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EXPONENTIAL CONVERGENCE OF THE STOCHASTIC MICROPOLAR AND MAGNETO-MICROPOLAR FLUID SYSTEMS

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Abstract. We study the micropolar and magneto-micropolar fluid systems with random forces in two-dimensional case. The additional terms on the equations that govern the time evolution of the velocity and micro-rotational velocity vector fields are more singular than many other equations that have been previously studied, for example Bénard or magnetic Bénard problem. Following the approach of [2] via a coupling method, we prove the existence and uniqueness of their solutions and the invariant measures as well as the exponential convergence of its trajectories to the unique invariant measure.

1. Introduction

The theory of micropolar fluids (MPF) was initially introduced in a series of papers by Eringen [15, 16], and subsequently the study of the magneto-micropolar fluids (MMPF) by Ahmadi and Shahinpoor in [1] followed suit. In particular, the MPF system models fluids consisting of bar-like elements such as liquid crystals made up of dumbbell molecules and animal blood. Due to diverse applications in real world, both systems have attracted much attention from engineers, physicists and mathematicians (e.g. [23, 31, 36, 37]).

Let us denote by u, w, b, p the velocity vector, the micro-rotational velocity vector, the magnetic vector and the hydrostatic pressure scalar fields respectively. Moreover, we let χ represent the vortex viscosity, μ the kinematic viscosity, j the microinertia, γ the spin viscosity, ν the reciprocal of the magnetic Reynolds number, all of which we assume to be positive. Under these notations, the two-dimensional micropolar fluid (MPF) and the magneto-micropolar fluid (MMPF) systems read as follows:

$$du + [(u \cdot \nabla)u + \nabla p - \chi \nabla \times w - (\mu + \chi)\Delta u]dt = \sqrt{Q_1} dW_1(t), \tag{1.1a}$$

$$jdw + [j(u \cdot \nabla)w + 2\chi w - \chi \nabla \times u - \gamma \Delta w]dt = \sqrt{Q_2} dW_2(t), \tag{1.1b}$$

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$$du + [(u\cdot\nabla)u - (b\cdot\nabla)b + \nabla p - \chi\nabla\times w - (\mu+\chi)\Delta u]dt = \sqrt{Q_1}dW_1(t), \quad (1.2a)$$

$$jdw + [j(u\cdot\nabla)w + 2\chi w - \chi\nabla\times u - \gamma\Delta w]dt = \sqrt{Q_2}dW_2(t), \quad (1.2b)$$

$$db + [(u\cdot\nabla)b - (b\cdot\nabla)u + \nu\nabla\times\nabla\times b]dt = \sqrt{Q_3}dW_3(t). \quad (1.2c)$$

The $\sqrt{Q_i}dW_i, i = 1, 2, 3$ represent random forces to be described in detail subsequently. In considering the two-dimensional case, we made the appropriate adjustment of

$$u = (u_1, u_2, 0), \quad w = (0, 0, w_3), \quad b = (b_1, b_2, 0)$$

(see pg. 185 [31]). We remark already that as we will see (e.g. estimates that led up to the bound in (3.16)), not only is χ a physically important quantity that plays the role of coupling u and w, manipulating estimates making use of χ lies at the heart of what distinguishes the MPF and the MMPF systems from many other systems of equations in fluid mechanics.

To the best of the author's knowledge, the existing results on the stochastic MPF and the MMPF systems with noise is only a few, namely [40, 42]. Even in the deterministic case with no noise, the MPF and the MMPF systems have a unique feature that represents a serious mathematical problem. Indeed, with same notations except the temperature scalar field θ, the Bénard problem which has been studied intensively both in the deterministic and stochastic cases (e.g. [3, 4, 13, 17]), is of the form, in the deterministic case for simplicity of discussion,

$$\partial_t u + (u\cdot\nabla)u + \nabla p - \theta e_2 - \mu\Delta u = 0, \quad (1.3a)$$

$$\partial_t \theta + (u\cdot\nabla)\theta - u_2 - \gamma\Delta\theta = 0, \quad (1.3b)$$

(see e.g. pg. 134 [38]) where we denote for brevity ∂_t for $\frac{\partial}{\partial t}$ and (e_1, e_2) the standard basis in $\mathbb{R}^2$. In particular, the $\chi\nabla\times w$ and $\chi\nabla\times u$ in (1.1a), (1.1b), (1.2a), (1.2b) are more singular than θe_2 and u_2 respectively and this leads to many known results that exist for the Boussinesq system (e.g. [25] in $\mathbb{R}^2$ case) to be out of reach for the MPF and the MMPF systems (cf. [11, 41]).

Throughout the rest of the manuscript, we only consider the MMPF system (1.2a)-(1.2c); the necessary modification for the MPF system (1.1a)-(1.1b) is clear (see e.g. Section 2.2 [40]) and analogous results for the MPF system (1.1a)-(1.1b) certainly hold, essentially just considering the case $b \equiv 0$.

2. Preliminaries and Statement of Main Results

We consider D, a bounded, simply connected and sufficiently smooth domain and the divergence-free, initial and boundary conditions of

$$\begin{cases} \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ (u, w, b)(x, 0) = (u_0, w_0, b_0)(x), \\ u|_{\partial D} = w|_{\partial D} = 0, \quad b\cdot n|_{\partial D} = 0, \quad \nabla\times b|_{\partial D} = 0. \end{cases} \quad (2.1)$$

For brevity, we write ∂_i for $\frac{\partial}{\partial x_i}, i = 1, 2$, $\int f$ for $\int_D f(x)dx$, and also assume $j = 1$ in (1.2b). Moreover, to emphasize the significance of the parameters on which the constant c depends, we write $c = c(a, b)$, while we may write $A \lesssim_{a,b} B$ to imply the existence of such a constant $c(a, b)$ such that $A \leq c(a, b)B$ and if c does not depend on any parameter of our interest, then we write $A \lesssim B$; analogously we write $A \approx_{a,b} B, A \approx B$. We let $Y \triangleq (u, w, b)$ denote the solution for the MMPF system (1.2a)-(1.2c) and list the standard notations in fluid mechanics literature as follows (see e.g. [5, 38]):

$$H_1 = H_3 \triangleq \{\phi \in \mathbb{L}^2 : \nabla \cdot \phi = 0, \phi \cdot n|_{\partial D} = 0\}, \quad H_2 \triangleq \mathbb{L}^2,$$
$$\mathcal{V}_1 \triangleq \{\phi \in C_c^\infty : \nabla \cdot \phi = 0\}, \quad V_1 \triangleq \{\phi \in \mathbb{H}_0^1 : \nabla \cdot \phi = 0\}, \quad V_2 \triangleq \mathbb{H}_0^1,$$
$$\mathcal{V}_3 \triangleq \{\phi \in C_c^\infty : \nabla \cdot \phi = 0, \phi \cdot n|_{\partial D} = 0\}, V_3 \triangleq \{\phi \in \mathbb{H}^2 : \nabla \cdot \phi = 0, \phi \cdot n|_{\partial D} = 0\},$$

where H_1, H_2, H_3 are endowed with inner products and norms of $(\phi, \psi) \triangleq \sum_{i=1}^2 \int_D \phi_i(x)\psi_i(x)dx, |\phi|^2 = (\phi, \phi)$, V_1 with $((\phi, \psi))_1 \triangleq \sum_{i=1}^2 (\partial_i \phi, \partial_i \psi)$, and similarly V_3 with $((\phi, \psi))_3 \triangleq (\nabla \times \phi, \nabla \times \psi)$. We denote by $H \triangleq H_1 \times H_2 \times H_3, V \triangleq V_1 \times V_2 \times V_3$ and also define

$$\begin{aligned}
&\langle A_1 X^1, X^2\rangle \triangleq -(\mu + \chi)\langle \Delta X^1, X^2\rangle, && D(A_1) \triangleq \mathbb{H}^2 \cap V_1,\\
&\langle A_2 Y^1, Y^2\rangle \triangleq \chi\langle Y^1, Y^2\rangle - \gamma\langle \Delta Y^1, Y^2\rangle, && D(A_2) \triangleq \mathbb{H}^2 \cap V_2,\\
&\langle A_3 Z^1, Z^2\rangle \triangleq \nu\langle \nabla \times \nabla \times Z^1, Z^2\rangle, && D(A_3) \triangleq H_1 \cap \{b \in \mathbb{H}^2 : \nabla \times b|_{\partial D} = 0\},\\
&B_1(u, b) \triangleq (u \cdot \nabla)u - (b \cdot \nabla)b,\ B_2(u, w) \triangleq (u \cdot \nabla)w,\ B_3(u, b) \triangleq (u \cdot \nabla)b - (b \cdot \nabla)u,\\
&R_1(w) \triangleq -\chi\nabla \times w, && R_2(w, u) \triangleq \chi w - \chi\nabla \times u,
\end{aligned} \tag{2.2}$$

so that we may consider instead of (1.2a)-(1.2c),

$$du + [A_1 u + B_1(u, b) + R_1(w)]dt = \sqrt{Q_1}dW_1(t), \tag{2.3a}$$
$$dw + [A_2 w + B_2(u, w) + R_2(w, u)]dt = \sqrt{Q_2}dW_2(t), \tag{2.3b}$$
$$db + [A_3 b + B_3(u, b)]dt = \sqrt{Q_3}dW_3(t). \tag{2.3c}$$

We denote by $W \triangleq (W_1, W_2, W_3)$ a cylindrical Wiener process defined for a fixed $T > 0$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ such that W_1, W_2, W_3 take values on H_1, H_2, H_3 respectively. Thus, $W_i = \sum_{k=1}^\infty \beta_k^i(t)e_k^i$ where $\{e_k^i\}_k$ is a complete orthonormal basis of eigenfunctions of $A_i, i = 1, 2, 3$ and $\{\beta_k^i\}_k$ is a sequence of independent one-dimensional Brownian motions. We denote by $C_W(0, T; H)$, the space of all functions continuous in $t \in [0, T]$ with values in $L^2(\Omega, \mathcal{F}, \mathbb{P}, H)$ that are $\mathcal{F}_t$-adapted; the spaces $L_W^2(0, T; V)$ and $L_W^2(0, T; V')$, where V' is the dual of V, are defined similarly.

We denote by $Q_i, i = 1, 2, 3$ linear, continuous, positive and symmetric operators on H of trace class $TrQ_i < \infty, i = 1, 2, 3$ satisfying

$$Q_i = A_i^{-\kappa}, \kappa \in \left(\frac{1}{2}, 1\right) \tag{2.4}$$

and also denote

$$A \triangleq \begin{pmatrix} A_1 & 0 & 0\\ 0 & A_2 & 0\\ 0 & 0 & A_3 \end{pmatrix}, \quad Q \triangleq \begin{pmatrix} Q_1 & 0 & 0\\ 0 & Q_2 & 0\\ 0 & 0 & Q_3 \end{pmatrix},$$

and consider the solution to

$$dW_A(t) + AW_A(t)dt = \sqrt{Q}dW(t), \quad W_A(0) = 0; \tag{2.5}$$

i.e.

$$dW_{A_1}(t) + A_1 W_{A_1} dt = \sqrt{Q_1} dW_1(t), \quad W_{A_1}(0) = 0, \tag{2.6a}$$

$$dW_{A_2}(t) + A_2 W_{A_2} dt = \sqrt{Q_2} dW_2(t), \quad W_{A_2}(0) = 0, \tag{2.6b}$$

$$dW_{A_3}(t) + A_3 W_{A_3} dt = \sqrt{Q_3} dW_3(t), \quad W_{A_3}(0) = 0, \tag{2.6c}$$

which is of the form

$$W_A(t) = (W_{A_1}(t), W_{A_2}(t), W_{A_3}(t))^T = \int_0^t e^{-A(t-s)} \sqrt{Q} dW(s). \tag{2.7}$$

It follows that (see (2.4) [2])

$$W_A \in C_W([0,T]; H) \cap (L_W^4([0,T] \times D))^2; \tag{2.8}$$

in fact, due to (2.4),

$$\mathbb{E}[\sup_{(x,t) \in D \times [0,T]} |W_{A_i}(x,t)|^4] < \infty \tag{2.9}$$

(see Theorem 2.13 [6], also (2.5) [2]). We now define a solution to the MMPF system (1.2a)-(1.2c).

Definition 2.1. Given $(u_0, w_0, b_0) \in H$, a stochastic process (u, w, b) is a *solution* to (1.2a) - (1.2c) on a time interval $[0,T]$ if $\mathbb{P}$-a.s.,

(1) $Y \triangleq (u, w, b) \in C([0,T]; H) \cap L_W^2(0,T;V)$,
(2) $\forall\, t \in [0,T]$,

$$u(t) + \int_0^t A_1 u(s) + B_1(u,b)(s) + R_1 w(s) ds = u_0 + \sqrt{Q_1} W_1(t), \tag{2.10a}$$

$$w(t) + \int_0^t A_2 w(s) + B_2(u,w)(s) + R_2(w,u)(s) ds = w_0 + \sqrt{Q_2} W_2(t), \tag{2.10b}$$

$$b(t) + \int_0^t A_3 b(s) + B_3(u,b)(s) ds = b_0 + \sqrt{Q_3} W_3(t). \tag{2.10c}$$

We state our first result:

Theorem 2.2. *Given any $Y_0 \triangleq Y(0) = (u_0, w_0, b_0) \in H, T > 0$, there exits a unique solution to (1.2a)-(1.2c) on $[0,T]$ such that the map $H \mapsto L^\infty(0,T;H) \cap L^2(0,T;V), (u_0, w_0, b_0) \mapsto (u(t), w(t), b(t))$ is continuous $\mathbb{P}$-almost surely.*

Next, we denote the solution to (1.2a)-(1.2c) with initial data $Y_0 = (x, y, z)$ by $Y(t, x, y, z) \triangleq (u(t,x,y,z), w(t,x,y,z), b(t,x,y,z)) \in L_W^2(0,T;V)$, and define the Markov semigroup P_t by

$$P_t g(x,y,z) \triangleq \mathbb{E}[g(Y(t,x,y,z))] \quad \forall\, g \in C_b(H) \tag{2.11}$$

in the space $C_b(H)$, the space of uniformly continuous and bounded mappings on H with a sup norm denoted by $\|\cdot\|_0$. This allows us to define a dual semigroup P_t^* in the space of probability measures on H, $\mathcal{P}(H)$, by $(g, P_t^* \beta) \triangleq (P_t g, \beta)$. We recall that a probability measure β is invariant or stationary with respect to $P_t, t \geq 0$, if and only if $P_t^* \beta = \beta$: i.e.

$$\int_H P_t g(x,y,z) \beta(dx,dy,dz) = \int_H g(x,y,z) \beta(dx,dy,dz) \quad \forall\, g \in C_b(H). \tag{2.12}$$

We now present our main result:

Theorem 2.3. *There exists a unique invariant measure β for P_t of (2.11) for (1.2a)-(1.2c) with support contained in V so that $\int_H \|(x,y,z)\|_V^2 \beta(dx,dy,dz) < \infty$.*

Remark 2.4. We follow the approach of the coupling method as illustrated by the authors in [2]. We also mention many important work in this direction of research from which the current work was inspired: [8, 17, 20] in which the authors showed the existence of an invariant measure for the stochastic Burgers' equation, Navier Stokes equations, and Bénard problem respectively, [18, 19, 22] in which the authors showed the uniqueness of such measures by using the classical Doob's theorem from [12] and notions of irreducibility and strong Feller property (see e.g. Theorem 4.2.1 [9]). We refer to the following important work concerning the coupling method [27, 28, 26, 34, 35] and [7, 21, 24, 33] for more work concerning ergodicity.

As a consequence of uniqueness, the invariant measure in Theorem 2.3 is ergodic (see Theorem 3.2.6 [9]).

3. Proof of Theorem 2.2

We define

$$\overline{u}(t) \triangleq u(t) - W_{A_1}(t), \quad \overline{w}(t) \triangleq w(t) - W_{A_2}(t), \quad \overline{b}(t) \triangleq b(t) - W_{A_3}(t), \tag{3.1}$$

where $(W_{A_1}, W_{A_2}, W_{A_3})$ is a solution of (2.6a)-(2.6c) and furthermore denote for $\phi, \Phi \in V_1, \psi, \Psi \in V_2, \theta, \Theta \in V_3$,

$$(F_1(\phi), \Phi) \triangleq b(\phi,\phi,\Phi), \quad (F_2(\theta),\Phi) \triangleq b(\theta,\theta,\Phi), \quad (F_3(\phi,\psi),\Psi) \triangleq b(\phi,\psi,\Psi), \tag{3.2a}$$

$$(F_4(\phi,\theta),\Theta) \triangleq b(\phi,\theta,\Theta), \quad (F_5(\theta,\phi),\Theta) \triangleq b(\theta,\phi,\Theta), \tag{3.2b}$$

$$(G_1(\phi),\Phi) \triangleq b(W_{A_1},\phi,\Phi) + b(\phi, W_{A_1},\Phi), \tag{3.2c}$$

$$(G_2(\theta),\Phi) \triangleq b(W_{A_3},\theta,\Phi) + b(\theta, W_{A_3},\Phi), \tag{3.2d}$$

$$(G_3(\phi),\Psi) \triangleq b(\phi, W_{A_2},\Psi), \quad (G_4(\psi),\Psi) \triangleq b(W_{A_1},\psi,\Psi), \tag{3.2e}$$

$$(G_5(\phi),\Theta) \triangleq b(\phi, W_{A_3},\Theta) - b(W_{A_3},\phi,\Theta), \tag{3.2f}$$

$$(G_6(\theta),\Theta) \triangleq b(\theta, W_{A_1},\Theta) - b(W_{A_1},\theta,\Theta) \tag{3.2g}$$

where e.g. $b(\phi,\phi,\Phi) \triangleq \int (\phi\cdot\nabla)\phi\cdot\Phi$ so that using (2.1), (2.2), (2.3a)-(2.3c), (2.5), (2.6a)-(2.6c), (3.1), (3.2a)-(3.2g), we obtain

$$\begin{aligned}\partial_t\overline{u} &+ A_1\overline{u} + F_1(\overline{u}) - F_2(\overline{b}) + G_1(\overline{u}) - G_2(\overline{b}) \\ &+ F_1(W_{A_1}) - F_2(W_{A_3}) + R_1(\overline{w}) + R_1(W_{A_2}) = 0,\end{aligned} \tag{3.3a}$$

$$\begin{aligned}\partial_t\overline{w} &+ A_2\overline{w} + F_3(\overline{u},\overline{w}) + G_3(\overline{u}) + G_4(\overline{w}) \\ &+ F_3(W_{A_1},W_{A_2}) + R_2(\overline{w},\overline{u}) + R_2(W_{A_2},W_{A_1}) = 0,\end{aligned} \tag{3.3b}$$

$$\begin{aligned}\partial_t\overline{b} &+ A_3\overline{b} + F_4(\overline{u},\overline{b}) - F_5(\overline{b},\overline{u}) + G_5(\overline{u}) - G_6(\overline{b}) \\ &+ F_4(W_{A_1},W_{A_3}) - F_5(W_{A_3},W_{A_1}) = 0,\end{aligned} \tag{3.3c}$$

$$(\overline{u},\overline{w},\overline{b})(0) = (u_0,w_0,b_0). \tag{3.3d}$$

Now we first prove the following proposition:

Proposition 3.1. *Let $Y_0 = (u_0, w_0, b_0) \in H, T > 0$. Then there exists a unique solution $(\overline{u}, \overline{w}, \overline{b}) \in L^2_W(0,T;V)$ to (3.3a)-(3.3d) such that $\mathbb{P}$-a.s.,*

1. *$(\overline{u}, \overline{w}, \overline{b}) : [0,T] \mapsto V'$ is absolutely continuous on $[0,T]$,*
2. *$(\partial_t \overline{u}, \partial_t \overline{w}, \partial_t \overline{b}) \in L^2(0,T;V')$,*
3. *$(\overline{u}, \overline{w}, \overline{b}) \in C([0,T];H)$.*

Proof. For a fixed $\omega \in \Omega$, we consider an approximation system of

$$\begin{aligned}\partial_t \overline{u}_\epsilon + A_1 \overline{u}_\epsilon + \Phi_1^\epsilon(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon) + G_1(\overline{u}_\epsilon) - G_2(\overline{b}_\epsilon) + R_1(\overline{w}_\epsilon)\\ = -F_1(W_{A_1}) + F_2(W_{A_3}) - R_1(W_{A_2}),\end{aligned} \tag{3.4a}$$

$$\begin{aligned}\partial_t \overline{w}_\epsilon + A_2 \overline{w}_\epsilon + \Phi_2^\epsilon(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon) + G_3(\overline{u}_\epsilon) + G_4(\overline{w}_\epsilon) + R_2(\overline{w}_\epsilon, \overline{u}_\epsilon)\\ = -F_3(W_{A_1}, W_{A_2}) - R_2(W_{A_2}, W_{A_1}),\end{aligned} \tag{3.4b}$$

$$\begin{aligned}\partial_t \overline{b}_\epsilon + A_3 \overline{b}_\epsilon + \Phi_3^\epsilon(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon) + G_5(\overline{u}_\epsilon) - G_6(\overline{b}_\epsilon)\\ = -F_4(W_{A_1}, W_{A_3}) + F_5(W_{A_3}, W_{A_1}),\end{aligned} \tag{3.4c}$$

$$(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)(0) = (u_0, w_0, b_0), \tag{3.4d}$$

where $\forall\ \phi \in V_1, \psi \in V_2, \theta \in V_3$, we define

$$\Phi_1^\epsilon(\phi,\psi,\theta) = \begin{cases} F_1(\phi) - F_2(\theta) & \text{if } \|(\phi,\psi,\theta)\|_V^2 \le \frac{1}{\epsilon^2}, \\ \frac{F_1(\phi)-F_2(\theta)}{\epsilon^2 \|(\phi,\psi,\theta)\|_V^2} & \text{if } \|(\phi,\psi,\theta)\|_V^2 > \frac{1}{\epsilon^2}, \end{cases} \tag{3.5a}$$

$$\Phi_2^\epsilon(\phi,\psi,\theta) = \begin{cases} F_3(\phi,\psi) & \text{if } \|(\phi,\psi,\theta)\|_V^2 \le \frac{1}{\epsilon^2}, \\ \frac{F_3(\phi,\psi)}{\epsilon^2 \|(\phi,\psi,\theta)\|_V^2} & \text{if } \|(\phi,\psi,\theta)\|_V^2 > \frac{1}{\epsilon^2}, \end{cases} \tag{3.5b}$$

$$\Phi_3^\epsilon(\phi,\psi,\theta) = \begin{cases} F_4(\phi,\theta) - F_5(\theta,\phi) & \text{if } \|(\phi,\psi,\theta)\|_V^2 \le \frac{1}{\epsilon^2}, \\ \frac{F_4(\phi,\theta)-F_5(\theta,\phi)}{\epsilon^2 \|(\phi,\psi,\theta)\|_V^2} & \text{if } \|(\phi,\psi,\theta)\|_V^2 > \frac{1}{\epsilon^2}. \end{cases} \tag{3.5c}$$

For simplicity of presentation, we only check the Lipschitz continuity in the case $\|(\phi^1,\psi^1,\theta^1)\|_V \le \frac{1}{\epsilon}, \|(\phi^2,\psi^2,\theta^2)\|_V \le \frac{1}{\epsilon}$. We compute $\mathbb{P}$-a.s.,

$$\|\Phi_1^\epsilon\|_{Lip(V,V_1')} \lesssim_\epsilon 1 \tag{3.6}$$

which follows from the estimate of

$$\begin{aligned}&\|\Phi_1^\epsilon(\phi^1,\psi^1,\theta^1) - \Phi_1^\epsilon(\phi^2,\psi^2,\theta^2)\|_{V_1'}\\ &\le \|(\phi^1\cdot\nabla)\phi^1 - (\phi^2\cdot\nabla)\phi^2\|_{V_1'} + \|(\theta^1\cdot\nabla)\theta^1 - (\theta^2\cdot\nabla)\theta^2\|_{V_1'}\\ &\lesssim \|(\phi^1-\phi^2)\otimes\phi^1\|_{L^2} + \|\phi^2\otimes(\phi^1-\phi^2)\|_{L^2}\\ &\quad + \|(\theta^1-\theta^2)\otimes\theta^1\|_{L^2} + \|\theta^2\otimes(\theta^1-\theta^2)\|_{L^2}\\ &\lesssim \|\phi^1-\phi^2\|_{L^4}(\|\phi^1\|_{L^4} + \|\phi^2\|_{L^4}) + \|\theta^1-\theta^2\|_{L^4}(\|\theta^1\|_{L^4} + \|\theta^2\|_{L^4})\\ &\lesssim_\epsilon \|(\phi^1-\phi^2, \psi^1-\psi^2, \theta^1-\theta^2)\|_V\end{aligned}$$

by (3.5a), (3.2a), Hölder's inequality and the following Gagliardo-Nirenberg inequality:

$$\|f\|_{L^4(D)} \lesssim_D |f|^{\frac{1}{2}} \|f\|_{H_0^1(D)}^{\frac{1}{2}} \tag{3.7}$$

(see Lemma 6.2 in Chapter 1 [30] and [32] for a comprehensive discussion). Similarly, if $\|(\phi^1,\psi^1,\theta^1)\|_V \leq \frac{1}{\epsilon}, \|(\phi^2,\psi^2,\theta^2)\|_V \leq \frac{1}{\epsilon}$, then $\mathbb{P}$-a.s.,

$$\|\Phi_2^\epsilon\|_{Lip(V,V_2')} \lesssim_\epsilon 1 \tag{3.8}$$

as

$$\begin{aligned}
&\|\Phi_2^\epsilon(\phi^1,\psi^1,\theta^1) - \Phi_2^\epsilon(\phi^2,\psi^2,\theta^2)\|_{V_2'}\\
&\lesssim \|(\phi^1-\phi^2)\otimes\psi^1\|_{L^2} + \|\phi^2\otimes(\psi^1-\psi^2)\|_{L^2}\\
&\lesssim \|\phi^1-\phi^2\|_{L^4}\|\psi^1\|_{L^4} + \|\phi^2\|_{L^4}\|\psi^1-\psi^2\|_{L^4}\\
&\lesssim_\epsilon \|(\phi^1-\phi^2,\psi^1-\psi^2,\theta^1-\theta^2)\|_V
\end{aligned}$$

by (3.5b), (3.2a), Hölder's inequality and (3.7). Finally, if $\|(\phi^1,\psi^1,\theta^1)\|_V \leq \frac{1}{\epsilon}$, $\|(\phi^2,\psi^2,\theta^2)\|_V \leq \frac{1}{\epsilon}$, then $\mathbb{P}$-a.s.,

$$\|\Phi_3^\epsilon\|_{Lip(V,V_3')} \lesssim_\epsilon 1 \tag{3.9}$$

as

$$\begin{aligned}
&\|\Phi_3^\epsilon(\phi^1,\psi^1,\theta^1) - \Phi_3^\epsilon(\phi^2,\psi^2,\theta^2)\|_{V_3'}\\
&\lesssim \|(\phi^1-\phi^2)\otimes\theta^1 + \phi^2\otimes(\theta^1-\theta^2) - (\theta^1-\theta^2)\phi^1 - \theta^2\otimes(\phi^1-\phi^2)\|_{L^2}\\
&\lesssim \|\phi^1-\phi^2\|_{L^4}(\|\theta^1\|_{L^4}+\|\theta^2\|_{L^4}) + \|\theta^1-\theta^2\|_{L^4}(\|\phi^1\|_{L^4}+\|\phi^2\|_{L^4})\\
&\lesssim_\epsilon \|(\phi^1-\phi^2,\psi^1-\psi^2,\theta^1-\theta^2)\|_V
\end{aligned}$$

by (3.5c), Hölder's inequality and (3.7). Next, $\mathbb{P}$-a.s., we can compute that

$$\|G_1(\cdot) - G_2(\cdot) + R_1(\cdot)\|_{Lip(V,V_1')} \lesssim_\epsilon 1 \tag{3.10}$$

as

$$\begin{aligned}
&\|G_1(\phi^1) - G_1(\phi^2) - (G_2(\theta^1) - G_2(\theta^2)) + R_1(\psi^1) - R_1(\psi^2)\|_{V_1'}\\
&\lesssim \|W_{A_1}\otimes(\phi^1-\phi^2)\|_{L^2} + \|W_{A_3}\otimes(\theta^1-\theta^2)\|_{L^2} + \|\psi^1-\psi^2\|_{L^2}\\
&\lesssim \|(\phi^1-\phi^2,\psi^1-\psi^2,\theta^1-\theta^2)\|_{L^4}(\|W_{A_1}\|_{L^4}+\|W_{A_3}\|_{L^4}+1)\\
&\lesssim \|(\phi^1-\phi^2,\psi^1-\psi^2,\theta^1-\theta^2)\|_V
\end{aligned}$$

by (3.2c), (3.2d), (2.2), (2.9) and (3.7). Next, $\mathbb{P}$-a.s.,

$$\|G_3(\cdot) + G_4(\cdot) + R_2(\cdot,\cdot)\|_{Lip(V,V_2')} \lesssim_\epsilon 1 \tag{3.11}$$

as

$$\begin{aligned}
&\|G_3(\phi^1) - G_3(\phi^2) + G_4(\psi^1) - G_4(\psi^2) + R_2(\psi^1,\phi^1) - R_2(\psi^2,\phi^2)\|_{V_2'}\\
&\lesssim \|(\phi^1-\phi^2)W_{A_2}\|_{L^2} + \|W_{A_1}(\psi^1-\psi^2)\|_{L^2} + \|\psi^1-\psi^2\|_{L^2} + \|\phi^1-\phi^2\|_{L^2}\\
&\lesssim \|\phi^1-\phi^2\|_{L^4}\|W_{A_2}\|_{L^4} + \|W_{A_1}\|_{L^4}\|\psi^1-\psi^2\|_{L^4} + \|\psi^1-\psi^2\|_{L^2} + \|\phi^1-\phi^2\|_{L^2}\\
&\lesssim_\epsilon \|(\phi^1-\phi^2,\psi^1-\psi^2,\theta^1-\theta^2)\|_V
\end{aligned}$$

by (3.2e), (2.2), Hölder's inequality, (2.9) and (3.7). Finally, we can also show similarly that $\mathbb{P}$-a.s.,

$$\|G_5(\cdot) - G_6(\cdot)\|_{Lip(V,V_3')} \lesssim_\epsilon 1 \tag{3.12}$$

as

$$\begin{aligned}
&\|G_5(\phi^1) - G_5(\phi^2) - [G_6(\theta^1) - G_6(\theta^2)]\|_{V_3'} \\
&\lesssim \|(\phi^1 - \phi^2) \otimes W_{A_3}\|_{L^2} + \|(\theta^1 - \theta^2) \otimes W_{A_1}\|_{L^2} \\
&\lesssim \|(\phi^1 - \phi^2, \theta^1 - \theta^2)\|_{L^4} \|(W_{A_3}, W_{A_1})\|_{L^4} \\
&\lesssim_\epsilon \|(\phi^1 - \phi^2, \psi^1 - \psi^2, \theta^1 - \theta^2)\|_V
\end{aligned}$$

by (3.2f), (3.2g), Hölder's inequality and (3.7).

Now we define

$$\mathcal{F}_\epsilon(\phi, \psi, \theta) \triangleq \begin{pmatrix} \Phi_1^\epsilon(\phi, \psi, \theta) + G_1(\phi) - G_2(\theta) + R_1(\psi) \\ \Phi_2^\epsilon(\phi, \psi, \theta) + G_3(\phi) + G_4(\psi) + R_2(\psi, \phi) \\ \Phi_3^\epsilon(\phi, \psi, \theta) + G_5(\phi) - G_6(\theta) \end{pmatrix}, \tag{3.13}$$

$$\mathcal{G}_\epsilon(t) \triangleq \begin{pmatrix} -F_1(W_{A_1}) + F_2(W_{A_3}) - R_1(W_{A_2}) \\ -F_3(W_{A_1}, W_{A_2}) - R_2(W_{A_2}, W_{A_1}) \\ -F_4(W_{A_1}, W_{A_3}) + F_5(W_{A_3}, W_{A_1}) \end{pmatrix}. \tag{3.14}$$

Due to (3.6), (3.8), (3.9), (3.10), (3.11), (3.12), we have already shown that $\mathcal{F}_\epsilon \in Lip(V, V')$. Moreover, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\|\mathcal{G}_\epsilon\|_{L^2(0,T;V')}^2 \leq & \int_0^T \|-(W_{A_1} \cdot \nabla) W_{A_1} + (W_{A_3} \cdot \nabla) W_{A_3} + \chi \nabla \times W_{A_2}\|_{V_1'}^2 \qquad (3.15) \\
& + \|-(W_{A_1} \cdot \nabla) W_{A_2} - \chi W_{A_2} + \chi \nabla \times W_{A_1}\|_{V_2'}^2 \\
& + \|-(W_{A_1} \cdot \nabla) W_{A_3} + (W_{A_3} \cdot \nabla) W_{A_1}\|_{V_3'}^2 dt \\
\lesssim & \int_0^T \|W_{A_1} \otimes W_{A_1}\|_{L^2}^2 + \|W_{A_3} \otimes W_{A_3}\|_{L^2}^2 + \|W_{A_2}\|_{L^2}^2 \\
& + \|W_{A_1} \otimes W_{A_2}\|_{L^2}^2 + \|W_{A_1}\|_{L^2}^2 + \|W_{A_1} \otimes W_{A_3}\|_{L^2}^2 dt \\
\lesssim & \int_0^T 1 + \|W_{A_1}\|_{L^4}^4 + \|W_{A_3}\|_{L^4}^4 + \|W_{A_2}\|_{L^2}^4 dt \lesssim 1
\end{aligned}$$

by (3.14), (3.2a), (2.2) and (3.2b).

Next, in case $\|(\phi^1, \psi^1, \theta^1)\|_V \leq \frac{1}{\epsilon}, \|(\phi^2, \psi^2, \theta^2)\|_V \leq \frac{1}{\epsilon}$, we can use the following cancellations

$$\begin{aligned}
&\int (\phi_2 \cdot \nabla)(\phi_1 - \phi_2) \cdot (\phi_1 - \phi_2) = 0, \quad \int (\phi_2 \cdot \nabla)(\psi_1 - \psi_2) \cdot (\psi_1 - \psi_2) = 0, \\
&\int (W_{A_1} \cdot \nabla)(\psi_1 - \psi_2) \cdot (\psi_1 - \psi_2) = 0, \quad \int (\phi_2 \cdot \nabla)(\theta_1 - \theta_2) \cdot (\theta_1 - \theta_2) = 0, \\
&\int (W_{A_1} \cdot \nabla)(\theta_1 - \theta_2) \cdot (\theta_1 - \theta_2) = 0, \\
&\int (\theta_2 \cdot \nabla)(\theta_1 - \theta_2)(\phi_1 - \phi_2) + (\theta_2 \cdot \nabla)(\phi_1 - \phi_2)(\theta_1 - \theta_2) = 0, \\
&\int (W_{A_3} \cdot \nabla)(\theta_1 - \theta_2)(\phi_1 - \phi_2) + (W_{A_3} \cdot \nabla)(\phi_1 - \phi_2)(\theta_1 - \theta_2) = 0,
\end{aligned}$$

to deduce

$$
\begin{aligned}
&\langle \mathcal{F}_\epsilon(\phi_1,\psi_1,\theta_1) - \mathcal{F}_\epsilon(\phi_2,\psi_2,\theta_2), (\phi_1-\phi_2,\psi_1-\psi_2,\theta_1-\theta_2)\rangle_{V,V'}\\
&= \begin{pmatrix}
\langle(\phi_1-\phi_2)\cdot\nabla\phi_1 + (\phi_2\cdot\nabla)(\phi_1-\phi_2) - (\theta_1-\theta_2)\cdot\nabla\theta_1\\
-(\theta_2\cdot\nabla)(\theta_1-\theta_2) + (W_{A_1}\cdot\nabla)(\phi_1-\phi_2) + (\phi_1-\phi_2)\cdot\nabla W_{A_1}\\
-(W_{A_3}\cdot\nabla)(\theta_1-\theta_2) - (\theta_1-\theta_2)\cdot\nabla W_{A_3} - \chi\nabla\times(\psi_1-\psi_2), \phi_1-\phi_2\rangle\\
\langle(\phi_1-\phi_2)\cdot\nabla\psi_1 + (\phi_2\cdot\nabla)(\psi_1-\psi_2) + (\phi_1-\phi_2)\cdot\nabla W_{A_2}\\
+(W_{A_1}\cdot\nabla)(\psi_1-\psi_2) + \chi(\psi_1-\psi_2) - \chi\nabla\times(\phi_1-\phi_2), \psi_1-\psi_2\rangle\\
\langle(\phi_1-\phi_2)\cdot\nabla\theta_1 + (\phi_2\cdot\nabla)(\theta_1-\theta_2) - (\theta_1-\theta_2)\cdot\nabla\phi_1\\
-(\theta_2\cdot\nabla)(\phi_1-\phi_2) + (\phi_1-\phi_2)\cdot\nabla W_{A_3} - (W_{A_3}\cdot\nabla)(\phi_1-\phi_2)\\
-(\theta_1-\theta_2)\cdot\nabla W_{A_1} - (W_{A_1}\cdot\nabla)(\theta_1-\theta_2), \theta_1-\theta_2\rangle
\end{pmatrix}
\end{aligned}
$$

by (3.13), (3.5a)-(3.5c), (3.2a)-(3.2g), (2.2) so that applications of Hölder's and Young's inequalities, (3.7) and (2.9) as in the previous estimates (e.g. (3.6)) lead to

$$
\begin{aligned}
&\langle \mathcal{F}_\epsilon(\phi_1,\psi_1,\theta_1) - \mathcal{F}_\epsilon(\phi_2,\psi_2,\theta_2), (\phi_1-\phi_2,\psi_1-\psi_2,\theta_1-\theta_2)\rangle_{V,V'}\\
&\lesssim_\epsilon \|(\phi_1-\phi_2,\psi_1-\psi_2,\theta_1-\theta_2)\|_{L^2}\\
&\quad\times(1+\|(\phi_1-\phi_2,\psi_1-\psi_2,\theta_1-\theta_2)\|_V) + \|\psi_1-\psi_2\|_{L^2}^2\\
&\leq \frac{\delta}{2}\|(\phi_1-\phi_2,\psi_1-\psi_2,\theta_1-\theta_2)\|_V^2 + c(\epsilon)(1+\|(\phi_1-\phi_2,\psi_1-\psi_2,\theta_1-\theta_2)\|_H^2).
\end{aligned}
$$

Moreover, in case $\|(\phi^1,\psi^1,\theta^1)\|_V \leq \frac{1}{\epsilon}, \|(\phi^2,\psi^2,\theta^2)\|_V \leq \frac{1}{\epsilon}$, we may compute

$$
\begin{aligned}
&\|\mathcal{F}_\epsilon(\phi,\psi,\theta)\|_{V'}\\
&\lesssim \|\phi\otimes\phi\|_{L^2} + \|\theta\theta\|_{L^2} + \|W_{A_1}\otimes\phi\|_{L^2} + \|W_{A_3}\otimes\theta\|_{L^2} + \|\psi\|_{L^2}\\
&\quad + \|\phi\otimes\psi\|_{L^2} + \|\phi\otimes W_{A_2}\|_{L^2} + \|W_{A_1}\otimes\psi\|_{L^2} + \|\phi\|_{L^2}\\
&\quad + \|\phi\theta\|_{L^2} + \|\phi\otimes W_{A_3}\|_{L^2} + \|\theta\otimes W_{A_1}\|_{L^2}\\
&\lesssim (1+\|W_{A_1}\|_{L^4} + \|W_{A_2}\|_{L^4} + \|W_{A_3}\|_{L^4} + \|(\phi,\psi,\theta)\|_H)\|(\phi,\psi,\theta)\|_V
\end{aligned}
$$

by (3.13), (3.5a)-(3.5c), (3.2a)-(3.2g), (2.2) and (3.7). Due to (2.8) and (2.9), these properties of $\mathcal{F}_\epsilon, \mathcal{G}_\epsilon$ in (3.13) and (3.14) are sufficient to guarantee the existence and uniqueness of the solution

$$(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon) \in C_W([0,T_\epsilon];H)\cap L_W^2(0,T_\epsilon;V) \text{ such that } \partial_t(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon) \in L_W^2(0,T_\epsilon;V')$$

to (3.4a)-(3.4c) on $[0,T_\epsilon]$ for some $T_\epsilon = T(\epsilon)$, (cf. [5, 30]).

Next, on (3.4a)-(3.4c), we obtain explicit bounds independent of $\epsilon > 0$ to subsequently take the limit $\epsilon \to 0$. We take L^2-inner products with $(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)$ respectively, sum and use cancellations of

$$
\begin{aligned}
&\int [F_1(\overline{u}_\epsilon) - F_2(\overline{b}_\epsilon)]\cdot\overline{u}_\epsilon + \int F_3(\overline{u}_\epsilon,\overline{w}_\epsilon)\cdot\overline{w}_\epsilon + \int [F_4(\overline{u}_\epsilon,\overline{b}_\epsilon) - F_5(\overline{b}_\epsilon,\overline{u}_\epsilon)]\cdot\overline{b}_\epsilon = 0,\\
&\int (W_{A_1}\cdot\nabla)\overline{u}_\epsilon\cdot\overline{u}_\epsilon = 0, \quad \int (W_{A_1}\cdot\nabla)\overline{w}_\epsilon\cdot\overline{w}_\epsilon = 0, \quad \int (W_{A_1}\cdot\nabla)\overline{b}_\epsilon\cdot\overline{b}_\epsilon = 0,\\
&\int (W_{A_3}\cdot\nabla)\overline{b}_\epsilon\cdot\overline{u}_\epsilon + (W_{A_3}\cdot\nabla)\overline{u}_\epsilon\cdot\overline{b}_\epsilon = 0,
\end{aligned}
$$

and that $\int \overline{u}_\epsilon \cdot (\nabla \times \overline{w}_\epsilon) = \int (\nabla \times \overline{u}_\epsilon)\overline{w}_\epsilon$ to obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t |(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)|^2 + (\mu+\chi)\|\overline{u}_\epsilon\|_{V_1}^2 + \min\{\chi,\gamma\}\|\overline{w}_\epsilon\|_{V_2}^2 + \nu\|\overline{b}_\epsilon\|_{V_3}^2 \\
\le & - b(\overline{u}_\epsilon, W_{A_1}, \overline{u}_\epsilon) + b(\overline{b}_\epsilon, W_{A_3}, \overline{u}_\epsilon) - b(\overline{u}_\epsilon, W_{A_2}, \overline{w}_\epsilon) - b(\overline{u}_\epsilon, W_{A_3}, \overline{b}_\epsilon) + b(\overline{b}_\epsilon, W_{A_1}, \overline{b}_\epsilon) \\
&+ 2\chi \int (\nabla \times \overline{u}_\epsilon)\cdot \overline{w}_\epsilon - \chi |\overline{w}_\epsilon|^2 \\
&- b(W_{A_1}, W_{A_1}, \overline{u}_\epsilon) + b(W_{A_3}, W_{A_3}, \overline{u}_\epsilon) + \int \chi(\nabla\times \overline{u}_\epsilon)\cdot W_{A_2} - b(W_{A_1}, W_{A_2}, \overline{w}_\epsilon) \\
&- \int \chi W_{A_2}\cdot \overline{w}_\epsilon + \chi(\nabla \times \overline{w}_\epsilon)\cdot W_{A_1} - b(W_{A_1}, W_{A_3}, \overline{b}_\epsilon) + b(W_{A_3}, W_{A_1}, \overline{b}_\epsilon) \\
\le & \|\overline{u}_\epsilon\|_{L^4}\|\overline{u}_\epsilon\|_{V_1}\|W_{A_1}\|_{L^4} + \|\overline{b}_\epsilon\|_{L^4}\|\overline{u}_\epsilon\|_{V_1}\|W_{A_3}\|_{L^4} + \|\overline{u}_\epsilon\|_{L^4}\|\overline{w}_\epsilon\|_{V_2}\|W_{A_2}\|_{L^4} \\
&+ \|\overline{u}_\epsilon\|_{L^4}\|\overline{b}_\epsilon\|_{V_3}\|W_{A_3}\|_{L^4} + \|\overline{b}_\epsilon\|_{L^4}\|\overline{b}_\epsilon\|_{V_3}\|W_{A_1}\|_{L^4} + 2\chi|\overline{w}_\epsilon|\|\overline{u}_\epsilon\|_{V_1} - \chi|\overline{w}_\epsilon|^2 \\
&+ \|W_{A_1}\|_{L^4}^2\|\overline{u}_\epsilon\|_{V_1} + \|W_{A_3}\|_{L^4}^2\|\overline{u}_\epsilon\|_{V_1} + \chi|W_{A_2}|\|\overline{u}_\epsilon\|_{V_1} + \|W_{A_1}\|_{L^4}\|W_{A_2}\|_{L^4}\|\overline{w}_\epsilon\|_{V_2} \\
&+ \chi|W_{A_2}||\overline{w}_\epsilon| + \chi|W_{A_1}|\|\overline{w}_\epsilon\|_{V_2} + \|W_{A_1}\|_{L^4}\|W_{A_3}\|_{L^4}\|\overline{b}_\epsilon\|_{V_3} \\
\le & (\frac{\mu}{2}+\chi)\|\overline{u}_\epsilon\|_{V_1}^2 + \frac{\min\{\chi,\gamma\}}{2}\|\overline{w}_\epsilon\|_{V_2}^2 + \frac{\nu}{2}\|\overline{b}_\epsilon\|_{V_3}^2 \\
&+ c(|\overline{u}_\epsilon|^2\|W_{A_1}\|_{L^4}^4 + |\overline{b}_\epsilon|^2\|W_{A_3}\|_{L^4}^4 + |\overline{u}_\epsilon|^2\|W_{A_2}\|_{L^4}^4 + |\overline{u}_\epsilon|^2\|W_{A_3}\|_{L^4}^4 \\
&+ |\overline{b}_\epsilon|^2\|W_{A_1}\|_{L^4}^4 + \|W_{A_1}\|_{L^4}^4 + \|W_{A_3}\|_{L^4}^4 + \|W_{A_2}\|_{L^4}^4 + \|W_{A_2}\|_{L^4}^2 + |\overline{w}_\epsilon|^2 + |W_{A_1}|^2)
\end{aligned}$$

by Hölder's inequalities, (3.7) and Young's inequalities. In particular we used that due to the vector calculus identity of $\nabla\times(\nabla\times f) = \nabla(\nabla\cdot f) - \Delta f$, $|\nabla \times \overline{u}_\epsilon|^2 = \|\overline{u}_\epsilon\|_{V_1}^2$, and $2\chi|\overline{w}_\epsilon|\|\overline{u}_\epsilon\|_{V_1} - \chi|\overline{w}_\epsilon|^2 \le \chi\|\overline{u}_\epsilon\|_{V_1}^2$ due to Young's inequality. After subtracting $(\frac{\mu}{2}+\chi)\|\overline{u}_\epsilon\|_{V_1}^2 + \frac{\min\{\chi,\gamma\}}{2}\|\overline{w}_\epsilon\|_{V_2}^2 + \frac{\nu}{2}\|\overline{b}_\epsilon\|_{V_3}^2$ from both sides and multiplying by 2, by Gronwall's inequality we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]} |(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)|^2(t) + \int_0^T \mu\|\overline{u}_\epsilon\|_{V_1}^2 + \min\{\chi,\gamma\}\|\overline{w}_\epsilon\|_{V_2}^2 + \nu\|\overline{b}_\epsilon\|_{V_3}^2 ds \\
&\lesssim |(u_0, w_0, b_0)|^2 e^{\int_0^T \|(W_{A_1}, W_{A_2}, W_{A_3})\|_{L^4}^4 ds}.
\end{aligned} \tag{3.16}$$

Moreover, we may estimate

$$\begin{aligned}
&\|\Phi_1^\epsilon(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)\|_{V_1'} + \|\Phi_2^\epsilon(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)\|_{V_2'} + \|\Phi_3^\epsilon(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)\|_{V_3'} \\
&+ \|G_1(\overline{u}_\epsilon)\|_{V_1'} + \|G_2(\overline{b}_\epsilon)\|_{V_1'} + \|G_3(\overline{u}_\epsilon)\|_{V_2'} \\
&+ \|G_4(\overline{w}_\epsilon)\|_{V_2'} + \|G_5(\overline{u}_\epsilon)\|_{V_3'} + \|G_6(\overline{b}_\epsilon)\|_{V_3'} + \|R_1(\overline{w}_\epsilon)\|_{V_1'} + \|R_2(\overline{w}_\epsilon, \overline{u}_\epsilon)\|_{V_2'} \\
\lesssim & 1 + \|\overline{u}_\epsilon\|_{L^4}^2 + \|\overline{b}_\epsilon\|_{L^4}^2 + \|\overline{w}_\epsilon\|_{L^4}^2 + \|W_{A_1}\|_{L^4}^2 + \|W_{A_3}\|_{L^4}^2 + \|W_{A_2}\|_{L^4}^2 \\
\lesssim & 1 + |\overline{u}_\epsilon|\|\overline{u}_\epsilon\|_{V_1} + |\overline{b}_\epsilon|\|\overline{b}_\epsilon\|_{V_3} + |\overline{w}_\epsilon|\|\overline{w}_\epsilon\|_{V_2} + \|(W_{A_1}, W_{A_2}, W_{A_3})\|_{L^4}^2 \in L^2(0,T)
\end{aligned} \tag{3.17}$$

by (3.5a)-(3.5c), (3.2a)-(3.2g), (2.2), (3.7), (3.16) and (2.9). The bounds of (3.16) and (3.17) imply that for a fixed $\omega \in \Omega$ with $\epsilon = \epsilon(\omega)$,

$$(\overline{u}^\epsilon, \overline{w}^\epsilon, \overline{b}^\epsilon) \to (u, w, b) \quad \text{weak}^* \text{ in } L^\infty(0,T;H), \text{ weakly in } L^2(0,T;V), \tag{3.18}$$

and for some Ψ_1, Ψ_2, Ψ_3,

$$\begin{aligned} &\Phi_i^\epsilon(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon) \to \Psi_i, \text{ weakly in } L^2(0,T;V'), i=1,2,3 \\ &G_1(\overline{u}_\epsilon) \to G_1(u) \text{ weakly in } L^2(0,T;V_1'), \\ &G_2(\overline{b}_\epsilon) \to G_2(b) \text{ weakly in } L^2(0,T;V_1'), \\ &G_3(\overline{u}_\epsilon) \to G_3(u) \text{ weakly in } L^2(0,T;V_2'), \\ &G_4(\overline{w}_\epsilon) \to G_4(w) \text{ weakly in } L^2(0,T;V_2'), \\ &G_5(\overline{u}_\epsilon) \to G_5(u) \text{ weakly in } L^2(0,T;V_3'), \\ &G_6(\overline{b}_\epsilon) \to G_6(b) \text{ weakly in } L^2(0,T;V_3'), \\ &R_1(\overline{w}_\epsilon) \to R_1(w) \text{ weakly in } L^2(0,T;V_1'), \\ &R_2(\overline{w}_\epsilon, \overline{u}_\epsilon) \to R_2(w,u) \text{ weakly in } L^2(0,T;V_2'), \end{aligned} \tag{3.19}$$

as $\epsilon \to 0$. Therefore, from (2.6a)-(2.6c), (3.18) and (3.19) we obtain for a.e. $t \in [0,T]$,

$$\partial_t u + A_1 u + \Psi_1 + G_1(u) - G_2(b) + R_1(w) = -F_1(W_{A_1}) + F_2(W_{A_3}) - R_1(W_{A_2}), \tag{3.20a}$$

$$\partial_t w + A_2 w + \Psi_2 + G_3(u) + G_4(w) + R_2(w,u) = -F_3(W_{A_1}, W_{A_2}) - R_2(W_{A_2}, W_{A_1}), \tag{3.20b}$$

$$\partial_t b + A_3 b + \Psi_3 + G_5(u) - G_6(b) = -F_4(W_{A_1}, W_{A_3}) + F_5(W_{A_3}, W_{A_1}). \tag{3.20c}$$

Moreover, from (3.4a)-(3.4c) we estimate

$$\begin{aligned} &\|(\partial_t \overline{u}_\epsilon, \partial_t \overline{w}_\epsilon, \partial_t \overline{b}_\epsilon)\|^2_{L^2(0,T;V')} \\ \lesssim &\|A_1 \overline{u}_\epsilon\|^2_{L^2(0,T;V_1')} + \|A_2 \overline{w}_\epsilon\|^2_{L^2(0,T;V_2')} + \|A_3 \overline{b}_\epsilon\|^2_{L^2(0,T;V_3')} \\ &+ \sum_{i=1}^3 \|\Phi_i^\epsilon(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)\|^2_{L^2(0,T;V_i')} + \|G_1(\overline{u}_\epsilon)\|^2_{L^2(0,T;V_1')} + \|G_2(\overline{b}_\epsilon)\|^2_{L^2(0,T;V_1')} \\ &+ \|R_1(\overline{w}_\epsilon)\|^2_{L^2(0,T;V_1')} + \|G_3(\overline{u}_\epsilon)\|^2_{L^2(0,T;V_2')} + \|G_4(\overline{w}_\epsilon)\|^2_{L^2(0,T;V_2')} \\ &+ \|R_2(\overline{w}_\epsilon, \overline{u}_\epsilon)\|^2_{L^2(0,T;V_2')} + \|G_5(\overline{u}_\epsilon)\|^2_{L^2(0,T;V_3')} + \|G_6(\overline{b}_\epsilon)\|^2_{L^2(0,T;V_3')} \\ &+ \int_0^T \|W_{A_1} \otimes W_{A_1}\|^2_{L^2} dt + \int_0^T \|W_{A_3} \otimes W_{A_3}\|^2_{L^2} dt + \int_0^T \|W_{A_2}\|^2_{L^2} dt \\ &+ \int_0^T \|W_{A_1} \otimes W_{A_2}\|^2_{L^2} dt + \int_0^T \|W_{A_1}\|^2_{L^2} dt + \int_0^T \|W_{A_1} \otimes W_{A_3}\|^2_{L^2} dt \lesssim 1 \end{aligned}$$

by (3.4a)-(3.4c), (3.16), (3.17) and (2.9). This implies that together with (3.16),

$$(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon) \in L^\infty(0,T;H) \cap L^2(0,T;V) \text{ and } (\partial_t \overline{u}_\epsilon, \partial_t \overline{w}_\epsilon, \partial_t \overline{b}_\epsilon) \in L^2(0,T;V')$$

and hence due to the well-known compact embedding result (cf. Lemma 8.6 [5]), we see that

$$(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon) \to (u,w,b) \text{ strongly in } L^2(0,T;H) \tag{3.21}$$

as $\epsilon \to 0$. Moreover, $\forall\, \phi \in C([0,T]; D(A_1))$, due to (3.5a) and (3.2a), we may write

$$\begin{aligned}
&\int_0^T (\Phi_1^\epsilon(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon), \phi) dt \\
=& \int_{\{t\in[0,T]:\|(\overline{u}_\epsilon,\overline{w}_\epsilon,\overline{b}_\epsilon)\|_V^2 \le \frac{1}{\epsilon^2}\}} \int [(\overline{u}_\epsilon \cdot \nabla)\overline{u}_\epsilon - (\overline{b}_\epsilon \cdot \nabla)\overline{b}_\epsilon] \cdot \phi dt \\
&+ \int_{\{t\in[0,T]:\|(\overline{u}_\epsilon,\overline{w}_\epsilon,\overline{b}_\epsilon)\|_V^2 > \frac{1}{\epsilon^2}\}} \int \frac{[(\overline{u}_\epsilon \cdot \nabla)\overline{u}_\epsilon - (\overline{b}_\epsilon \cdot \nabla)\overline{b}_\epsilon]}{\epsilon^2 \|(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)\|_V^2} \cdot \phi dt \triangleq I_\epsilon^1 + I_\epsilon^2 \qquad (3.22)
\end{aligned}$$

where we may compute

$$\begin{aligned}
&\int_0^T |\int [(\overline{u}_\epsilon \cdot \nabla)\overline{u}_\epsilon - (\overline{b}_\epsilon \cdot \nabla)\overline{b}_\epsilon] \cdot \phi - \int [(u \cdot \nabla)u - (b\cdot\nabla)b] \cdot \phi | dt \\
\lesssim& \|\phi\|_{C([0,T];D(A_1))} [(\int_0^T \|u_\epsilon - u\|_{L^2}^2 dt)^{\frac{1}{2}} (\int_0^T \|\overline{u}_\epsilon\|_{V_1}^2 + \|u\|_{V_1}^2 dt)^{\frac{1}{2}} \\
&\qquad + (\int_0^T \|\overline{b}_\epsilon - b\|_{L^2}^2 dt)^{\frac{1}{2}} (\int_0^T \|\overline{b}_\epsilon\|_{V_3}^2 + \|b\|_{V_3}^2)^{\frac{1}{2}}] \to 0
\end{aligned}$$

as $\epsilon \to 0$ by Hölder's inequality, embeddings of $H^1 \hookrightarrow L^4, V_1 \hookrightarrow L^4, V_3 \hookrightarrow L^4$ due to the well-known facts that V_1, V_2 have equivalent norms as H^1-norm (cf. [14] Theorem VII 6.1 for the V_2-norm case), (3.18) and because $\overline{u}_\epsilon \to u$ strongly in $L^2(0,T;H_1), \overline{b}_\epsilon \to b$ strongly in $L^2(0,T;H_3)$ as $\epsilon \to 0$ due to (3.21). Thus, we see that after relabeling a subsequence if necessary,

$$\int [(\overline{u}_\epsilon \cdot \nabla)\overline{u}_\epsilon - (\overline{b}_\epsilon \cdot \nabla)\overline{b}_\epsilon] \cdot \phi \to \int [(u\cdot\nabla)u - (b\cdot\nabla)b] \cdot \phi$$

as $\epsilon \to 0$ a.e. $t \in [0,T]$. Moreover,

$$\left| \int [(\overline{u}_\epsilon \cdot \nabla)\overline{u}_\epsilon - (\overline{b}_\epsilon \cdot \nabla)\overline{b}_\epsilon] \cdot \phi \right| \lesssim |\overline{u}_\epsilon| \|\overline{u}_\epsilon\| + |\overline{b}_\epsilon| \|\overline{b}_\epsilon\|_{V_3} \in L^1([0,T])$$

by Hölder's inequality, (3.7) and (3.16) where the integrability is independent of ϵ. Thus, by the dominated convergence theorem, we obtain

$$I_\epsilon^1 \to \int_0^T \int [(u\cdot\nabla)u - (b\cdot\nabla)b] \cdot \phi dt, \qquad (3.23)$$

as $\epsilon \to 0$. On the other hand,

$$\begin{aligned}
|I_\epsilon^2| \lesssim& (\sup_{t\in[0,T]} |\overline{u}_\epsilon(t)| + |\overline{b}_\epsilon(t)|) \int_{\{t\in[0,T]:\|(\overline{u}_\epsilon,\overline{w}_\epsilon,\overline{b}_\epsilon)\|_V^2 > \frac{1}{\epsilon^2}\}} \|\overline{u}_\epsilon\|_{V_1} + \|\overline{b}_\epsilon\|_{V_3} dt \\
\lesssim& \epsilon \int_0^T \|(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)\|_V^2 dt \lesssim \epsilon \to 0 \qquad (3.24)
\end{aligned}$$

as $\epsilon \to 0$ where we used Hölder's inequality and that $(\overline{u}_\epsilon, \overline{b}_\epsilon) \in C([0,T]; H_1 \times H_3), (\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon) \in L^2(0,T;V)$ by (3.16). Thus, $\Psi_1 = F_1(u) - F_2(b)$.

Similarly, $\forall\, \psi \in C([0,T]; D(A_2))$, due to (3.5b), (3.2a),

$$\int_0^T (\Phi_2^\epsilon(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon), \psi) dt = \int_{\{t\in[0,T]: \|(\overline{u}_\epsilon,\overline{w}_\epsilon,\overline{b}_\epsilon)\|_V^2 \le \frac{1}{\epsilon^2}\}} \int (\overline{u}_\epsilon \cdot \nabla)\overline{w}_\epsilon \psi dt$$
$$+ \int_{\{t\in[0,T]: \|(\overline{u}_\epsilon,\overline{w}_\epsilon,\overline{b}_\epsilon)\|_V^2 > \frac{1}{\epsilon^2}\}} \int \frac{(\overline{u}_\epsilon \cdot \nabla)\overline{w}_\epsilon}{\epsilon^2 \|(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)\|_V^2} \psi dt \triangleq I_\epsilon^3 + I_\epsilon^4 \tag{3.25}$$

where

$$\int_0^T \left| \int (\overline{u}_\epsilon \cdot \nabla)\overline{w}_\epsilon \psi - \int (u\cdot\nabla) w\psi \right| dt$$
$$\lesssim \|\psi\|_{C([0,T];D(A_2))} \Bigg[\left(\int_0^T \|\overline{u}_\epsilon - u\|_{L^2}^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \|\overline{w}_\epsilon\|_{V_2}^2 dt\right)^{\frac{1}{2}}$$
$$+ \left(\int_0^T \|u\|_{V_1}^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \|\overline{w}_\epsilon - w\|_{L^2}^2 dt\right)^{\frac{1}{2}} \Bigg] \to 0$$

as $\epsilon \to 0$ by Hölder's inequalities, and embeddings of $V_2 \hookrightarrow L^4, V_1 \hookrightarrow L^4$, (3.16) and (3.21). By relabeling subsequence if necessary, this implies

$$\int (\overline{u}_\epsilon \cdot \nabla)\overline{w}_\epsilon \psi \to \int (u\cdot\nabla) w\psi$$

as $\epsilon \to 0$ for a.e. $t \in [0,T]$. Moreover,

$$\left| \int (\overline{u}_\epsilon \cdot \nabla)\overline{w}_\epsilon \psi \right| \lesssim |\overline{u}_\epsilon|^{\frac{1}{2}} \|\overline{u}_\epsilon\|_{V_1}^{\frac{1}{2}} |\overline{w}_\epsilon|^{\frac{1}{2}} \|\overline{w}_\epsilon\|_{V_2}^{\frac{1}{2}} \in L^1([0,T])$$

by Hölder's inequality, (3.7) and (3.16). Thus, by the dominated convergence theorem,

$$I_\epsilon^3 \to \int_0^T \int (u\cdot\nabla) w\psi dt \tag{3.26}$$

as $\epsilon \to 0$. On the other hand, similarly to (3.24), it can be shown that

$$|I_\epsilon^4| \le \int_{\{t\in[0,T]: \|(\overline{u}_\epsilon,\overline{w}_\epsilon,\overline{b}_\epsilon)\|_V^2 > \frac{1}{\epsilon^2}\}} \|\overline{u}_\epsilon\|_{L^4} \|\nabla\psi\|_{L^2} \|\overline{w}_\epsilon\|_{L^4} dt \lesssim \epsilon \to 0 \tag{3.27}$$

as $\epsilon \to 0$ by Hölder's inequality, (3.7) and (3.16). Thus, $\Psi_2 = F_3(u,w)$.

Finally, $\forall\, \theta \in C([0,T]; D(A_3))$, due to (3.5c) and (3.2b), we may write

$$\int_0^T (\Phi_3^\epsilon(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon), \theta) dt$$
$$= \int_{\{t\in[0,T]: \|(\overline{u}_\epsilon,\overline{w}_\epsilon,\overline{b}_\epsilon)\|_V^2 \le \frac{1}{\epsilon^2}\}} \int [(\overline{u}_\epsilon \cdot \nabla)\overline{b}_\epsilon - (\overline{b}_\epsilon \cdot \nabla)\overline{u}_\epsilon] \cdot \theta dt$$
$$+ \int_{\{t\in[0,T]: \|(\overline{u}_\epsilon,\overline{w}_\epsilon,\overline{b}_\epsilon)\|_V^2 > \frac{1}{\epsilon^2}\}} \int \frac{[(\overline{u}_\epsilon \cdot \nabla)\overline{b}_\epsilon - (\overline{b}_\epsilon \cdot \nabla)\overline{u}_\epsilon]}{\epsilon^2 \|(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)\|_V^2} \cdot \theta dt \triangleq I_\epsilon^5 + I_\epsilon^6 \tag{3.28}$$

where

$$\int_0^T |\int [(\overline{u}_\epsilon \cdot \nabla)\overline{b}_\epsilon - (\overline{b}_\epsilon \cdot \nabla)\overline{u}_\epsilon] \cdot \theta - \int [(u\cdot\nabla)b - (b\cdot\nabla)u]\cdot\theta| dt$$

$$\lesssim \|\theta\|_{C([0,T];D(A_3))}\Bigg[\left(\int_0^T \|\overline{u}_\epsilon - u\|_{L^2}^2 dt\right)^{\frac{1}{2}} \left(\left(\int_0^T \|\overline{b}_\epsilon\|_{V_3}^2 dt\right)^{\frac{1}{2}} + \left(\int_0^T \|b\|_{V_3}^2 dt\right)^{\frac{1}{2}}\right)$$

$$+ \left(\int_0^T \|\overline{b}_\epsilon - b\|_{L^2}^2 dt\right)^{\frac{1}{2}} \left(\left(\int_0^T \|u\|_{V_1}^2 dt\right)^{\frac{1}{2}} + \left(\int_0^T \|\overline{u}_\epsilon\|_{V_1}^2 dt\right)^{\frac{1}{2}}\right)\Bigg] \to 0$$

as $\epsilon \to 0$ by Hölder's inequalities and embeddings of $V_3 \hookrightarrow L^4, V_1 \hookrightarrow L^4$. By relabeling subsequence if necessary, we obtain

$$\int [(\overline{u}_\epsilon \cdot \nabla)\overline{b}_\epsilon - (\overline{b}_\epsilon \cdot \nabla)\overline{u}_\epsilon] \cdot \theta \to \int [(u\cdot\nabla)b - (b\cdot\nabla)u]\cdot\theta$$

as $\epsilon \to 0$ for a.e. $t \in [0,T]$. Moreover,

$$\left|\int [(\overline{u}_\epsilon \cdot \nabla)\overline{b}_\epsilon - (\overline{b}_\epsilon \cdot \nabla)\overline{u}_\epsilon] \cdot \theta\right| \lesssim |\overline{u}_\epsilon|^{\frac{1}{2}} \|\overline{u}_\epsilon\|_{V_1}^{\frac{1}{2}} |\overline{b}_\epsilon|^{\frac{1}{2}} \|\overline{b}_\epsilon\|_{V_3}^{\frac{1}{2}} \in L^1([0,T])$$

by Hölder's inequalities, (3.7) and (3.16). Thus, by the dominated convergence theorem,

$$I_\epsilon^5 \to \int_0^T \int [(u\cdot\nabla)b - (b\cdot\nabla)u]\cdot\theta dt \tag{3.29}$$

as $\epsilon \to 0$. On the other hand, similar computations to (3.24) shows that

$$|I_\epsilon^6| \lesssim \int_{\{t\in[0,T]: \|(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)\|_V^2 > \frac{1}{\epsilon^2}\}} \|\overline{u}_\epsilon\|_{L^4} \|\overline{b}_\epsilon\|_{L^4} \|\nabla\theta\|_{L^2} dt \lesssim \epsilon \to 0 \tag{3.30}$$

as $\epsilon \to 0$ where we used Hölder's inequality, (3.7) and (3.16). Thus, $\Psi_3 = F_4(u,b) - F_5(b,u)$.

Therefore, (u,w,b) solves the equation of $(\overline{u}, \overline{w}, \overline{b})$; i.e. we have for fixed $\omega \in \Omega$,

$$\partial_t u + A_1 u + F_1(u) - F_2(b) + G_1(u) - G_2(b) + F_1(W_{A_1}) - F_2(W_{A_3}) + R_1(w) + R_1(W_{A_2}) = 0, \tag{3.31a}$$

$$\partial_t w + A_2 w + F_3(u,w) + G_3(u) + G_4(w) + F_3(W_{A_1}, W_{A_2}) + R_2(w,u) + R_2(W_{A_2}, W_{A_1}) = 0, \tag{3.31b}$$

$$\partial_t b + A_3 b + F_4(u,b) - F_5(b,u) + G_5(u) - G_6(b) + F_4(W_{A_1}, W_{A_3}) - F_5(W_{A_3}, W_{A_1}) = 0, \tag{3.31c}$$

from (3.4a)-(3.4c). On the other hand, for any $\omega \in \Omega$, (3.20a)-(3.20c) has at most one solution in the regularity class of $C([0,T];H) \cap L^2(0,T;V)$. Thus, $\mathbb{P}$-a.s., $(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon) \to (u,w,b)$ weakly in $L^2(0,T;V)$ as $\epsilon \to 0$. Hence, (u,w,b) is adapted with respect to $\mathcal{F}_t$ because $(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)$ is adapted to $\mathcal{F}_t$. Therefore, $(u,w,b) \in L_W^2(0,T;V)$ and $(\partial_t u, \partial_t w, \partial_t b) \in L_W^2(0,T;V')$. This completes the proof of Proposition 3.1. □

3.1. Conclusion of the Proof of Theorem 2.2. For brevity, we consider the case $\|(\overline{u}_\epsilon, \overline{w}_\epsilon, \overline{b}_\epsilon)\|_V^2 \leq \frac{1}{\epsilon^2}$, $\|(u_\epsilon, w_\epsilon, b_\epsilon)\|_V^2 \leq \frac{1}{\epsilon^2}$. Defining

$$u_\epsilon \triangleq \overline{u}_\epsilon + W_{A_1}, \quad w_\epsilon \triangleq \overline{w}_\epsilon + W_{A_2}, \quad b_\epsilon \triangleq \overline{b}_\epsilon + W_{A_3}, \tag{3.32}$$

due to (3.32), (3.4a)-(3.4d), (2.6a)-(2.6c), (3.5a)-(3.5c), (2.2), (2.5), we obtain

$$\begin{cases} du_\epsilon + [A_1 u_\epsilon + \Phi_1^\epsilon(u_\epsilon, w_\epsilon, b_\epsilon) + R_1(w_\epsilon)]dt = \sqrt{Q_1}dW_1, \\ dw_\epsilon + [A_2 w_\epsilon + \Phi_2^\epsilon(u_\epsilon, w_\epsilon, b_\epsilon) + R_2(w_\epsilon, u_\epsilon)]dt = \sqrt{Q_2}dW_2, \\ db_\epsilon + [A_3 b_\epsilon + \Phi_3^\epsilon(u_\epsilon, w_\epsilon, b_\epsilon)]dt = \sqrt{Q_3}dW_3, \\ u_\epsilon(0) = u_0, \quad w_\epsilon(0) = w_0, \quad b_\epsilon(0) = b_0. \end{cases} \tag{3.33}$$

By Ito's formula with $f(t,x) = x^2$, summing and taking expectations gives

$$\begin{aligned} &\mathbb{E}[|(u_\epsilon, w_\epsilon, b_\epsilon)(t)|^2] + 2\mathbb{E}[\int_0^t (\mu+\chi)\|u_\epsilon\|_{V_1}^2 + \min\{\chi,\gamma\}\|w_\epsilon\|_{V_2}^2 + \nu\|b_\epsilon\|_{V_3}^2 ds] \\ \leq & |(u_0, w_0, b_0)|^2 - 2\mathbb{E}[\int_0^t \langle u_\epsilon, \Phi_1^\epsilon(u_\epsilon, w_\epsilon, b_\epsilon) + R_1(w_\epsilon)\rangle ds \\ &+ \int_0^t \langle w_\epsilon, \Phi_2^\epsilon(u_\epsilon, w_\epsilon, b_\epsilon) + R_2(w_\epsilon, u_\epsilon)\rangle ds + \int_0^t \langle b_\epsilon, \Phi_3^\epsilon(u_\epsilon, w_\epsilon, b_\epsilon)\rangle ds] \\ &+ tTr(Q_1 + Q_2 + Q_3) \end{aligned}$$

where we may estimate

$$\begin{aligned} &-2\mathbb{E}[\int_0^t \langle u_\epsilon, \Phi_1^\epsilon(u_\epsilon, w_\epsilon, b_\epsilon) + R_1(w_\epsilon)\rangle ds \\ &+ \int_0^t \langle w_\epsilon, \Phi_2^\epsilon(u_\epsilon, w_\epsilon, b_\epsilon) + R_2(w_\epsilon, u_\epsilon)\rangle ds + \int_0^t \langle b_\epsilon, \Phi_3^\epsilon(u_\epsilon, w_\epsilon, b_\epsilon)\rangle ds] \\ =& -2\chi\mathbb{E}[\int_0^t |w_\epsilon|^2 ds] + 4\chi\mathbb{E}[\int_0^t \langle \nabla \times u_\epsilon, w_\epsilon\rangle ds] \leq 2\chi\mathbb{E}[\int_0^t \|u_\epsilon\|_{V_1}^2 ds] \end{aligned}$$

due to (3.5a)-(3.5c), (3.2a)-(3.2b), (2.2), that $\langle u_\epsilon, \chi\nabla \times w_\epsilon\rangle = \chi\langle \nabla \times u_\epsilon, w_\epsilon\rangle$ and Young's inequality as we estimated to obtain (3.16). Thus, subtracting $2\chi\mathbb{E}[\int_0^t \|u_\epsilon\|_{V_1}^2 ds]$ from both sides, we obtain

$$\begin{aligned} &\mathbb{E}[|(u_\epsilon, w_\epsilon, b_\epsilon)(t)|^2] + 2\mathbb{E}[\mu\int_0^t \|u_\epsilon\|_{V_1}^2 + \min\{\chi,\gamma\}\|w_\epsilon\|_{V_2}^2 + \nu\|b_\epsilon\|_{V_3}^2 ds] \\ \leq & |(u_0, w_0, b_0)|^2 + tTr(Q_1 + Q_2 + Q_3). \end{aligned} \tag{3.34}$$

Therefore, by weak compactness we obtain the convergence of $(u_\epsilon, w_\epsilon, b_\epsilon) \to (u, w, b)$ weakly in $L_W^2(0, T; V)$ as $\epsilon \to 0$ and by (3.1) this implies $(u_\epsilon, w_\epsilon, b_\epsilon) \to (\overline{u} + W_{A_1}, \overline{w} + W_{A_2}, \overline{b} + W_{A_3})$ weakly in $L_w^2(0, T; V)$ as $\epsilon \to 0$ where (u, w, b) solves (1.2a)-(1.2c).

Finally, the uniqueness of (u, w, b) as a solution to (1.2a)-(1.2c) is straightforward: suppose $(u^1, w^1, b^1), (u^2, w^2, b^2)$ both solve (1.2a)-(1.2c). Then taking L^2-inner products with $(u^1 - u^2), (w^1 - w^2), (b^1 - b^2)$ on the difference of equations respectively gives in sum

$$\frac{1}{2}\partial_t|(u^1-u^2,w^1-w^2,b^1-b^2)|^2$$
$$+(\mu+\chi)\|u^1-u^2\|_{V_1}^2+\min\{\chi,\gamma\}\|w^1-w^2\|_{V_2}^2+2\chi|w^1-w^2|^2+\nu\|b^1-b^2\|_{V_3}^2$$
$$\leq(\frac{\mu}{2}+\chi)\|u^1-u^2\|_{V_1}^2+\frac{\min\{\chi,\gamma\}}{2}\|w^1-w^2\|_{V_2}^2+\frac{\nu}{2}\|b^1-b^2\|_{V_3}^2$$
$$+c\|(u^1,b^1,w^1)\|_V^2|(u^1-u^2,w^1-w^2,b^1-b^2)|^2$$

by Hölder's inequalities, (3.7) and Young's inequalities, in particular,

$$2\chi|w^1-w^2|\|u^1-u^2\|_{V_1}-\chi|w^1-w^2|^2\leq\chi\|u^1-u^2\|_{V_1}^2.$$

After subtracting $\frac{\mu}{2}\|u^1-u^2\|_{V_1}^2+\frac{\min\{\chi,\gamma\}}{2}\|w^1-w^2\|_{V_2}^2+\frac{\nu}{2}\|b^1-b^2\|_{V_3}^2$ from both sides, the uniqueness follows due to Gronwall's inequality. This completes the proof of Theorem 2.2.

4. Proof of Theorem 2.3

4.1. Existence of Invariant Measure. We let $(u,w,b)(t,x,y,z)\in L_W^2(0,T;V)$ be the solution to (1.2a)-(1.2c) with initial value $(u,w,b)(0)=(x,y,z)$. Here, for example $u(t,x,y,z)$ is a solution u at time t that initiated from (x,y,z) at time $t=0$. As we obtained the estimate (3.34) for the system (3.33), identical computations show that for (1.2a)-(1.2c), $\forall\,(x_0,y_0,z_0)\in H$,

$$\begin{aligned}&\mathbb{E}[|(u(t,x_0,y_0,z_0),w(t,x_0,y_0,z_0),b(t,x_0,y_0,z_0))|^2]\\&+2\mathbb{E}[\int_0^t\mu\|u(s,x_0,y_0,z_0)\|_{V_1}^2+\min\{\chi,\gamma\}\|w(s,x_0,y_0,z_0)\|_{V_2}^2\\&\quad+\nu\|b(s,x_0,y_0,z_0)\|_{V_3}^2ds]\leq|(x_0,y_0,z_0)|^2+tTr(Q_1+Q_2+Q_3).\end{aligned}\tag{4.1}$$

We let $\Pi_t(Y(x,y,z),\cdot)$ be the law of the process $Y(t)=(u(t,x,y,z),w(t,x,y,z),b(t,x,y,z))$. Then $\forall\,g\in C_b(H)$,

$$P_tg(x,y,z)=\int_H g(x_1,y_1,z_1)\Pi_t(Y(x,y,z),dx_1,dy_1,dz_1)\quad\forall\,(x,y,z)\in H.$$

In order to prove the existence of an invariant measure, it suffices in view of the corollary of Krylov-Bogoliubov theorem (see Corollary 3.1.2 [9], also Corollary 11.8 [10]), that the family of measures $\{\beta_T\}_{T\geq1}\triangleq\{\frac{1}{T}\int_0^T\Pi_t(Y(x,y,z),\cdot)dt\}_{T\geq1}$ is tight in $\mathcal{B}(H)$. Let us fix $(x_0,y_0,z_0)\in H$ and denote by B_R the ball in V of radius R so that $\forall\,R>0$, $T\geq1$,

$$\beta_T(B_R^c)\leq\frac{C}{R^2}(|(x_0,y_0,z_0)|^2+Tr(Q_1+Q_2+Q_3))$$

by Chebyshev's inequality and (4.1). Therefore, $\{\beta_T\}_{T\geq1}$ is tight. Next, denoting by β the cluster point of $\{\beta_T\}_{T\geq1}$, integrating (4.1) over H with respect to β gives

$$\int_H\|(x,y,z)\|_V^2\beta(dx,dy,dz)$$
$$\leq\lim_{T\to\infty}\frac{1}{T}\int_H[\frac{1}{2\min\{\mu,\chi,\gamma,\nu\}}(|(x,y,z)|^2+TTr(Q_1+Q_2+Q_3))]\beta(dx,dy,dz)<\infty.$$

4.2. Uniqueness of invariant measure. In order to prove the uniqueness, we start with the following lemma:

Lemma 4.1. *For the solution (u, w, b) to (1.2a)-(1.2c) subjected to (2.1), there exists $c^* > 0$ such that*

$$\min\{\mu, \chi, \gamma, \nu\}\mathbb{E}[\int_0^t \|(u, w, b)\|_V^2 ds] \leq \frac{1}{2}|(u_0, w_0, b_0)|^2 + \frac{t}{2}Tr(Q_1 + Q_2 + Q_3), \quad (4.2)$$

$$\mathbb{E}[|(u, w, b)(t)|^2] \leq e^{-c^* t}|(u_0, w_0, b_0)|^2 + \frac{Tr(Q_1 + Q_2 + Q_3)}{c^*}. \quad (4.3)$$

Proof. From (2.3a)-(2.3c), by Ito's formula with $f(t, x) = x^2$, we obtain in sum

$$\begin{aligned} & d|(u, w, b)|^2 + [2(\mu + \chi)\|u\|_{V_1}^2 + 4\chi|w|^2 + 2\gamma|\nabla w|^2 + 2\nu\|b\|_{V_3}^2]dt \quad (4.4) \\ =& 4\chi\langle\nabla \times u, w\rangle dt + 2\langle u, \sqrt{Q_1}dW_1(t)\rangle + 2\langle w, \sqrt{Q_2}dW_2(t)\rangle \\ & + 2\langle b, \sqrt{Q_3}dW_3(t)\rangle + Tr(Q_1 + Q_2 + Q_3)dt \end{aligned}$$

as $\langle u, (b \cdot \nabla)b\rangle + \langle b, (b \cdot \nabla)u\rangle = 0$. We integrate over $[0, t]$, estimate $4\chi \int_0^t \langle\nabla \times u, w\rangle ds \leq 2\chi \int_0^t \|u\|_{V_1}^2 + |w|^2 ds$, subtract $\int_0^t 2\chi\|u\|_{V_1}^2 + 2\chi|w|^2 ds$ from both sides and take expectations to obtain

$$\begin{aligned} & \mathbb{E}[|(u, w, b)(t)|^2] + 2\min\{\mu, \chi, \gamma, \nu\}\mathbb{E}[\int_0^t \|(u, w, b)\|_V^2 ds] \\ \leq & |(u_0, w_0, b_0)|^2 + tTr(Q_1 + Q_2 + Q_3). \end{aligned}$$

In particular, this implies

$$\min\{\mu, \chi, \gamma, \nu\}\mathbb{E}[\int_0^t \|(u, w, b)\|_V^2 ds] \leq \frac{1}{2}|(u_0, w_0, b_0)|^2 + \frac{t}{2}Tr(Q_1 + Q_2 + Q_3);$$

thus, (4.2). Next, let us fix $c^* > 0$ such that $c^*|(u, w, b)|^2 \leq 2\min\{\mu, \chi, \gamma, \nu\}$ $\|(u, w, b)\|_V^2$, which is due to Poincare's inequality. We use Ito's formula with $f(t, x) = e^{c^* t}x$ on (4.4) to obtain

$$\begin{aligned} & d(e^{c^* t}|(u, w, b)|^2) \\ \leq & e^{c^* t}[2\langle u, \sqrt{Q_1}dW_1\rangle + 2\langle w, \sqrt{Q_2}dW_2\rangle + 2\langle b, \sqrt{Q_3}dW_3\rangle + Tr(Q_1 + Q_2 + Q_3)dt] \end{aligned}$$

where we used that

$$c^*|(u, w, b)|^2 - 2(\mu\|u\|_{V_1}^2 + \min\{\chi, \gamma\}\|w\|_{V_2}^2 + \nu\|b\|_{V_3}^2) \leq 0$$

due to our choice of c^*. Integrating over $[0, t]$ and taking expectations allow us to obtain (4.3), completing the proof of Lemma 4.1. □

Lemma 4.2. *For any $\rho_0, \rho_1 > 0$ such that $|x|, |y|, |z| \leq \rho_0$, there exists $\alpha = \alpha(\rho_0, \rho_1) > 0$ and $T = T(\rho_0, \rho_1) > 0$ such that*

$$\begin{aligned} \mathbb{P}(\max\{ & \sup_{t\in[T,2T]} |u(t, x, y, z)|, \\ & \sup_{t\in[T,2T]} |w(t, x, y, z)|, \sup_{t\in[T,2T]} |b(t, x, y, z)|\} \leq \rho_1) \geq \alpha(\rho_0, \rho_1). \quad (4.5) \end{aligned}$$

Proof. We go back to (1.2a)-(1.2c) and (2.6a)-(2.6c) and define

$$\overline{U} \triangleq u - W_{A_1}, \quad \overline{W} \triangleq w - W_{A_2}, \quad \overline{B} \triangleq b - W_{A_3} \tag{4.6}$$

so that deriving the equations of $\partial_t \overline{U}, \partial_t \overline{W}, \partial_t \overline{B}$, and taking L^2-inner products with $(\overline{U}, \overline{W}, \overline{B})$ respectively gives $\mathbb{P}$-a.s.

$$\begin{aligned}
&\frac{1}{2}\partial_t |(\overline{U}, \overline{W}, \overline{B})|^2 + (\mu+\chi)\|\overline{U}\|_{V_1}^2 + \min\{\chi,\gamma\}\|\overline{W}\|_{V_2}^2 + \nu\|\overline{B}\|_{V_3}^2 \\
\leq& \|\overline{U}\|_{L^4}\|\overline{U}\|_{V_1}\|W_{A_1}\|_{L^4} + \|W_{A_1}\|_{L^4}^2\|\overline{U}\|_{V_1} + \|\overline{B}\|_{L^4}\|\overline{U}\|_{V_1}\|W_{A_3}\|_{L^4} \\
&+ \|W_{A_3}\|_{L^4}^2\|\overline{U}\|_{V_1} + \|\overline{U}\|_{L^4}\|\overline{W}\|_{V_2}\|W_{A_2}\|_{L^4} + \|W_{A_1}\|_{L^4}\|\overline{W}\|_{V_2}\|W_{A_2}\|_{L^4} \\
&+ \|\overline{U}\|_{L^4}\|\overline{B}\|_{V_3}\|W_{A_3}\|_{L^4} + \|W_{A_1}\|_{L^4}\|\overline{B}\|_{V_3}\|W_{A_3}\|_{L^4} \\
&+ \|\overline{B}\|_{L^4}\|\overline{B}\|_{V_3}\|W_{A_1}\|_{L^4} + \|W_{A_3}\|_{L^4}\|\overline{B}\|_{V_3}\|W_{A_1}\|_{L^4} + \chi|\overline{W}|\|\overline{U}\|_{V_1} \\
&+ \chi\|\overline{U}\|_{V_1}|W_{A_2}| - \chi|\overline{W}|^2 + 2\chi|W_{A_2}||\overline{W}| + \chi\|\overline{U}\|_{V_1}|\overline{W}| + \chi\|\overline{W}\|_{V_2}|W_{A_1}| \\
\leq& (\frac{\mu}{2}+\chi)\|\overline{U}\|_{V_1}^2 + \frac{\min\{\chi,\gamma\}}{2}\|\overline{W}\|_{V_2}^2 + \frac{\nu}{2}\|\overline{B}\|_{V_3}^2 \\
&+ c(|(\overline{U}, \overline{W}, \overline{B})|^2 + 1)(\|(W_{A_1}, W_{A_2}, W_{A_3})\|_{L^4}^4 + |(W_{A_1}, W_{A_2})|^2)
\end{aligned}$$

where we used Hölder's inequalities, (3.7) and Young's inequalities. Subtracting $(\frac{\mu}{2}+\chi)\|\overline{U}\|_{V_1}^2 + \frac{\min\{\chi,\gamma\}}{2}\|\overline{W}\|_{V_2}^2 + \frac{\nu}{2}\|\overline{B}\|_{V_3}^2$ from both sides, we obtain for some $c_0 \geq 0$,

$$\begin{aligned}
&\partial_t |(\overline{U}, \overline{W}, \overline{B})|^2 + \mu\|\overline{U}\|_{V_1}^2 + \min\{\chi,\gamma\}\|\overline{W}\|_{V_2}^2 + \nu\|\overline{B}\|_{V_3}^2 \\
\leq& c_0(|(\overline{U}, \overline{W}, \overline{B})|^2 + 1)(\|(W_{A_1}, W_{A_2}, W_{A_3})\|_{L^4}^4 + |(W_{A_1}, W_{A_2})|^2).
\end{aligned}$$

Using the fact that $\forall\, \eta > 0$,

$$\mathbb{P}(\sup_{t\in[0,2T]} \|(W_{A_1}, W_{A_2}, W_{A_3})(t)\|_{L^4}^2 + |(W_{A_1}, W_{A_2}, W_{A_3})(t)| \leq \eta) > 0 \tag{4.7}$$

(see (6.5) [2]), we obtain on some Γ_η with $\mathbb{P}(\Gamma_\eta) > 0$,

$$\partial_t |(\overline{U}, \overline{W}, \overline{B})|^2 + \mu\|\overline{U}\|_{V_1}^2 + \min\{\chi,\gamma\}\|\overline{W}\|_{V_2}^2 + \nu\|\overline{B}\|_{V_3}^2 \leq c_0(|(\overline{U}, \overline{W}, \overline{B})|^2 + 1)\eta^2$$

for a.e. $t \in [0, 2T]$. Taking $\eta > 0$ sufficiently small so that by Poincare's inequality, there exists $\delta > 0$ such that

$$\delta|(\overline{U}, \overline{W}, \overline{B})|^2 \leq \mu\|\overline{U}\|_{V_1}^2 + \min\{\chi,\gamma\}\|\overline{W}\|_{V_2}^2 + \nu\|\overline{B}\|_{V_3}^2 - c_0|(\overline{U}, \overline{W}, \overline{B})|^2\eta^2$$

with which it follows that

$$|(\overline{U}, \overline{W}, \overline{B})(t)|^2 \leq e^{-\delta t}|(u_0, w_0, b_0)|^2 + \frac{c_0\eta^2}{\delta} \tag{4.8}$$

$\forall\, t \in [0, 2T]$ $\mathbb{P}$-a.s. on Γ_η where $\delta > 0$ is independent of T and $\mathbb{P}(\Gamma_\eta) > 0$ due to (2.1), (2.6a)-(2.6c). Thus, for any fixed $\rho_0, \rho_1 > 0$ such that $|u_0|^2, |w_0|^2, |b_0|^2 \leq \rho_0$, taking $T > 0$ large and $\eta > 0$ small so that

$$e^{-\delta T}3\rho_0 + \frac{c_0\eta^2}{\delta} < \rho_1 \tag{4.9}$$

implies by (4.8) and (4.9) that

$$\sup_{t\in[T,2T]} |(\overline{U}, \overline{W}, \overline{B})(t)|^2 \leq \sup_{t\in[T,2T]} e^{-\delta t}|(u_0, w_0, b_0)|^2 + \frac{c_0\eta^2}{\delta} < \rho_1$$

on a set Γ_η of positive probability. Taking $\eta > 0$ small enough, this implies

$$\sup_{t\in[T,2T]} |u(t)|^2 \le \rho_1, \quad \sup_{t\in[T,2T]} |w(t)|^2 \le \rho_1, \quad \sup_{t\in[T,2T]} |b(t)|^2 \le \rho_1,$$

due to (4.6) and (4.7) on a set Γ_η of positive probability. This completes the proof of Lemma 4.2. □

Lemma 4.3. *Suppose $g \in C_b(H)$ and $\|g\|_0 \le 1$. Then for any $t > 0$, there exists $\delta > 0$ such that for P_t defined in (2.11),*

$$|P_t g(x,y,z) - P_t g(x_1,y_1,z_1)| \le \frac{1}{2} \tag{4.10}$$

for any $(x,y,z),(x_1,y_1,z_1) \in H$ such that $\max\{|x|,|y|,|z|,|x_1|,|y_1|,|z_1|\} < \delta$.

Proof. We denote by $Y = (u,w,b)$ the solution to (1.2a)-(1.2c) with initial data $(x,y,z) \in H$ and by DY, the Gateaux derivative of Y:

$$DY = \begin{pmatrix} D_x u & D_x w & D_x b \\ D_y u & D_y w & D_y b \\ D_z u & D_z w & D_z b \end{pmatrix} = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_4 & \eta_5 & \eta_6 \\ \eta_7 & \eta_8 & \eta_9 \end{pmatrix} \tag{4.11}$$

where D_x, D_y, D_z are Gateaux derivatives with respect to x, y, z respectively. From (2.3a)-(2.3c) with the notations of F_1, F_2, F_3, F_4, F_5 from (3.2a), (3.2b), we have

$$\partial_t \eta_1 + A_1\eta_1 + F_1'(u)\eta_1 - F_2'(b)\eta_3 + R_1'(w)\eta_2 = 0, \tag{4.12a}$$
$$\partial_t \eta_2 + A_2\eta_2 + [D_u F_3(u,w) + D_u R_2(w,u)]\eta_1 + [D_w F_3(u,w) + D_w R_2(w,u)]\eta_2 = 0, \tag{4.12b}$$
$$\partial_t \eta_3 + A_3\eta_3 + [D_u F_4(u,b) - D_u F_5(b,u)]\eta_1 + [D_b F_4(u,b) - D_b F_5(b,u)]\eta_3 = 0, \tag{4.12c}$$
$$\eta_1(0) = 1, \eta_2(0) = 0, \eta_3(0) = 0, \tag{4.12d}$$

$$\partial_t \eta_4 + A_1\eta_4 + F_1'(u)\eta_4 - F_2'(b)\eta_6 + R_1'(w)\eta_5 = 0, \tag{4.13a}$$
$$\partial_t \eta_5 + A_2\eta_5 + [D_u F_3(u,w) + D_u R_2(w,u)]\eta_4 + [D_w F_3(u,w) + D_w R_2(w,u)]\eta_5 = 0, \tag{4.13b}$$
$$\partial_t \eta_6 + A_3\eta_6 + [D_u F_4(u,b) - D_u F_5(b,u)]\eta_4 + [D_b F_4(u,b) - D_b F_5(b,u)]\eta_6 = 0, \tag{4.13c}$$
$$\eta_4(0) = 0, \eta_5(0) = 1, \eta_6(0) = 0, \tag{4.13d}$$

$$\partial_t \eta_7 + A_1\eta_7 + F_1'(u)\eta_7 - F_2'(b)\eta_9 + R_1'(w)\eta_8 = 0, \tag{4.14a}$$
$$\partial_t \eta_8 + A_2\eta_8 + [D_u F_3(u,w) + D_u R_2(w,u)]\eta_7 + [D_w F_3(u,w) + D_w R_2(w,u)]\eta_8 = 0, \tag{4.14b}$$
$$\partial_t \eta_9 + A_3\eta_9 + [D_u F_4(u,b) - D_u F_5(b,u)]\eta_7 + [D_b F_4(u,b) - D_b F_5(b,u)]\eta_9 = 0, \tag{4.14c}$$
$$\eta_7(0) = 0, \eta_8(0) = 0, \eta_9(0) = 1. \tag{4.14d}$$

Taking L^2-inner products of (4.12a)-(4.12c) with (η_1, η_2, η_3) respectively, we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t|(\eta_1,\eta_2,\eta_3)|^2 + (\mu+\chi)\|\eta_1\|_{V_1}^2 + \min\{\chi,\gamma\}\|\eta_2\|_{V_2}^2 + \nu\|\eta_3\|_{V_3}^2 \\
&\leq -\int(\eta_1\cdot\nabla)u\cdot\eta_1 + \int(\eta_3\cdot\nabla)b\cdot\eta_1 + 2\chi\int(\nabla\times\eta_1)\cdot\eta_2 \\
&\quad -\int(\eta_1\cdot\nabla)w\cdot\eta_2 - \chi|\eta_2|^2 - \int(\eta_1\cdot\nabla)b\cdot\eta_3 + \int(\eta_3\cdot\nabla)u\cdot\eta_3 \\
&\leq \|\eta_1\|_{L^4}^2\|u\|_{V_1} + \|\eta_3\|_{L^4}\|b\|_{V_3}\|\eta_1\|_{L^4} + 2\chi\|\eta_1\|_{V_1}|\eta_2| + \|\eta_1\|_{L^4}\|w\|_{V_2}\|\eta_2\|_{L^4} - \chi|\eta_2|^2 \\
&\quad + \|\eta_1\|_{L^4}\|b\|_{V_3}\|\eta_3\|_{L^4} + \|\eta_3\|_{L^4}^2\|u\|_{V_1} \\
&\leq c(|\eta_1|\|\eta_1\|_{V_1}\|u\|_{V_1} + |\eta_3|^{\frac{1}{2}}\|\eta_3\|_{V_3}^{\frac{1}{2}}\|b\|_{V_3}|\eta_1|^{\frac{1}{2}}\|\eta_1\|_{V_1}^{\frac{1}{2}} + |\eta_1|^{\frac{1}{2}}\|\eta_1\|_{V_1}^{\frac{1}{2}}\|w\|_{V_2}|\eta_2|^{\frac{1}{2}}\|\eta_2\|_{V_2}^{\frac{1}{2}} \\
&\quad + |\eta_1|^{\frac{1}{2}}\|\eta_1\|_{V_1}^{\frac{1}{2}}\|b\|_{V_3}|\eta_3|^{\frac{1}{2}}\|\eta_3\|_{V_3}^{\frac{1}{2}} + |\eta_3|\|\eta_3\|_{V_3}\|u\|_{V_1}) + \chi\|\eta_1\|_{V_1}^2 \\
&\leq (\frac{\mu}{2}+\chi)\|\eta_1\|_{V_1}^2 + \frac{\min\{\chi,\gamma\}}{2}\|\eta_2\|_{V_2}^2 + \frac{\nu}{2}\|\eta_3\|_{V_3}^2 + c|(\eta_1,\eta_2,\eta_3)|^2\|(u,w,b)\|_V^2
\end{aligned}$$

where e.g. η_{1i} is the ith component of $\eta_1 = D_x u$ and we used that in particular,

$$\int(u\cdot\nabla)\eta_3\cdot\eta_3 = 0, \quad \int(b\cdot\nabla)\eta_3\cdot\eta_1 + \int(b\cdot\nabla)\eta_1\cdot\eta_3 = 0,$$

Hölder's inequalities, (3.7) and Young's inequalities. Thus, after subtracting $(\frac{\mu}{2} + \chi)\|\eta_1\|_{V_1}^2 + \frac{\min\{\chi,\gamma\}}{2}\|\eta_2\|_{V_2}^2 + \frac{\nu}{2}\|\eta_3\|_{V_3}^2$ from both sides, we obtain

$$|(\eta_1,\eta_2,\eta_3)(t)|^2 + \int_0^t \|(\eta_1,\eta_2,\eta_3)\|_V^2 ds \lesssim e^{c\int_0^t \|(u,w,b)\|_V^2 ds} \tag{4.15}$$

as $|(\eta_1,\eta_2,\eta_3)(0)|^2 = 1$. Similarly, we obtain

$$|(\eta_4,\eta_5,\eta_6)(t)|^2 + \int_0^t \|(\eta_4,\eta_5,\eta_6)\|_V^2 ds \lesssim e^{c\int_0^t \|(u,w,b)\|_V^2 ds}, \tag{4.16}$$

$$|(\eta_7,\eta_8,\eta_9)(t)|^2 + \int_0^t \|(\eta_7,\eta_8,\eta_9)\|_V^2 ds \lesssim e^{c\int_0^t \|(u,w,b)\|_V^2 ds}. \tag{4.17}$$

In the rest of the computations, the distinct structure of the MMPF system in comparison to the MHD system is rarely used; hence, the computation in [2] goes through via a straight-forward modification; we sketch it for completeness. To estimate $\mathbb{E}[g(Y(t,x,y,z)) - g(Y(t,x_1,y_1,z_1))]$, we let Ψ_k be a smooth function with compact support that satisfies

$$\Psi_k(r)\begin{cases} = 1 & \text{if } r \in [0,k], \\ \in [0,1] & \text{if } r \in [k,2k], \\ = 0 & \text{if } r \in [2k,\infty], \end{cases} \tag{4.18}$$

with k to be determined subsequently. We write

$$\begin{aligned}
&\mathbb{E}[g(Y(t,x,y,z)) - g(Y(t,x_1,y_1,z_1))] \qquad (4.19)\\
=&\mathbb{E}[g(Y(t,x,y,z))\Psi_k(\int_0^t \|Y(s,x,y,z)\|_V^2 ds)]\\
&-\mathbb{E}[g(Y(t,x_1,y_1,z_1))\Psi_k(\int_0^t \|Y(s,x_1,y_1,z_1)\|_V^2 ds)]\\
&+\mathbb{E}[g(Y(t,x,y,z))\left(1-\Psi_k(\int_0^t \|Y(s,x,y,z)\|_V^2 ds)\right)]\\
&-\mathbb{E}[g(Y(t,x_1,y_1,z_1))\left(1-\Psi_k(\int_0^t \|Y(s,x_1,y_1,z_1)\|_V^2 ds)\right)]\\
\triangleq&\mathcal{H}_1(t)+\mathcal{H}_2(t)+\mathcal{H}_3(t)
\end{aligned}$$

where

$$\begin{aligned}
&|\mathcal{H}_2(t)|+|\mathcal{H}_3(t)| \qquad (4.20)\\
\leq&\|g\|_0 \frac{1}{2k\min\{\mu,\chi,\gamma,\nu\}}\left(|(x,y,z)|^2+|(x_1,y_1,z_1)|^2+2tTr(Q_1+Q_2+Q_3)\right)
\end{aligned}$$

due to (4.18), Chebyshev's inequality and (4.2). To estimate $\mathcal{H}_1(t)$, we denote $x_\lambda \triangleq \lambda x + (1-\lambda)x_1, y_\lambda \triangleq \lambda y + (1-\lambda)y_1, z_\lambda \triangleq \lambda z + (1-\lambda)z_1$ and rewrite

$$\mathcal{H}_1(t) = \int_0^1 \frac{d}{d\lambda}\mathbb{E}[g(Y(t,x_\lambda,y_\lambda,z_\lambda))\Psi_k(\int_0^t \|Y(s,x_\lambda,y_\lambda,z_\lambda)\|_V^2 ds)]d\lambda. \qquad (4.21)$$

Now we denote $h \triangleq (x-x_1, y-y_1, z-z_1)$. Then, using Bismut-Elworthy-Li formula (see e.g. [10]), that $\frac{d}{d\lambda}(x_\lambda,y_\lambda,z_\lambda) = h$, letting

$$\sigma_{\lambda,k} \triangleq \inf\{t>0 : \int_0^t \|Y(s,x_\lambda,y_\lambda,z_\lambda)\|_V^2 ds \geq 2k\} \qquad (4.22)$$

so that for k large, $t\wedge\sigma_{\lambda,k} = t$, we can compute

$$\begin{aligned}
|\mathcal{H}_1(t\wedge\sigma_{\lambda,k})| \lesssim& \int_0^1 \frac{1}{t\wedge\sigma_{\lambda,k}}\left(\mathbb{E}[\int_0^{t\wedge\sigma_{\lambda,k}} |Q^{-\frac{1}{2}}DY(s,x_\lambda,y_\lambda,z_\lambda)\cdot h|^2 ds]\right)^{\frac{1}{2}} d\lambda\\
&+\|\Psi_k'\|_0 \int_0^1 \left(\mathbb{E}[\int_0^{t\wedge\sigma_{\lambda,k}} \|Y(s,x_\lambda,y_\lambda,z_\lambda)\|_V^2 ds]\right)^{\frac{1}{2}}\\
&\times\left(\mathbb{E}[\int_0^{t\wedge\sigma_{\lambda,k}} \|DY(s,x_\lambda,y_\lambda,z_\lambda)\cdot h\|_V^2 ds]\right)^{\frac{1}{2}} d\lambda
\end{aligned}$$

due to Burkholder-Davis-Gundy and Hölder's inequalities. Thus, using

$$\int_0^t |Q^{-\frac{1}{2}}DY\cdot h|^2 ds \lesssim |h|^2 \qquad (4.23)$$

which is due to (2.4), (4.11), (4.15), (4.16), (4.17) and (4.2), we obtain

$$|\mathcal{H}_1(t\wedge\sigma_{\lambda,k})| \lesssim \int_0^1 \frac{1}{t\wedge\sigma_{\lambda,k}}|h|d\lambda + \|\Psi_k'\|_0 \left(\mathbb{E}[\int_0^{t\wedge\sigma_{\lambda,k}} \|DY(s,x_\lambda,y_\lambda,z_\lambda)\cdot h\|_V^2 ds]\right)^{\frac{1}{2}} \times \left(|(x_\lambda,y_\lambda,z_\lambda)| + \sqrt{t\wedge\sigma_{\lambda,k}}\sqrt{Tr(Q_1+Q_2+Q_3)}\right) d\lambda.$$

Hence, together with (4.19) and (4.20), using that $\max\{|x|,|y|,|z|,|x_1|,|y_1|,|z_1|\} < \delta$, and that

$$\int_0^{t\wedge\sigma_{\lambda,k}} \|DY(s,x_\lambda,y_\lambda,z_\lambda)\cdot h\|_V^2 ds \lesssim e^{ck}\delta$$

due to (4.11), (4.15), (4.16) and (4.17), for $\delta > 0$ small and $k > 0$ appropriately chosen, we obtain

$$|\mathbb{E}[g(Y(t,x,y,z)) - g(Y(t,x_1,y_1,z_1))]| \leq |\mathcal{H}_1(t)| + |\mathcal{H}_2(t)| + |\mathcal{H}_3(t)| \leq \frac{1}{2}.$$

This completes the proof of Lemma 4.3. □

We now let

$$\tau \triangleq \inf\{t = kT, k\in\mathbb{N} : |Y(kT,x,y,z)|^2 \geq MK_1\},\ K_1 \triangleq \frac{1}{c^*}Tr(Q_1+Q_2+Q_3). \quad (4.24)$$

Then from Lemma 4.1 and the Markov property of $\{Y(kT)\}_{k\in\mathbb{N}}$, we obtain $\delta_0 > 0, c_0 > 0$ such that

$$\mathbb{E}[e^{\delta_0\tau}] \leq c_0(1 + |(x,y,z)|^2) \quad \forall\ (x,y,z)\in H \quad (4.25)$$

(see (6.15) [2], also [35]).

Lemma 4.4. *For P_t defined by (2.11) for the system (1.2a)-(1.2c), for any (x,y,z), $(x_1,y_1,z_1)\in H$, there exists $\xi > 0$ such that for any $T > 0, k\in\mathbb{N}$ and any $g\in C_b(H)$,*

$$|P_{kT}g(x,y,z) - P_{kT}g(x_1,y_1,z_1)| \lesssim \|g\|_0 e^{-\xi kT}(1 + |(x,y,z,x_1,y_1,z_1)|^2).$$

Proof. We fix $T > 0, \delta > 0$ as in Lemma 4.3. We let $v \triangleq (x,y,z)$, $v_1 \triangleq (x_1,y_1,z_1)$, $Y(t,v) \triangleq (u,w,b)(t,v)$, $Y(t,v_1) \triangleq (u,w,b)(t,v_1)$. Then, $\forall\ v,v_1\in B_\delta$, a ball of radius $\delta > 0$ in H, by (2.11) and Lemma 4.3, we obtain

$$\begin{aligned}&\|\Pi_T(Y(v,\cdot)) - \Pi_T(Y(v_1,\cdot))\|_{TV} \quad (4.26)\\ \triangleq &\sup_{\|g\|_0\leq 1, g\in C_b^1(H)} |\mathbb{E}[g(Y(T,v))] - \mathbb{E}[g(Y(T,v_1))]| \leq \frac{1}{2}\end{aligned}$$

where we recall that $\Pi_t(Y(x,y,z,\cdot))$ is the law of the process $(u,w,b)(t)$ with initial data (x,y,z). It can be shown (see Appendix [2]) using Kantorovich-Rubinstein Theorem (e.g. pg. 34 [39]) that there exists a maximal coupling $(Z_1(v,v_1), Z_2(v,v_1))$ of $(Y(T,v), Y(T,v_1))$ which depend measurably on v, v_1; that is,

$$\mathcal{D}(Z_1(v,v_1)) = \Pi_T(Y(v,\cdot)), \quad \mathcal{D}(Z_2(v,v_1)) = \Pi_T(Y(v_1,\cdot)), \quad (4.27)$$

$$\mathbb{P}(Z_1(v,v_1) \neq Z_2(v,v_1)) = \|\Pi_T(Y(v,\cdot)) - \Pi_T(Y(v_1,\cdot))\|_{TV} \quad (4.28)$$

(see [28, 29] for definition of coupling) and hence by (4.26),

$$\mathbb{P}(Z_1(v,v_1) \neq Z_2(v,v_1)) \leq \frac{1}{2}. \tag{4.29}$$

Moreover, by (2.11) and (4.27),

$$P_T g(v) - P_T g(v_1) = \mathbb{E}[g(Z_1(v,v_1)) - g(Z_2(v,v_1))]. \tag{4.30}$$

We define

$$(X_1^1(v,v_1), X_2^1(v,v_1)) \triangleq \begin{cases} (Z_1^1(v,v_1), Z_2^1(v,v_1)) & \text{if } v, v_1 \in B_\delta, v \neq v_1, \\ (Y(T,v), Y(T,v_1)) & \text{if } v = v_1, \\ (Y(T,v), \tilde{Y}(T,v_1)) & \text{otherwise}, \end{cases} \tag{4.31}$$

where $\tilde{Y}(T,v)$ is a solution to the system (1.2a)-(1.2c) with a Wiener process $\tilde{W} \triangleq (\tilde{W}_1, \tilde{W}_2, \tilde{W}_3)$ independent of $W \triangleq (W_1, W_2, W_3)$. We define recursively,

$$\begin{aligned} &(X_1^{k+1}(v,v_1), X_2^{k+1}(v,v_1)) \\ =&(X_1^1(X_1^k(v,v_1), X_2^k(v,v_1)), X_2^1(X_1^k(v,v_1), X_2^k(v,v_1))) \end{aligned} \tag{4.32}$$

so that $\forall\, k \in \mathbb{N}, (X_1^k(v,v_1), X_2^k(v,v_1))$ is a coupling of $(Y(kT,v), Y(kT,v_1))$. Thus,

$$|\mathbb{E}[g(Y(kT,v))] - \mathbb{E}[g(Y(kT,v_1))]| \leq 2\|g\|_0 \mathbb{P}(X_1^k(v,v_1) \neq X_2^k(v,v_1)). \tag{4.33}$$

We furthermore define

$$\mathcal{K}_1 \triangleq \inf\{k \in \mathbb{N} : X_1^k, X_2^k \in B_{\rho_0}\}, \quad \rho_0 \triangleq MK_1, \tag{4.34}$$

$$\mathcal{K}_{l+1} \triangleq \inf\{k > \mathcal{K}_l : X_1^k, X_2^k \in B_{\rho_0}\}. \tag{4.35}$$

Due to (4.32), (4.3) and (4.24), it follows that

$$\mathbb{E}[|X_1^{k+1}|^2 | \mathcal{F}_{kT}] \lesssim e^{-c^* T}|X_1^k|^2 + K_1, \tag{4.36}$$

and similarly

$$\mathbb{E}[|X_2^{k+1}|^2 | \mathcal{F}_{kT}] \lesssim e^{-c^* T}|X_2^k|^2 + K_1. \tag{4.37}$$

By (4.24), (4.25), (4.34), (4.35), we obtain $\forall\, \alpha < \frac{1}{2}c^* T$,

$$\mathbb{E}[e^{\alpha \mathcal{K}_1 T}] \leq c(\alpha, T)(1 + |(v,v_1)|^2). \tag{4.38}$$

Moreover, by Lemma 4.2, (4.32), we obtain $K_0 \in \mathbb{N}$, $\alpha(\rho_0, \delta) > 0$ such that

$$\mathbb{P}(|X_1^{K_0 + \mathcal{K}_1}| \leq \delta, |X_2^{K_0 + \mathcal{K}_1}| \leq \delta) \geq \alpha(\rho_0, \delta). \tag{4.39}$$

By the strong Markov property, (4.38) and (4.32) we obtain

$$\mathbb{E}[e^{\alpha(\mathcal{K}_{l+1} - \mathcal{K}_l)T} | \mathcal{F}_{\mathcal{K}_l T}] \leq c(\alpha, T)(1 + |X_1^{\mathcal{K}_l}|^2 + |X_2^{\mathcal{K}_l}|^2). \tag{4.40}$$

Thus, there exists $K_2 > 0$ such that

$$\mathbb{E}[e^{\alpha \mathcal{K}_l T}] \leq K_2^l \mathbb{E}[e^{\alpha \mathcal{K}_1 T}] \leq c(\alpha, T) K_2^l (1 + |(v,v_1)|^2) \tag{4.41}$$

by (4.38). Now we define recursively a sequence of stopping times to enter B_δ,

$$\tilde{\mathcal{K}}_{l+1} \triangleq \inf\{k \geq \tilde{\mathcal{K}}_l : |X_1^k| \leq \delta, |X_2^k| \leq \delta\}.$$

As we found $\alpha(\rho_0, \delta) > 0$, we see that there exists $\tilde{\xi} > 0, K_3 > 0$ such that

$$\mathbb{E}[e^{\tilde{\xi} \tilde{\mathcal{K}}_l T}] \leq K_3^l (1 + |(v,v_1)|^2). \tag{4.42}$$

We set $\tilde{l}_0 \triangleq \inf\{l \in \mathbb{N} : X_1^{\tilde{\mathcal{K}}_l+1} = X_2^{\tilde{\mathcal{K}}_{l+1}}\}$. Again, due to (4.32) and (4.26) we get

$$\mathbb{P}(X_1^{\tilde{\mathcal{K}}_l} \neq X_2^{\tilde{\mathcal{K}}_l}) \leq \frac{1}{2}. \tag{4.43}$$

Moreover, $\mathbb{P}(\tilde{l}_0 > l) \leq 2^{-l}$ and hence

$$\mathbb{P}(X_1^k \neq X_2^k) \lesssim e^{-\tilde{\xi}kT}(1 + |(v, v_1)|^2) \tag{4.44}$$

by Chebyshev's inequality. Therefore,

$$|P_{kT}g(v) - P_{kT}g(v_1)| \lesssim \|g\|_0 e^{-\tilde{\xi}kT}(1 + |(x, y, z, x_1, y_1, z_2)|^2)$$

by (2.11), (4.32) and (4.44). This completes the proof of Lemma 4.4. □

The uniqueness of the invariant measure as claimed in the statement of Theorem 2.3 follows from the asymptotic behavior of Lemma 4.4.

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