonic set if either $V = M$ or M is an edge monophonic set with the subgraph $G[V - M]$ induced by $V - M$ has no isolated vertices. The minimum cardinality of a restrained edge monophonic set of G is the restrained edge monophonic number of G, and is denoted by $em_r(G)$.

Theorem 1 Each extreme vertex of a connected graph G belongs to every restrained edge monophonic set of G.

2. FORCING RESTRAINED EDGE MONOPHONIC NUMBER OF A GRAPH

Definition 2. Let G be a connected graph and M is a minimum restrained edge monophonic set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum restrained edge monophonic set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing restrained edge monophonic number of M, denoted by $f_{em_r}(M)$, is the cardinality of a minimum forcing subset of M. The forcing restrained edge monophonic number of G, denoted by $f_{em_r}(G)$, is $f_{em_r}(G) = \min\{f_{em_r}(M)\}$, Where the minimum is taken over all minimum restrained edge monophonic sets M in G.

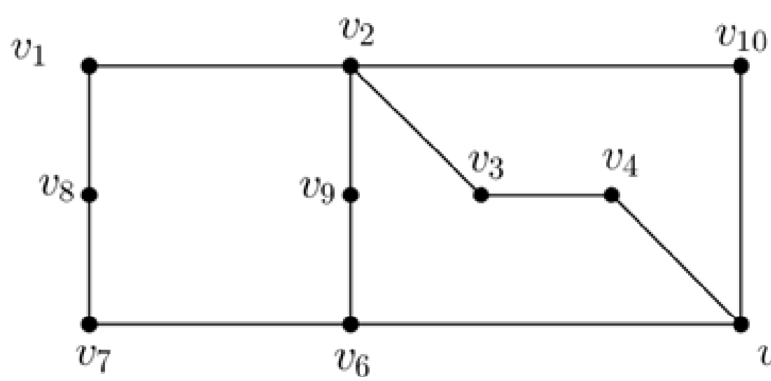

Figure 1: The graph G is Forcing Restrained Edge Monophonic Number of a Graph

Example 3. For the graph G given in Figure 1, $M_1 = \{v_1, v_7, v_8\}$, $M_2 = \{v_1, v_2, v_8\}$, $M_3 = \{v_2, v_6, v_9\}$, $M_4 = \{v_2, v_5, v_{10}\}$, $M_5 = \{v_3, v_4, v_5\}$, and $M_6 = \{v_6, v_7, v_8\}$, are the only six minimum restrained edge monophonic sets of G. It is clear that $f_{em_r}(M_1) = 1$, $f_{em_r}(M_2) = 1$, $f_{em_r}(M_3) = f_{em_r}(M_4) = f_{em_r}(M_5) = f_{em_r}(M_6) = 2$ so that $f_{em_r}(G) = 1$.

Theorem 4. For any connected graph G, $0 \leq f_{em_r}(G) \leq em_r(G) \leq p$

Theorem 5. Let G be a connected graph. Then

(a) f_{em_r} (G) = 0 if and only if G has a unique minimum restrained edge monophonic set.

(b) $f_{em_r}(G) = 1$ if and only if G has atleast two minimum restrained edge monophonic sets, one of which is a unique minimum restrained edge monophonic set containing one of its elements,

(c) $f_{em_r}(G) = f_{em_r}(G)$ if and only if no minimum restrained edge monophonic set of G is the unique minimum restrained edge monophonic set of containing any of its proper subsets.

Proof. (a) Let $f_{em_r}(G) = 0$. Then, by definition, $f_{em_r}(M) = 0$ for some minimum restrained edge monophonic set M of G so that the empty set φ is the minimum forcing subset for M. Since the empty set φ is a subset of every set, it follows that M is the unique minimum restrained edge monophonic set of G. The converse is clear.

(b) Let $f_{em_r}(G) = 1$. Then by Theorem 5(a), G has atleast two minimum restrained edge monophonic sets. Also since $f_{em_r}(G) = 1$ there is a singleton subset T of a minimum restrained edge monophonic set M of G such that T is not a subset of any other minimum restrained edge monophonic set containing one of its elements. The converse is clear.

(c) $f_{em_r}(G) = em_r(G)$. Then $f_{em_r}(M) = em_r(G)$. for every minimum restrained edge monophonic set M in G. Also, by Theorem 5, $em_r(G) \geq 2$ and hence $f_{em_r}(G) \geq 2$. Then by Theorem 5(a), G has atleast two minimum restrained edge monophonic sets and so the empty set φ is not a forcing for any minimum restrained edge monophonic set of G. Since $f_{em_r}(M) = em_r(G)$ no proper subset of M is a forcing subset of M. Thus, no minimum restrained edge monophonic set of G is the unique minimum restrained edge monophonic set containing any of its proper subsets. Conversely, G contains more than one minimum restrained edge monophonic set and no subset of any minimum restrained edge monophonic set M other than M is a forcing subset for M. Hence it follows that $f_{em_r}(G) = em_r(G)$.

Definition 6. A vertex v of a connected graph G is said to be a restrained edge monophonic vertex of G if v belongs to every minimum restrained edge monophonic set of G.

Example 7. For the graph G given in Figure 2, $M_1 = \{v_1, v_4, v_6\}$ and $M_2 = \{v_1, v_4, v_7\}$ are the only two restrained edge monophonic sets of G. It is clear that v_1 and v_4 are restrained edge monophonic vertices of G.

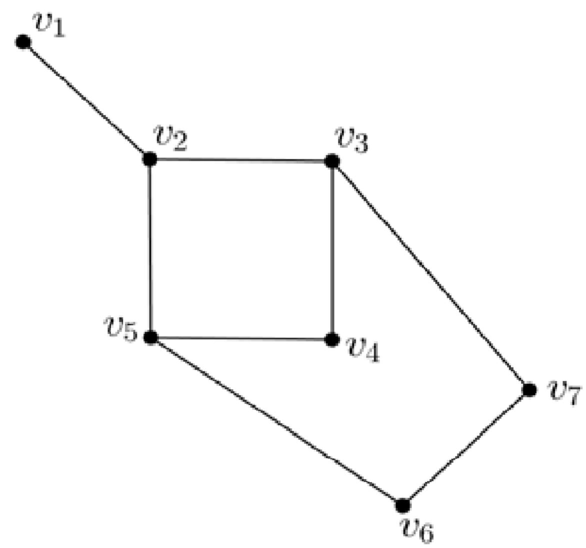

Figure 2: The graph G is Restrained Edge Monophonic Vertices of a Graph

Theorem 8. Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective minimum restrained edge monophonic sets in G. Then $\cap_{F\in\Im} F$ is the set of restrained edge monophonic vertices of G.

Proof. Let W be the set of all restrained edge monophonic vertices of G. We show that $W = \cap_{F\in\Im}$ F. Let v is a restrained edge monophonic vertex of G that belongs to every minimum restrained edge monophonic set M of G. Let $T \subseteq M$ be any minimum forcing subset for any minimum restrained edge monophonic set M of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that M is the unique minimum restrained edge monophonic set containing T' so that T' is a forcing subset for M with $|T'| < |T|$, which is a contradiction to T is a minimum forcing subset for M. Thus $v \notin T$ and so $v \in F$, where F in the relative complement of T in M. Hence $v \in \cap_{F\in\Im} F$ so that $W \subseteq \cap_{F\in\Im} F$. Conversely, let $v \in \cap_{F\in\Im} F$. Then v belongs to the relative complement of T in M for every T and every M such that $T \subseteq M$, where T is a minimum forcing subset for M. Since F is the relative complement of T in M, we have $F \subseteq M$ for every M, which implies that v is a restrained edge monophonic vertex of G. Thus $v \in W$ and so $\cap_{F\in\Im} F \subseteq W$. Hence $W = \cap_{F\in\Im} F$.

Theorem 9. Let G be a connected graph and W be the set of all restrained edge monophonic vertices of G. Then $f_{em_r}(G) \le em_r(G) - |W|$.

Proof. Let M be any minimum restrained edge monophonic set M of G. Then $em_r(G) = |M|$, $W \subseteq M$ and M is the unique minimum restrained edge monophonic set containing $M - W$. Thus $f_{em_r}(G) \le |M - W| = |M| - |W| = em_r(G) - |W|$.

Theorem 10. For a cycle $G = C_p$ $(p \geq 6)$, $f_{em_r}(G) = 2$.

Proof. Let $em_r(G) = 2$ and by Theorem 4, $0 \leq f_{em_r}(G) \leq 2$. Suppose $0 \leq f_{em_r}(G) \leq 1$. Since $em_r(G) = 2$, the restrained edge monophonic set of G is not the unique. Hence by Theorem 5(b), $f_{em_r}(G) = 1$. Let $M = \{u, v\}$, be a restrained edge monophonic set of G. Let us assume that $f_{em_r}(M) = 1$. By Theorem 5(b), M is the only restrained edge monophonic set containing u or v. Let us assume that M is the only restrained edge monophonic set containing u. u is adjacent to more than two vertices of G, which is a contradiction to G is a cycle. $f_{em_r}(G) = 2$.

3. REALIZATION RESULTS

Theorem 11. For every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $f_{em_r}(G) = f_{em}(G) = 0$, $em(G) = a$ and $em_r(G) = b$.

Proof. Let $P_i : u_i, v_i$, be a copy of order 2, and let $K_{2,b-a}$ be the complete bipartite graph with bipartite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, ..., y_{b-a}\}$. Let G be the graph obtained from adding the new verices $z_1, z_2, ..., z_{a-2}$ and join each z_i $(1 \leq i \leq a-1)$ to v_a, and join x_1 and x_2 to u_1. The graph G is given in Figure 3.

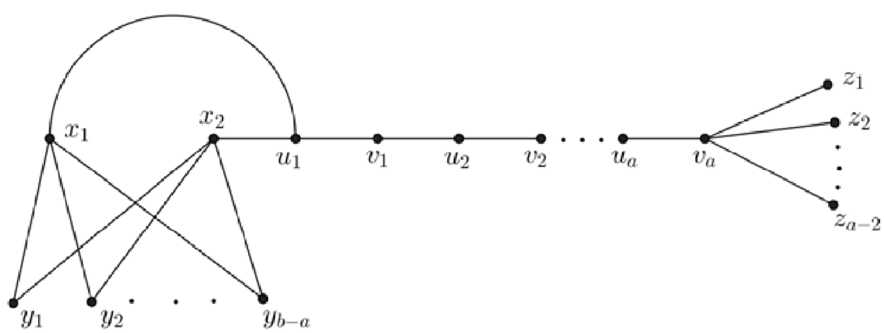

Figure 3: The graph G in Theorem 11

Let $Z = \{z_1, z_2, ..., z_{a-2}\}$ be the set of all end vertices of G. Z is a not an edge monophonic set of G. $Z_1 = Z \cup \{x_1, x_2\}$. Then it is clear that Z is the unique edge monophonic set of G so that $em(G) = a$ and $f_{em}(G) = 0$ by Theorem 5(a). By Theorem 1, Z_1 is a subset of every restrained edge monophonic set of G. We see that Z_1 is not a restrained edge monophonic set of G. Now, it is easily seen that $W = Z_1 \cup \{y_1, y_2, ..., y_{b-a}\}$ is the unique restrained monophonic set of G so that $em_r(G)$ = band $f_{em_r}(G) = 0$ by Theorem 5(a).

Theorem 12. For every integers a, b and c with $0 \leq a < b \leq c$ and $b > a + 1$, there exists a connected graph G such that $f_{em_r}(G) = 0$, $f_{em}(G) = a$, $em(G) = b$ and $em_r(G) = c$.

Proof. let $C_i : u_i, v_i, w_i, y_i$ be a cycle of order 4, and let $K_{1,b-a}$ be a star with the cut vertex x and $K_{1,b-a} = \{x, z_1, z_2,..., z_{b-a}\}$. Let G be the graph obtained from adding the new vertices $l_1, l_2,..., l_{c-b}$ and join each l_i $(1 \le i \le c-b)$ to each y_i $(1 \le i \le a)$ and join each u_i $(1 \le i \le a)$ to x. The graph G is shown in Figure 4.

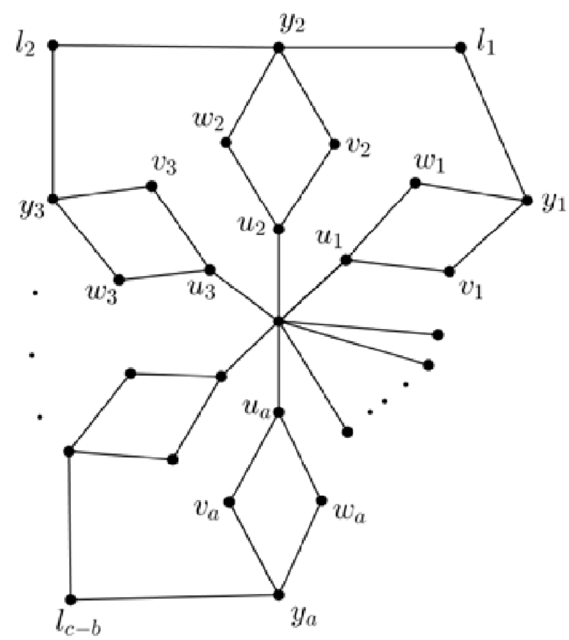

Figure 4: The graph G in Theorem 12

Let $Z = \{z_1, z_2,..., z_{b-a})\}$ be the set of all end vertices of G. It is clear that Z is not an edge monophonic set of G. For $1 \le i \le a$, let $N_i = \{v_i\ u_i, w_i\}$. We observe that every edge monophonic set of G must contain at least one vertex from each N_i so that $em(G) \ge b - a + a = b$. Now, $W = Z \cup \{y_1, y_2,..., y_a\}$ is an edge monophonic set of G so that $em(G) = b$. Next, we show that $f_{em}(G) = a$. Since every restrained monophonic set contains Z, it follows from Theorem 9 that $f_{em}(G) \le em(G) - (b - a) = a$. Now, since $em(G) = b$ and every edge monophonic set contains Z, it is easily seen that every edge monophonic set M of the $Z \cup \{d_1, d_2,..., d_a\}$, Where $d_i \in N_i$ $(1 \le i \le a)$. Let T be any proper subset of M with $|T| < a$. Then there exists d_j $(1 \le j \le a)$ such that $d_j \in T$. Let ej be the vertex of N_j distinct from d_j. Then $W = (M - \{d_j\}) \cup \{e_j\}$ is an edge monophonic set properly containing T. Thus M is not the unique edge monophonic set containing T so that T is not a forcing subset of M. This is true for all edge monophonic sets of G so that $f_{em}(G) = a$. Next, we show that $em_r(G) = c$. Now, $W = Z \cup \{l_1, l_2,...,l_{c-b}\} \cup \{y_1, y_2,..., y_a\}$ is the unique restrained edge monophonic set of G so that $em_r(G) = c$ and $f_{em_r}(G) = 0$, by Theorem 5(a).

Theorem 13. For every integers a, b and c with $0 \le a < b < c$ and $c > a + b$, there exists a connected graph G such that $f_{em_r}(G) = f_{em}(G) = a$, $em(G) = b$ and $em_r(G) = c$.

Proof. let $C_i : u_i, v_i, w_i$, be a cycle of order 3. and let $K_{2,c-b-a}$ be the complete bipartite graph with bipartite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2,..., y_{c-b-a}\}$. Let G be the graph obtained from adding the new verices $z_1, z_2,..., z_{b-a-2}, t_1, t_2,..., t_a$, and join each y_i $(1 \le i \le a)$ to each v_i $(1 \le i \le a)$ and join each z_i $(1 \le i \le b - a - 2)$ to v_a and join x_1 and x_2 with u_1. The graph G is given in Figure 5.

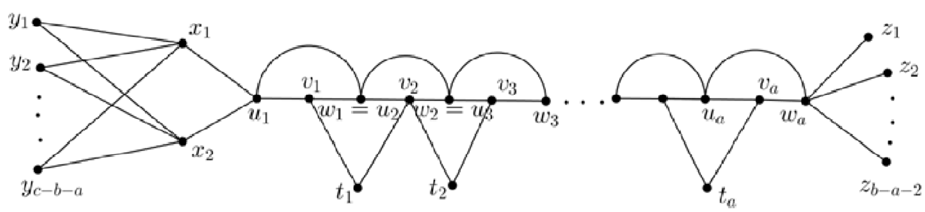

Figure 5: The graph G in Theorem 13

Let $Z = \{z_1, z_2,..., z_{b-a-2}\}$ be the extreme vertices of G. Then $Z_1 = Z \cup \{x_1, x_2\}$, it is clear that Z_1 is not an edge monophonic set of G. For $1 \le i \le a$, let $N_i = \{v_i\}$. We observe that every edge monophonic set of G must contain a vertex from N_i so that $em(G) \ge b - a + a = b$. Now, $W = Z_1 \cup \{v_1, v_2,..., v_a\}$ is an edge monophonic set of G so that $em(G) = b$. Next, we show that $f_{em}(G) = a$. Since every edge monophonic set containing $Z \cup \{v_1, v_2,..., v_a\}$, it follows from Theorem 1, that $f_{em}(G) \le em(G) - (b - a) = a$. Now, since $em(G) = b$ and every edge monophonic set contains Z, it is easily seen that every edge monophonic set M of the $Z \cup \{d_1, d_2,..., d_a\}$, Where $d_i \in N_i$ $(1 \le i \le a)$. Let T be any proper subset of M with $|T| < a$. Then there exists d_j $(1 \le j \le a)$ such that $d_j \in T$. Let ej be the vertex of N_j distinct from d_j. Then $W = (M - \{d_j\}) \cup \{e_j\}$ is an edge monophonic set properly containing T. Thus M is not the unique edge monophonic set containing T so that T is not a forcing subset of M. This is true for all edge monophonic sets of G so that $f_{em}(G) = a$.

Next, we show that $em_r(G) = c$. Z_1 is a subset of every restrained monophonic set of G. It is clear that Z_1 is not a restrained edge monophonic set of G. We observe that every edge monophonic set of G so that $em_r(G) \ge b - a + c - b - a = c$. Now, $W_1 = W \cup \{y_1, y_2,..., y_{c-b-a}\} \cup \{v_1, v_2,..., v_a\}$ is a restrained edge monophonic set of G so that $em_r(G) = c$. Next, we show that $f_{em_r}(G) = a$. Since every restrained monophonic set containing $W \cup \{y_1, y_2,..., y_{c-b-a}\}$, it follows from Theorem 1, that $f_{em_r}(G) \le f_{em_r}(G) - (b - a + c - b) = c - b + a - c + b = a$. Now since $em_r(G) = c$ and every restrained edge monophonic set contains W it is easily seen that every restrained edge monophonic set M is of the form $W \cup \{y_1, y_2,..., y_{c-b-a}\} \cup \{p_1, p_2, ..., p_a\}$, where $p_i \in M_i$ $(1 \le i \le a)$. Let T be any proper subset of M with $|T| < a$. Then there exists p_j such that $p_j \in T$. Let e_j be the vertex of M_j distinct from p_j. Then $W = (M - \{p_j\}) \cup \{e_j\}$ is a restrained edge monophonic set properly containing T. Thus M is not the unique restrained edge monophonic set containing T so that T is not a forcing subset of M. This is true for all restrained edge monophonic sets containing G so that $f_{em_r}(G) = a$.

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