THE PRODUCT OF DISTRIBUTIONS AND WHITE NOISE DISTRIBUTION-VALUED STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

In this paper we introduce a new locally convex space of distributions, as a generalization of the space from [12], in which we have the product of any distributions as a series expansion. Then we discuss higher powers of the complex white noise on the space consisting of distributions without any renormalization. We also extend the Lévy and Voltera Laplacians to operators on a locally convex space taking the completion of the set of all distribution-coefficient polynomials on distributions with respect to some topology, and give an infinite dimensional Brownian motion generated by the Lévy Laplacian with a divergent part as a distribution. Based on those results, we obtain white noise distribution-valued stochastic differential equations, for the delta distribution centered at the infinite dimensional Brownian motion and also a sum of delta distributions centered at one dimensional Brownian motions.

1. Introduction

Let $\mathcal{S}'(\mathbb{R})$ be the Schwartz space of tempered distributions. The Gross Laplacian generates an infinite dimensional Brownian motion

$$\mathbb{B}(t) = \sum_{k=1}^{\infty} B_k(t) e_k,$$

as an $\mathcal{S}'(\mathbb{R})$-valued stochastic process with a sequence $\{B_k(t)\}_{k=1}^{\infty}$ of independent one dimensional Brownian motions and an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of the real Hilbert space $L^2(\mathbb{R})$, which is contained in $\mathcal{S}(\mathbb{R})$. Obviously, we can not consider any power of $\mathbb{B}(t)$ in $\mathcal{S}'(\mathbb{R})$. However, in the paper [12] we introduced some closed subspace H of a complex Hilbert space $L^2([0,1])$ based on functions $e_n(u) := e^{2\pi i n u}$, $n = 0, 1, 2, 3, \ldots$, and constructed a Gel'fand triple $E \subset H \subset E^*$ with a nuclear space E and its dual space E^*, of which we can compute the usual product of any elements as series expansions. We introduced an E^*-valued Brownian motion $\{\mathbf{B}(t); \ t \geq 0\}$ and discussed the Itô formula for powers of the Brownian motion in the paper [12].

In this paper we introduce a new locally convex space E_∞^* of distributions, based on all of functions consisting of a basis for $L^2([0,1])$:

$$e_n(u) := e^{2\pi i n u}, \ n \in \mathbb{Z},$$

in which we have the product of any distributions naturally defined as a series expansion. This is a generalization of the construction of E^* from [12]. Based on this space, we can construct a space of white noise distributions in which we have powers of white noise without any renormalization. We also discuss the complex version of the Gross, Lévy and Volterra Laplacians, and an important formula among those Laplacians, with the divergent part $\delta(0)$ being realized as a distribution in E_∞^*. Moreover, we define an infinite dimensional Brownian motion $\{\mathbf{B}_m(t) \equiv \sum_{k=-m}^\infty B_k(t) e_k; \ t \geq 0\}$ on E_∞^* for each $m \in \mathbb{N}$ and obtain stochastic differential equations induced from a delta distribution $\delta_{\mathbf{B}(t)}$ of $\mathbf{B}(t)$ and a series $\sum_{k=-m}^\infty \delta_{B_k(t)}$ of delta distributions $\delta_{B_k(t)}$, $k = 1, 2, 3, \ldots$, on some spaces of white noise distributions.

The paper is organized as follows.

In section 2 we introduce some spaces E_∞^* of distributions, of which any product of elements is included in itself.

In Section 3 we discuss powers of the delta distribution in E^* and give a representation of higher powers of the delta distribution by derivatives of the delta distribution.

In Section 4 we construct a space of white noise distributions in which the power of white noise can be defined without any renormalization connecting to the space E_∞^*.

In Section 5 we generalize the definition of LV functionals as a domain of the Lévy and Volterra Laplacians from [8] and define those Laplacians on the generalized domain with realizing $\delta(0) = \sum_{k=-\infty}^\infty e_{2k}$ as a distribution. Moreover, we introduce a set $\text{Poly}(E_\infty^*)$ of all polynomials on E_∞^* with E_∞^*-coefficients and taking the completion of $\text{Poly}(E_\infty^*)$ with respect to some topology, we define a locally convex space $\mathbb{D}_{\infty, -\infty}$ on which the Lévy Laplacian is extended to operators as the generator of an infinite dimensional stochastic process consisting of an infinite dimensional Brownian motion and the inverse of the distribution $\delta_m(0) = \sum_{k=-m}^\infty e_{2k}$ for every $m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

In Section 6 we introduce some class $\mathbb{A}_\infty$ of entire functions on E_∞^* as an analytic version of the space of the test white noise functionals, and define an $\mathbb{A}_\infty^*$-valued stochastic process $X = \{X(t); \ 0 \leq t < 0\}$ as the delta distribution centered at the infinite dimensional Brownian motion. Then we obtain a stochastic differential equation of which X is a solution.

In Section 7 we introduce another class $\mathbb{A}_\infty^\star$ of entire functions on E_∞^* smaller than $\mathbb{A}_\infty$, in order to define a sum of delta distributions centered at one dimensional Brownian motions as an $(\mathbb{A}_\infty^\star)^*$-valued stochastic process. We also obtain a stochastic differential equation from the stochastic process as a solution. This is a different approach to get the infinite dimensional stochastic equation from the sum of delta distributions in [5].

Finally, in the last section we discuss a stochastic differential equation which has differential operators with a random direction, as concluding remarks.

2. Basic Spaces of Distributions

Let $e_n(u) := e^{2\pi i n u}$ for $n \in \mathbb{Z}$. Then the set

$$\{e_n;\ n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2([0,1])$. For any $n \in \mathbb{Z}$ let H_n be the closed subspace of $L^2([0,1])$ generated by the functions $e_k, k \geq -n, k \in \mathbb{Z}$. Then for any $n \in \mathbb{Z}$ and sequence $\{\ell_k\}_{k=-\infty}^{\infty}$ satisfying

$$1 = \ell_0 < \ell_1 < \ell_2 < \ell_3 < \cdots, \quad \sum_{k=0}^{\infty} \ell_k^{-2} < \infty,$$

and

$$\ell_{-k} = \ell_k, \ k \in \mathbb{N},$$

and introducing a densely defined selfadjoint operator A_n on H_n by

$$A_n \xi = \sum_{k=-n}^{\infty} \ell_k \alpha_k e_k, \ \xi = \sum_{k=-n}^{\infty} \alpha_k e_k \in H_n,$$

we can define norms $|\cdot|_{n,p}, \ p \in \mathbb{R}$ and spaces $E_{n,p}$ by

$$|\xi|_{n,p}^2 \ := \ |A_n^p \xi|_0^2 = \sum_{k=-n}^{\infty} \ell_k^{2p} |\langle \xi, e_k \rangle_*|^2;$$

$$E_{n,p} \ := \ \{\xi \in H_n;\ |\xi|_{n,p} < \infty\} \text{ for } p \geq 0;$$

$$E_{n,p} \ := \ \text{completion of } H_n \text{ with respect to } |\cdot|_{n,p} \text{ for } p < 0,$$

where $|\cdot|_0$ is the norm of $L^2([0,1])$ and $\langle \cdot, \cdot \rangle_*$ is the conjugate bilinear form of E_n^* and E_n. Here E_n and E_n^* are the projective limit space of $E_{n,p}, \ p \in \mathbb{R}$ and the inductive limit space of $E_{n,p}, \ p \in \mathbb{R}$, respectively. We note that $H_n = E_{n,0}$ for $n \in \mathbb{Z}$.

We also assume the following conditions:

(1)   For any $\alpha > 0$, there exists $\beta > 0$ such that $(k+1)^\alpha \leq \ell_k^\beta, \ k = 1, 2, 3, \ldots$.
(2) $\rho \equiv \sup_{k \in \mathbb{N}_0} \ell_{k+1}/\ell_k < \infty$.
(3)   For any $a > 0$, there exists $p \geq 1$ such that $\sum_{k=0}^{\infty} \ell_k^{-2p} \rho^{ak} < \infty$.

For any $n \in \mathbb{N}_0$ and $a > 1$, the sequence $\{(|k|+1)^n a^{|k|}\}_{k=-\infty}^{\infty}$ is an example satisfying the above three conditions.

Define spaces E_∞, H_∞ and E_∞^* by inductive limit spaces of $E_n, \ n \in \mathbb{Z}$, $H_n, \ n \in \mathbb{Z}$ and $E_n^*, \ n \in \mathbb{Z}$, respectively.

We can get the product of distributions in E_∞^* as follows:

$$x \cdot y = \sum_{j=-n}^{\infty} \overline{\langle x, e_j \rangle_*} e_j \cdot \sum_{j=-n}^{\infty} \overline{\langle y, e_j \rangle_*} e_j = \sum_{j=-2n}^{\infty} \sum_{k=-n}^{j+n} \overline{\langle x, e_k \rangle_*} \overline{\langle y, e_{j-k} \rangle_*} e_j$$

if $x = \sum_{j=-n}^{\infty} \overline{\langle x, e_j \rangle_*} e_j \in E_n^*$ and $y = \sum_{j=-n}^{\infty} \overline{\langle y, e_j \rangle_*} e_j \in E_n^*$ for some $n \in \mathbb{N}$.

Theorem 2.1. *For any x and y in E_∞^*, the above product $x \cdot y$ is also in E_∞^*. Moreover, the product operator $\cdot : \ E_\infty^* \times E_\infty^* \to E_\infty^*$ is a continuous bilinear operator.*

Proof. Let x, y be elements of E_∞^*. Then there exist $n \in \mathbb{Z}$ and $q > 0$ such that x and y are in $E_{n,-q}$. For some $r > 0$ and any $p \geq 2q + r + 1$ we can estimate the $(-p)$-norm $|x \cdot y|_{2n,-p}$ of $x \cdot y$ as follows:

$$
\begin{aligned}
|x \cdot y|_{2n,-p}^2 &= \sum_{\nu=-2n}^{\infty} \ell_\nu^{-2p} \left| \sum_{k=-2n}^{\infty} \sum_{j=-n}^{k+n} \overline{\langle x, e_j \rangle_* \langle y, e_{k-j} \rangle_*} \delta_{\nu,k} \right|^2 \\
&= \sum_{\nu=-2n}^{\infty} \ell_\nu^{-2p} \left| \sum_{j=-n}^{\nu+n} \overline{\langle x, e_j \rangle_* \langle y, e_{\nu-j} \rangle_*} \right|^2 \\
&\leq \sum_{\nu=-2n}^{\infty} \ell_\nu^{-2p} \sum_{j=-n}^{\nu+n} |\langle x, e_j \rangle_*|^2 \sum_{j=1}^{\nu+n} |\langle y, e_{\nu-j} \rangle_*|^2 \\
&\leq \sum_{\nu=-2n}^{\infty} \ell_\nu^{-2p} \sum_{j=-n}^{\nu+n} |e_j|_{n,q}^2 |x|_{n,-q}^2 \sum_{j=-n}^{\nu+n} |e_{\nu-j}|_{n,q}^2 |y|_{n,-q}^2 \\
&\leq \sum_{\nu=-2n}^{\infty} \ell_\nu^{-2p} \sum_{j=-n}^{\nu+n} \ell_j^{2q} \sum_{j=-n}^{\nu+n} \ell_{\nu-j}^{2q} |x|_{n,-q}^2 |y|_{n,-q}^2 \\
&\leq \sum_{\nu=-2n}^{-1} \ell_\nu^{-2p} \sum_{j=-n}^{\nu+n} \ell_j^{2q} \sum_{j=-n}^{\nu+n} \ell_{\nu-j}^{2q} |x|_{n,-q}^2 |y|_{n,-q}^2 \\
&\quad + \sum_{\nu=0}^{\infty} \ell_\nu^{-2p} \sum_{j=-n}^{\nu+n} \ell_j^{2q} \sum_{j=-n}^{\nu+n} \ell_{\nu-j}^{2q} |x|_{n,-q}^2 |y|_{n,-q}^2 \\
&\leq \sum_{\nu=-2n}^{-1} \ell_\nu^{-2p} (\nu + 2n + 1)^2 \ell_{-n}^{4q} |x|_{n,-q}^2 |y|_{n,-q}^2 \\
&\quad + \sum_{\nu=0}^{\infty} \ell_\nu^{-2p} (\nu + 2n + 1)^2 \ell_{\nu+n}^{4q} |x|_{n,-q}^2 |y|_{n,-q}^2 \\
&\leq \sum_{\nu=-2n}^{-1} \ell_\nu^{-2p} \ell_\nu^{2r} \ell_n^{4q} |x|_{n,-q}^2 |y|_{n,-q}^2 \\
&\quad + \sum_{\nu=0}^{\infty} \ell_\nu^{-2p} \ell_{\nu+2n}^{2r} \ell_{\nu+2n}^{4q} |x|_{n,-q}^2 |y|_{n,-q}^2 .
\end{aligned}
$$

Since conditions (1) and (2) on $\{\ell_j\}$, there exists a positive constant M_n depending only on n such that

$$
|x \cdot y|_{2n,-p}^2 \leq M_n \rho^{4n(r+2q)} |x|_{n,-q}^2 |y|_{n,-q}^2. \tag{2.1}
$$

This implies $x \cdot y \in E_{2n}^*$. The bilinearity follows from properties of the conjugate bilinear form between E_∞^* and E_∞.

Suppose that $x_\ell \to x_0$, $y_\ell \to y_0$ in E_∞^* as $\ell \to \infty$. Then there exist $n \in \mathbb{Z}$ and $q > 0$ such that $|x_\ell - x_0|_{n,-q} \to 0$, $|y_\ell - y_0|_{n,-q} \to 0$. For $p \geq 2q + r + 1$ we can estimate the $(-p)$-norm $|x_\ell \cdot y_\ell - x_0 \cdot y_0|_{2n,-p}$ for each $\ell \in \mathbb{N}$ as follows:

$$|x_\ell \cdot y_\ell - x_0 \cdot y_0|_{2n,-p}$$
$$= |(x_\ell - x_0) \cdot y_\ell - x_0 \cdot (y_\ell - y_0)|_{2n,-p}$$
$$\leq |(x_\ell - x_0) \cdot y_\ell|_{2n,-p} + |x_0 \cdot (y_\ell - y_0)|_{2n,-p}$$
$$\leq M_n \rho^{4n(r+2q)}(|x_\ell - x_0|_{n,-q}|y_\ell|_{n,-q} + |x_0|_{n,-q}|y_\ell - y_0|_{n,-q}).$$

This implies that $x_\ell \cdot y_\ell \to x_0 \cdot y_0$ in E_∞^* as $\ell \to \infty$. Thus we obtain the continuity of the product operator. $\qquad\square$

Remark 2.2. We note that the product operator is also a continuous bilinear operator from $E_n^* \times E_n^*$ into E_{2n}^* for each $n \in \mathbb{N}_0$ by the above proof of Theorem 2.1.

3. Powers of the Delta Distribution in E^*

For $t \in [0,1]$ and $n \in \mathbb{N}_0$, we define $\delta_{t,n}$ by

$$\delta_{t,n} := \sum_{k=-n}^{\infty} \overline{\langle \delta_{t,n}, e_k \rangle_*} e_k = \sum_{k=-n}^{\infty} \overline{e_k(t)} e_k. \tag{3.1}$$

Then $\delta_{t,n}$ is in E_n^* and the equality $\delta_{t,n}(\xi) = \xi(t)$, $\xi \in E_n$ holds. For any $n \in \mathbb{N}_0$ the square of the delta distribution $\delta_{t,n} \in E_n^*$ is given by

$$\delta_{t,n}^2 = \sum_{j,k=-n}^{\infty} e^{-2\pi i(j+k)t} e_{j+k}$$
$$= \sum_{\ell=-2n}^{\infty} \sum_{k=-n}^{\ell+n} e^{-2\pi i\ell t} e_\ell = \sum_{\ell=-2n}^{\infty} (\ell + 2n - 1) e^{-2\pi i\ell t} e_\ell$$

in E_{2n}^*. Since

$$\delta_{t,2n}' = 2\pi i \sum_{\ell=-2n}^{\infty} \ell e^{-2\pi i\ell t} e_\ell$$

for $\delta_{t,2n} \in E_{2n}^*$, we have

$$\delta_{t,n}^2 = \frac{1}{2\pi i} \delta_{t,2n}' - (2n-1)\delta_{t,2n}$$

in E_{2n}^*. Here we define the derivative $\partial x \equiv x'$ of the distribution $x \in E_m^*, m \in \mathbb{N}_0$, by

$$\partial x := \sum_{\ell=-m}^{\infty} \overline{\langle x, e_\ell \rangle_*} \partial e_\ell \left(= 2\pi i \sum_{\ell=-m}^{\infty} \ell \overline{\langle x, e_\ell \rangle_*} e_\ell \right).$$

For any $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, the m-th power of $\delta_{t,n} \in E_n^*$ is given by

$$\delta_{t,n}^m = \sum_{k_1,\ldots,k_m=-n}^{\infty} \overline{e_{k_1+\cdots+k_m}(t)} e_{k_1+\cdots+k_m} = \sum_{k=-mn}^{\infty} \sum_{k_1+\cdots+k_m=k} \overline{e_k(t)} e_k$$
$$= \sum_{k=-mn}^{\infty} \binom{k+mn+m-1}{m-1} \overline{e_k(t)} e_k \tag{3.2}$$

in E^*_{mn}. In general we have the following:

Proposition 3.1. (cf. [12])   *For any $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $t \in [0,1]$, we have $\delta_{t,n} \in E^*_\infty$ and the equality*

$$\delta^m_{t,n} = \frac{1}{(m-1)!} \sum_{j=0}^{m-1} \frac{1}{(2\pi i)^{m-j-1}} c_j \delta^{(m-j-1)}_{t,(m-j)n} \qquad (3.3)$$

holds in E^*_∞, where constants c_j, $j = 0,1,2,\ldots,m$ are determined by the equality

$$(k+mn+1)(k+mn+2)\cdots(k+mn+m-1) = \sum_{j=0}^{m-1} c_j k^{m-j-1}. \qquad (3.4)$$

We can define δ_t by $\delta_t = \sum_{k=-\infty}^{\infty} \overline{e_k(t)} e_k$ as an element of E^*_∞. Then we note that $\delta_t = \delta_{t,n}$ on E_n for each $n \in \mathbb{N}_0$. Therefore, we can write the equation (3.3) by

$$\delta^m_t = \frac{1}{(m-1)!} \sum_{j=0}^{m-1} \frac{1}{(2\pi i)^{m-j-1}} c_j \delta^{(m-j-1)}_t.$$

Remark 3.2. Since the delta distribution $\delta_{t,n}$ has an expansion given by

$$\delta_{t,n} = \sum_{k=-n}^{\infty} \overline{e_k(t)} e_k,$$

the formulas

$$\int_{[0,1]} \delta_{t,n} dt = e_0 \in E^*_n \subset E^*_\infty$$

and

$$\int_{[0,1]} \delta^2_{t,n} dt = (2n+1) e_0 \in E^*_n \subset E^*_\infty$$

hold in the distribution sense.

4. Powers of White Noise

Let $e_{\mathbb{R},0}$, $e_{\mathbb{R},2k}$, $e_{\mathbb{R},2k-1}$, $k = 1,2,3,\ldots$ be functions on $[0,1]$ given by

$$e_{\mathbb{R},0}(u) = 1, \ e_{\mathbb{R},2k}(u) := \sqrt{2}\sin(2k\pi u), \ e_{\mathbb{R},2k-1}(u) := \sqrt{2}\cos(2k\pi u),$$

$$u \in [0,1], \ k = 1,2,3,\ldots,$$

which form an orthonormal basis for $L^2_\mathbb{R}([0,1])$. Then we can construct spaces $E_{n,\mathbb{R}}$, $E^*_{n,\mathbb{R}}$ using the sequence $\{\ell_k\}$ and an operator $A_{n,\mathbb{R}}$ given by

$$A_\mathbb{R}\xi = \sum_{k=0}^{2n} \ell_k \alpha_k e_{\mathbb{R},k}, \ \xi = \sum_{k=0}^{\infty} \alpha_k e_{\mathbb{R},k} \in L^2_\mathbb{R}([0,1])$$

as in Section 2.

By the Bochner-Minlos theorem for any $n \in \mathbb{Z}$ and $\sigma > 0$, there exists a probability measure $\mu_{n,\sigma}$ on the Borel field $\mathcal{B}(E^*_{n,\mathbb{R}})$ of $E^*_{n,\mathbb{R}}$ such that

$$\int_{E^*_{n,\mathbb{R}}} e^{i\langle x,\xi\rangle_n} d\mu_{n,\sigma}(x) = e^{-\frac{\sigma^2}{2}\langle \xi,\xi\rangle_n}$$

holds for all $\xi \in E_{n,\mathbb{R}}$, where $\langle \cdot, \cdot \rangle_n$ is the canonical bilinear form on $E_{n,\mathbb{R}}^* \times E_{n,\mathbb{R}}$. For any $n \in \mathbb{N}$, we define a projection p_n from $E_{\infty,\mathbb{R}}^*$ into $E_{n,\mathbb{R}}^*$ by

$$p_n(x) := \sum_{k=-n}^{\infty} \overline{\langle x, e_{\mathbb{R},k} \rangle_n} e_{\mathbb{R},k}, \ x \in E_{\infty,\mathbb{R}}^*.$$

Then the canonical bilinear form $\langle \cdot, \cdot \rangle$ on $E_{\infty,\mathbb{R}}^* \times E_{\infty,\mathbb{R}}$ is given by

$$\langle x, \xi \rangle := \lim_{n \to \infty} \langle p_n(x), p_n(\xi) \rangle_n, \ x \in E_{\infty,\mathbb{R}}^*, \ \xi \in E_{\infty,\mathbb{R}}.$$

We define a probability measure $\mu_{\infty,\sigma}$ on the Borel field $\mathcal{B}(E_{\infty,\mathbb{R}}^*)$ of $E_{\infty,\mathbb{R}}^*$ by

$$\mu_{\infty,\sigma}(B) := \sum_{n=1}^{\infty} \mu_{n,\sigma}(B \cap (E_{n,\mathbb{R}}^* \setminus E_{n-1,\mathbb{R}}^*))$$

for any $B \in \mathcal{B}(E_{\infty,\mathbb{R}}^*)$, where $E_{0,\mathbb{R}}^* = \phi$. Then

$$\int_{E_{\infty,\mathbb{R}}^*} e^{i\langle x, \xi \rangle} d\mu_{\infty,\sigma}(x) = e^{-\frac{\sigma^2}{2} \langle \xi, \xi \rangle}$$

holds for all $\xi \in E_{\infty,\mathbb{R}}$.

Let $\mu = \mu_{\infty,1/\sqrt{2}} \times \mu_{\infty,1/\sqrt{2}}$. Then we have a probability space (E_∞^*, μ) and the equality

$$\int_{E_\infty^*} e^{\langle \bar{z}, \xi \rangle + \langle z, \eta \rangle} d\mu(z) = e^{\langle \xi, \eta \rangle}$$

holds for $\xi, \eta \in E_\infty$.

Proposition 4.1. *For any $n \in \mathbb{N}_0$, the following equalities hold:*

$$E_n = E_{n,\mathbb{R}} + iE_{n,\mathbb{R}} + E_{-(n+1)}, \ E_n^* = E_{n,\mathbb{R}}^* + iE_{n,\mathbb{R}}^* + E_{-(n+1)}^*.$$

Consequently we have

$$E_\infty = E_{\infty,\mathbb{R}} + iE_{\infty,\mathbb{R}}, \ E_\infty^* = E_{\infty,\mathbb{R}}^* + iE_{\infty,\mathbb{R}}^*.$$

For each $n \in \mathbb{N}$, we can construct spaces $\mathcal{W}_n$ of white noise test functionals and $\mathcal{W}_n^*$ of white noise distributions with

$$\mathcal{W}_n \subset L^2(E_\infty^*, \mu_{X_n}) \subset \mathcal{W}_n^*,$$

where X_n is a continuous operator from E_∞^* into E_∞^* given by

$$X_n(x) := x^n, \ x \in E_\infty^*$$

and μ_{X_n} is a probability measure defined by

$$\mu_{X_n}(B) := \mu(X_n^{-1}(B)), \ B \in \mathcal{B}(E_\infty^*).$$

Since

$$x^n(t) = \sum_{j_1, \ldots, j_n = -m}^{\infty} \overline{\langle x, e_{j_1} \rangle_*} \cdots \overline{\langle x, e_{j_n} \rangle_*} e_{j_1 + \cdots + j_n}(t) = x(t)^n$$

for $x \in E_m^*$ and $n \in \mathbb{N}_0$, we can define $x(t)^n$ as an element of $\mathcal{W}_n^*$ with the measure μ_{X_n}.

5. The Lévy and Volterra Laplacians, and Associated Stochastic Processes

Let $\mathcal{L}(\mathcal{X}; \mathcal{Y})$ denote the set of continuous linear operators from $\mathcal{X}$ into $\mathcal{Y}$ for locally convex linear topological spaces $\mathcal{X}$ and $\mathcal{Y}$. We denote $\mathcal{L}(\mathcal{X}; \mathcal{X})$ by $\mathcal{L}(\mathcal{X})$ simply. For $p \geq 0$ let $C^p(\mathcal{X}; \mathcal{Y})$ be the set of $\mathcal{Y}$-valued C^p-functions on $\mathcal{X}$ and $C^\infty(\mathcal{X}; \mathcal{Y})$ the set of $\mathcal{Y}$-valued C^∞-functions on $\mathcal{X}$. We also denote $C^0(\mathcal{X}; \mathcal{Y})$ and $C^p(\mathcal{X}; \mathbb{C})$ simply by $C(\mathcal{X}; \mathcal{Y})$ and $C^p(\mathcal{X})$, respectively.

The Lévy Laplacian and Volterra Laplacian are introduced in [9] and [14], respectively. By Theorem 2.1 we can define an *LV functional* φ as a functional in $C^\infty(E_\infty^*)$, such that for any $x \in E_\infty^*$, its second derivative $\varphi''(x)(y, z), y, z \in E_\infty^*$ is given by the form

$$\varphi''(x)(y, z) = \varphi_L''(x)(yz) + \varphi_V''(x)(y, z), \ \ y, z \in E_\infty^*,$$

where $\varphi_L''(x) \in \mathcal{L}(E_\infty^*)$ and $\varphi_V''(x) \in \mathcal{L}(E_\infty^* \times E_\infty^*)$ that is a trace class operator of H_∞.

For any $n \in \mathbb{N}_0$, we define the operators $\mathcal{A}$ and $\mathcal{A}_n$ on $C^\infty(E_\infty^*)$ by

$$\mathcal{A}\varphi(x) = \sum_{k=-\infty}^{\infty} \varphi''(x)(e_k, e_k), \ \ \varphi \in C^\infty(E_\infty^*),$$

and

$$\mathcal{A}_n\varphi(x) = \sum_{k=-n}^{\infty} \varphi''(x)(e_k, e_k)$$

for $\varphi \in C^\infty(E_\infty^*)$. Since $\sum_{k=-\infty}^{\infty} e_k \otimes e_k \in E_\infty^* \otimes E_\infty^*$, for any $\varphi \in C^\infty(E_\infty^*)$, $\mathcal{A}_n\varphi$ exists and can be written by

$$\mathcal{A}_n\varphi(x) = \sum_{k=-n}^{\infty} \varphi''(x)(e_k \otimes e_k) = \varphi''(x)\left(\sum_{k=-n}^{\infty} e_k \otimes e_k\right), \ x \in E_\infty^*$$

for each $n \in \mathbb{N}_0$.

We also define the *Lévy Laplacian* $\Delta_L\varphi$ of $\varphi \in C^\infty(E_\infty^*)$ by

$$\Delta_L\varphi(x) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi''(x)(J(e_n), e_n), \ x \in E_\infty^*,$$

if the limit exists, where J is the conjugate operator. Then for an *LV* functional φ we have

$$\Delta_L\varphi(x) = \varphi_L''(x)(e_0).$$

Any function $\varphi \in C^\infty(E_{\infty,\mathbb{R}}^*)$ can be regarded as $\widetilde{\varphi} \in C^\infty(E_\infty^*)$ given by

$$\widetilde{\varphi}(x) = \varphi(\mathrm{Re}(x)), \ x \in E_\infty^*.$$

Therefore, the Lévy Laplacian $\Delta_L\varphi$ of φ is defined by

$$\Delta_L\varphi(x_r) := \Delta_L\widetilde{\varphi}(x_r + i0), \ x_r \in E_{\infty,\mathbb{R}}^*.$$

This means the Laplacian Δ_L acting on $C^\infty(E_\infty^*)$ is an extension of its action on $C^\infty(E_{\infty,\mathbb{R}}^*)$.

Since $\sum_{k=-\infty}^{\infty} e_{2k} \in E_\infty^*$ we denote this distribution by $\delta(0)$. This is the divergent part in the usual infinite dimensional analysis. However, we can get the part by a distribution in E_∞^*. Therefore, we can introduce an operator $\tau_{\delta(0)}$ defined on the range of the Lévy Laplacian by

$$\tau_{\delta(0)}\Delta_L\varphi(x) = \varphi_L''(x)(\delta(0)e_0).$$

The Volterra Laplacian Δ_V for an LV functional is defined as a trace of $\varphi_V''(x)$ for $x \in E_\infty^*$:

$$\Delta_V\varphi(x) := \text{trace } \varphi_V''(x)$$

$$= \sum_{k=-\infty}^{\infty} \varphi_V''(x)(e_k, e_k).$$

The Volterra Laplacian Δ_V acting on $C^\infty(E_\infty^*)$ is also an extension of its action on $C^\infty(E_{\infty,\mathbb{R}}^*)$. Then we have an interesting formula:

$$\mathcal{A}\varphi = \tau_{\delta(0)}\Delta_L\varphi + \Delta_V\varphi$$

for any LV functional φ. Since the operator $\mathcal{A}$ means the complex version of the Gross Laplacian and $\tau_{\delta(0)}$ is implied from the divergent part $\frac{1}{dx}$, this formula is important with the setup of $\tau_{\delta(0)}\Delta_L\varphi$ as an element of $\mathcal{L}(E_\infty^*)$ instead of the formal expression

$$\Delta_G = \frac{1}{dx}\Delta_L + \Delta_V.$$

Lemma 5.1. *For any $x \in E_\infty^*$, let*

$$e^x := \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!}.$$

Then $e^x \in E_\infty^*$.

Proof. We may prove that for any $x \in E_\infty^*$, there exist $m \in \mathbb{N}_0$ and $q > 0$ such that $x \in E_{m,-q}$. We can estimate the norm $|e^x|_{n,-p}$ for any $n \in \mathbb{N}_0$ and some $p \geq 1$ as follows:

$$|e^x|_{n,-p}^2 = \sum_{k=-n}^{\infty} \ell_k^{-2p} \left| \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \langle x^\nu, e_k \rangle_* \right|^2$$

$$= \sum_{k=-n}^{\infty} \ell_k^{-2p} \left| \langle 1, e_k \rangle_* + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \sum_{\substack{i_1 + \cdots + i_\nu = k \\ -m \leq i_1, \ldots, i_\nu < \infty}} \langle x, e_{i_1} \rangle_* \cdots \langle x, e_{i_\nu} \rangle_* \right|^2$$

$$
\begin{aligned}
\leq \quad & 1 + e \sum_{k=-n}^{\infty} \ell_k^{-2p} \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \sum_{\substack{i_1 + \cdots + i_\nu = k \\ -m \leq i_1, \ldots, i_\nu < \infty}} 1 \cdot \\
& \qquad \cdot \sum_{\substack{i_1 + \cdots + i_\nu = k \\ -m \leq i_1, \ldots, i_\nu < \infty}} |\langle x, e_{i_1} \rangle_*|^2 \cdots |\langle x, e_{i_\nu} \rangle_*|^2 \\
\leq \quad & 1 + e \sum_{k=-n}^{\infty} \ell_k^{-2p} \sum_{\nu=1}^{\infty} \frac{1}{\nu!} |x|_{m,-q}^{2\nu} \sum_{\substack{i_1 + \cdots + i_\nu = k \\ -m \leq i_1, \ldots, i_\nu < \infty}} 1 \cdot \\
& \qquad \cdot \sum_{\substack{i_1 + \cdots + i_\nu = k \\ -m \leq i_1, \ldots, i_\nu < \infty}} (\ell_{i_1} \cdots \ell_{i_\nu})^{2q\nu} \\
\leq \quad & 1 + e \sum_{k=-n}^{\infty} \ell_k^{-2p} \rho^{2q|k|} \sum_{\nu=1}^{\infty} \frac{1}{\nu!} |x|_{m,-q}^{2\nu} \rho^{4mq\nu} \binom{|k| + (m+1)\nu - 1}{\nu - 1}^2 .
\end{aligned}
$$

Let $a_\nu := \dbinom{|k| + (m+1)\nu - 1}{\nu - 1}^2$. Then since

$$
\begin{aligned}
\frac{a_{\nu+1}}{a_\nu} &= \left(\frac{(|k| + (m+1)\nu + m) \cdots (|k| + (m+1)\nu)}{\nu(|k| + m\nu + m) \cdots (|k| + m\nu + 1)} \right)^2 \\
&= \left(\frac{|k|}{\nu} + m + 1 \right)^2 \left(1 + \frac{\nu}{|k| + (m+1)\nu} \right)^2 \cdots \left(1 + \frac{\nu}{|k| + m\nu + 1} \right)^2 \\
&\leq 2^m \left(\frac{|k|}{\nu} + m + 1 \right)^2 ,
\end{aligned}
$$

we obtain

$$
a_{\nu+1} \leq 2^{\nu m} (m+2)^{2(\nu+1)} a_{|k|} .
$$

Since similarly the inequality

$$
a_{|k|+1} \leq (m+2)^2 2^{2(m+1)} a_{|k|}
$$

holds, we have

$$
a_{|k|} \leq ((m+2)^2 2^{2(m+1)})^{|k|-1} .
$$

Hence, there exists a constant $r > 0$ such that $a_{|k|} \leq \rho^{r|k|}$ holds. Then we have an estimation of the norm $|e^x|_{n,-p}$ by

$$
|e^x|_{n,-p}^2 \leq 1 + e \sum_{k=-n}^{\infty} \ell_k^{-2p} \rho^{2(q+r)|k|} \sum_{\nu=1}^{\infty} \frac{1}{\nu!} |x|_{m,-q}^{2\nu} \rho^{4mq\nu} 2^{2m\nu} (m+2)^{2(\nu+1)} .
$$

Thus we obtain

$$
|e^x|_{n,-p}^2 \leq C e^{K|x|_{m,-q}^2} ,
$$

for some constants $C > 0$ and $K > 0$. $\qquad \qquad \square$

Remark 5.2. By the inequality (2.1) we see that for any $m \in \mathbb{N}_0$ and $q > 0$, there exist a constant $M > 0$ and $p > 2q + 1$, such that

$$|yx|^2_{2m,-p} \leq M|y|^2_{m,-q}|x|^2_{m,-q}.$$

Therefore, by the proof of Theorem 5.1 there exist positive constants $C, K, n \in \mathbb{N}_0$ and $r \geq 1$, such that

$$|e^{yx}|^2_{n,-r} \leq Ce^{K|yx|^2_{2m,-p}} \leq Ce^{KM|y|^2_{m,-q}|x|^2_{m,-q}}.$$

We give two examples to show the calculation of operators $\mathcal{A}_n$, the Lévy and Volterra Laplacians acting on certain functions.

Example 5.3. [1]   Let $\varphi(x) = x^p(f)$, $f \in E_\infty$, $p \geq 2$. Then φ is in $C^\infty(E^*_\infty)$ and

$$\mathcal{A}_n\varphi(x) = p(p-1)x^{p-2}\sum_{k=-n}^{\infty} e_{2k}(f).$$

[2]   Let $\varphi(x) = e^{x(f)}$, $f \in E_\infty$. Then φ is in $C^\infty(E^*_\infty)$ and

$$\mathcal{A}_n\varphi(x) = \left(\sum_{k=-n}^{\infty} e_k(f)\right)^2 \varphi(x).$$

[3]   Let $\varphi(x) = e^{x^2(f)}$, $f \in E_\infty$. Then φ is in $C^\infty(E^*_\infty)$ and

$$\mathcal{A}_n\varphi(x) = \left(2\sum_{k=-n}^{\infty} e_{2k}(f) + \left(2x\sum_{k=-n}^{\infty} e_k(f)\right)^2\right)\varphi(x).$$

Example 5.4. [1]   Let $\varphi(x) = x^\alpha(f)$, $f \in E_\infty$, $\alpha \geq 2$. Then φ is in $C^\infty(E^*_\infty)$ and

$$\Delta_L\varphi(x) = \alpha(\alpha-1)x^{\alpha-2}(f), \ \Delta_V\varphi(x) = 0.$$

[2]   Let $\varphi(x) = e^{x(f)}$, $f \in E_\infty$. Then φ is in $C^\infty(E^*_\infty)$ and

$$\Delta_L\varphi(x) = 0, \ \Delta_V\varphi(x) = \sum_{k=-\infty}^{\infty} e_k(f)^2\varphi(x).$$

[3]   Let $\varphi(x) = e^x(f)$, $f \in E_\infty$. Then φ is in $C^\infty(E^*_\infty)$ and

$$\Delta_L\varphi(x) = \varphi(x), \ \Delta_V\varphi(x) = 0.$$

[4]   Let $\varphi(x) = e^{x^\alpha}(f)$, $f \in E_\infty$, $\alpha \geq 2$. Then φ is in $C^\infty(E^*_\infty)$ and

$$\Delta_L\varphi(x) = (\alpha(\alpha-1)x^{\alpha-2} + \alpha^2 x^{2(\alpha-1)})\varphi(x), \ \Delta_V\varphi(x) = 0.$$

[5]   Let $\varphi(x) = e^{iyx}(f)$, $f \in E_\infty$, $y \in E^*_\infty$. Then φ is in $C^\infty(E^*_\infty)$ and

$$\Delta_L\varphi(x) = -y^2\varphi(x), \ \Delta_V\varphi(x) = 0.$$

Remark 5.5. In the white noise analysis a normal monomial φ is defined by

$$\varphi(x) = \int_{\mathbb{R}} f(u)x^{\diamond\alpha}(u)du$$

using the Wick product $\diamond$ as a white noise distribution. This is realized as part [1] in Example 5.4 without the Wick product.

For $n \in \mathbb{N}_0$ and $p \geq 1$, let $L^2(E_\infty^*; E_{n,p}^*)$ be the set of $E_{n,p}^*$-valued functions on E_∞^* with

$$||\Phi||_{n,-p}^2 := \int_{E_\infty^*} |\Phi(x)|_{n,-p}^2 d\mu(x) < \infty.$$

Let $\mathcal{P}_k(x) := x^k$, $x \in E_\infty^*$ for $k \in \mathbb{N}_0$ and denote the set of all elements Φ expressed in the form

$$\Phi = \sum_{k=0}^n c_k \mathcal{P}_k, \ c_k \in E_\infty^*, \ k = 0, 1, 2, \ldots, n, \ n \in \mathbb{N}_0$$

by $\mathrm{Poly}(E_\infty^*; E_\infty^*)$ or simply $\mathrm{Poly}(E_\infty^*)$. This is the set of all polynomials on E_∞^* with E_∞^*-coefficients and is in the space $C^\infty(E_\infty^*; E_\infty^*)$ of C^∞-functions from E_∞^* into itself. In general let $\mathrm{Poly}(\mathcal{X}; \mathcal{Y})$ be the set of all polynomials on $\mathcal{X}$ with $\mathcal{Y}$-coefficients for locally convex linear topological spaces $\mathcal{X}$ and $\mathcal{Y}$. Then $L^2(E_\infty^*; E_{n,p}^*)$ is the Hilbert space with norm $|| \cdot ||_{L^2(E_\infty^*; E_{n,p}^*)}$ and we have

$$\mathrm{Poly}(E_\infty^*) \subset C^\infty(E_\infty^*; E_\infty^*) \subset \bigcup_{n \in \mathbb{N}_0} \bigcup_{p \geq 1} L^2(E_\infty^*; E_{n,p}^*).$$

Let $\{B_k(t); \ t \geq 0\}$ be a sequence of independent one-dimensional Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$ and set

$$\mathbf{B}_n(t) := \sum_{k=-n}^\infty B_k(t) e_k, \ t \geq 0,$$

for any $n \in \mathbb{N}_0$. Then we have the following.

Lemma 5.6. *For any $n \in \mathbb{N}_0$ and $t \geq 0$, we have $\mathbf{B}_n(t) \in E_n^*$ (a.e.).*

Proof. For any $n \in \mathbb{N}_0$ and $p \geq 1$, we can check that

$$E[|\mathbf{B}_n(t)|_{n,-p}^2] = E\left[\sum_{j=-n}^\infty \ell_j^{-2p} \left| \left\langle \sum_{k=-\infty}^\infty B_k(t) e_k, e_j \right\rangle_* \right|^2 \right]$$

$$= \sum_{j=-n}^\infty \ell_j^{-2p} E[B_k(t)^2] = t \sum_{j=-n}^\infty \ell_j^{-2p} < \infty$$

which implies the assertion. □

We can define $\mathbf{B}(t)$ by $\mathbf{B}(t) = \sum_{k=-\infty}^\infty B_k(t) e_k$ as an element of E_∞^*. Then we note that $\mathbf{B}(t) = \mathbf{B}_n(t)$ on E_n for each $n \in \mathbb{N}_0$.

From Theorem 2.1 we can define $\mathbf{B}_n(t)^m \in E_\infty^*$ for any $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $t \geq 0$. The distribution $\mathbf{B}_n(t)^m$ is given by

$$\mathbf{B}_n(t)^m = \sum_{k_1, k_2, \ldots, k_m = -mn}^\infty B_{k_1}(t) B_{k_2}(t) \cdots B_{k_m}(t) e_{k_1 + k_2 + \cdots + k_m}.$$

Remark 5.7. We can calculate that

$$E[\mathbf{B}_n(t)^{2m}] = (2m-1)!! t^m \left(\sum_{k=-n}^\infty e_{2k} \right)^m, \ E[\mathbf{B}_n(t)^{2m-1}] = 0$$

for any $m \in \mathbb{N}$. (See [12].)

Let
$$\mathcal{D}_{poly} := LS\{\Phi(\cdot)(f)|\ f \in E_\infty,\ \Phi \in \mathrm{Poly}(E_\infty^*)\},$$
where LS means the linear span. Then we have the following:

Theorem 5.8. (cf. [12]) *For any $\varphi \in \mathcal{D}_{poly}$ and $t \geq 0$, we have*
$$e^{\frac{t}{2}\mathcal{A}_n}\varphi(x) = E[\varphi(x + \mathbf{B}_n(t))],\ x \in E_\infty^*.$$
for some $n \in \mathbb{N}_0$.

Let D_y be an operator on $\mathrm{Poly}(E_\infty^*)$ defined by
$$(D_y\Phi)(\cdot) = \Phi'(\cdot)(y),\ \Phi \in \mathrm{Poly}(E_\infty^*),$$
for $y \in E_\infty^*$. Since for any $\Phi \in \mathrm{Poly}(E_\infty^*)$ there exist $n \in \mathbb{N}$ and $p > 0$, such that
$$\sum_{k=1}^\infty \sum_{\alpha_1,\dots,\alpha_k=0}^\infty \sum_{j_1,\dots,j_k=-\infty}^\infty ||D_{e_\mathbf{j}}^\alpha \Phi||_{n,-p}^2 < \infty,$$
for any $n \in \mathbb{N}$ and $p > 1$, we define a norm $|||\cdot|||_{n,-p}$ on $\mathrm{Poly}(E_\infty^*)$ by
$$|||\Phi|||_{n,-p} := \left(\sum_{k=1}^\infty \sum_{\alpha_1,\dots,\alpha_k=0}^\infty \sum_{j_1,\dots,j_k=-\infty}^\infty ||D_{e_\mathbf{j}}^\alpha \Phi||_{n,-p}^2\right)^{1/2} \in [0,\infty]$$
for $\Phi \in \mathrm{Poly}(E_\infty^*)$, where $D_{e_\mathbf{j}}^\alpha = D_{e_{j_1}}^{\alpha_1} \cdots D_{e_{j_k}}^{\alpha_k}$ for $k \in \mathbb{N}$ and $\mathbf{j} = (j_1,\dots,j_k) \in \mathbb{N}_0^k$. We also define spaces $\mathbb{D}_{n,-p}$, $\mathbb{D}_{n,-\infty}$ and $\mathbb{D}_{\infty,-\infty}$ by the completion of
$$\{\Phi \in \mathrm{Poly}(E_\infty^*);\ |||\Phi|||_{n,-p} < \infty\}$$
with respect to $|||\cdot|||_{n,-p}$, the inductive limit space of $\mathbb{D}_{n,-p}$, $p > 0$, and the inductive limit space of $\mathbb{D}_{n,-p}$, $n \in \mathbb{N}, p > 0$, respectively.

Lemma 5.9. *Let $\Phi \in \mathrm{Poly}(E_\infty^*)$. Then there exist $n \in \mathbb{N}_0$ and $p \geq 1$ such that for any $m \in \mathbb{N}_0$ the inequality*
$$|||D_{e_m}\Phi|||_{n,-p} \leq |||\Phi|||_{n,-p}$$
holds.

Proof. Let $p \geq 1$. Then for any $n \in \mathbb{N}_0$ and $\Phi \in \mathrm{Poly}(E_\infty^*)$ we can check that
$$
\begin{aligned}
|||D_{e_m}\Phi|||_{n,-p}^2 &= \sum_{k=1}^\infty \sum_{\alpha_1,\dots,\alpha_k=0}^\infty \sum_{j_1,\dots,j_k=-\infty}^\infty ||D_{e_\mathbf{j}}^\alpha D_{e_m}\Phi||_{n,-p}^2 \\
&\leq \sum_{k=1}^\infty \sum_{\alpha_1,\dots,\alpha_k=0}^\infty \sum_{j_1,\dots,j_k=-\infty}^\infty \sum_{\nu=-\infty}^\infty ||D_{e_\mathbf{j}}^\alpha D_{e_\nu}\Phi||_{n,-p}^2 \\
&\leq |||\Phi|||_{n,-p}^2.
\end{aligned}
$$
Thus the assertion holds. $\qquad\square$

Theorem 5.10. *Let $\Phi \in \mathbb{D}_{n,-p}$ for any $n \in \mathbb{N}_0$ and $p \geq 1$. Then for any $m \in \mathbb{N}_0$ the inequality*

$$|||D_{e_m}\Phi|||_{n,-p} \leq |||\Phi|||_{n,-p}$$

holds.

Theorem 5.11. *Let $\Phi \in \mathbb{D}_{n,-p}$ for any $n \in \mathbb{N}_0$ and $p \geq 1$. Then the inequality*

$$|||\Delta_L\Phi|||_{n,-p} \leq |||\Phi|||_{n,-p}$$

holds. This means that the Lévy Laplacian is a contraction on $\mathbb{D}_{n,-p}$.

Let $\delta_m(0) := \sum_{k=-m}^{\infty} e_{2k}$ for $m \in \mathbb{N}_0$. We can easily check $\delta_m(0) \in E_m^*$. Since

$$
\begin{aligned}
(e_{2m} - e_{2(m+1)})\delta_m(0) &= \delta_m(0)(e_{2m} - e_{2(m+1)}) \\
&= \sum_{k=-m}^{\infty} e_{2(k+m)} - \sum_{k=-m}^{\infty} e_{2(k+m+1)} = e_0,
\end{aligned}
$$

we regard $e_{2m} - e_{2(m+1)}$ as the inverse element $\delta_m(0)^{-1}$ of $\delta_m(0)$ in E_m^*.

Let $c(\cdot)$ be a function on $\mathbb{R}$ given by

$$
c(u) := \begin{cases} \sqrt{-2\cos(2\pi u)}, & 1/4 \leq u < 3/4, \\ i\sqrt{2\cos(2\pi u)}, & 0 \leq u < 1/4, \ 3/4 \leq u \leq 1. \end{cases}
$$

Then, for $m \in \mathbb{N}_0$, we have

$$\delta_m(0)^{-1} = e^{2(2m+1)\pi(\cdot)}c(\cdot)^2$$

and therefore, we can regard $e^{(2m+1)\pi(\cdot)}c(\cdot)$ as $\delta_m(0)^{-1/2}$.

Lemma 5.12. *For any $m \in \mathbb{N}_0$ we have $\delta_m(0)^{-1/2} \in E_m^*$.*

Theorem 5.13. *Let $\Phi \in \mathbb{D}_{\infty,-\infty}$. Then the equality*

$$e^{\frac{t}{2}\Delta_L}\Phi(x) = E[\Phi(x + \delta_m(0)^{-1/2}\mathbf{B}_m(t))], \ x \in E_{2m}^*,$$

holds for any $m \in \mathbb{N}_0$.

Theorem 5.14. *Let $\Phi \in \mathbb{D}_{\infty,-\infty}^{\otimes n}$. Then the equality*

$$e^{\frac{t}{2}d\Gamma(\Delta_L)}\Phi(\mathbf{x}) = E[\Phi(\mathbf{x} + \widetilde{\mathbf{B}}_m(t))], \ \mathbf{x} = (x_1, x_2, \ldots, x_n) \in (E_{2m}^*)^{\otimes n},$$

holds for any $m \in \mathbb{N}_0$, where

$$\widetilde{\mathbf{B}}_m(t) = (\delta_m(0)^{-1/2}\mathbf{B}_m^{[1]}(t), \delta_m(0)^{-1/2}\mathbf{B}_m^{[2]}(t), \ldots, \delta_m(0)^{-1/2}\mathbf{B}_m^{[n]}(t))$$

belongs to $(E_{2m}^*)^{\otimes n}$ with independent E_m^*-valued dimensional Brownian motions $\mathbf{B}_m^{[1]}(t)$, $\mathbf{B}_m^{[2]}(t), \ldots,$ and $\mathbf{B}_m^{[n]}(t)$.

6. The Delta Distribution Centered at an Infinite Dimensional Brownian Motion

We endow a base probability space $(\Omega, \mathcal{F}, P)$ with a reference family $\{\mathcal{F}_{n,t}; \ t \geq 0\}$ of σ-subalgebras of $\mathcal{F}$, and assume that $\{\mathbf{B}_n(t); \ t \geq 0\}$ is adapted to $(\mathcal{F}_{n,t})$, i.e., for each $t \geq 0$, $\mathbf{B}_n(t)$ is $\mathcal{F}_{n,t}$-measurable.

Let $\{X(t); \ t \geq 0\}$ be an $\mathcal{L}(E_\infty^*; \mathbb{C})$-valued stochastic process. If a function $[0, \infty) \times \Omega \ni (t, \omega) \mapsto X(t)(\omega)(f) \in \mathbb{C}$ is an $(\mathcal{F}_{n,t})$-adapted process for every $f \in E_\infty^*$, then $\{X(t); \ t \geq 0\}$ is called an $(\mathcal{F}_{n,t})$-*adapted* $\mathcal{L}(E_\infty^*; \mathbb{C})$-*valued process.*

We define the stochastic integral $\int_0^t X(t) d\mathbf{B}_n(t)$ by

$$\int_0^t X(t) d\mathbf{B}_n(t) := \sum_{k=-n}^\infty \int_0^t X(t)(e_k) dB_k(t)$$

under the following conditions:

(1): $\{X(t); \ t \geq 0\}$ is an $(\mathcal{F}_{n,t})$-adapted $\mathcal{L}(E_\infty^*; \mathbb{C})$-valued process.

(2): $\sum_{k=-n}^\infty \int_0^t E[|X(s)(e_k)|^2] ds < \infty, \ t \geq 0$.

For any $\varphi \in C^\infty(E_\infty^*)$ with $\sup_{x \in E_\infty^*} |\varphi''(x)|_{\mathcal{L}(E_{n,-p}, \mathcal{L}(E_{n,-p}))} < \infty$ for some $p \geq 1, n \in \mathbb{N}_0$ and any $t \geq 0$ the stochastic integral $\int_0^t \varphi'(\mathbf{B}_n(u))(d\mathbf{B}_n(u))$ is given by

$$\int_0^t \varphi'(\mathbf{B}_n(u))(d\mathbf{B}_n(u)) = \sum_{k=-\infty}^\infty \int_0^t \varphi'(\mathbf{B}_n(u))(e_k) dB_k(u)$$

since

$$E\left[\left|\sum_{k=-\infty}^\infty \int_0^t \varphi'(\mathbf{B}_n(u))(e_k) dB_k(u)\right|^2\right] = \sum_{k=-\infty}^\infty \int_0^t E[|\varphi'(\mathbf{B}_n(u))(e_k)|^2] du$$

$$\leq \text{constant} \sum_{k=-\infty}^\infty \ell_k^{-2p} \int_0^t E[|\mathbf{B}_n(u)|_{-p}^2] du \leq \text{constant} \left(\sum_{k=-\infty}^\infty \ell_k^{-2p}\right)^2 t^2.$$

With the first exit time

$$\sigma_{x,n,-p}^{(r)} := \inf\{t \geq 0; \ x + \mathbf{B}_n(t) \notin B_r(x)\}$$

for $x + \mathbf{B}_n(t)$ from the open ball $B_r(x) := \{y \in E_{n,-p}; \ |y - x|_{n,-p} < r\}$ in $E_{n,-p}$ for $p \geq 1$ and $n \in \mathbb{N}_0$, we have an extension of the Itô formula using the similar method in [7] with changing the time parameter t by $t \wedge \sigma_{x,n,-p}^{(r)}$.

Theorem 6.1. (cf. [12]) *Let* $\varphi \in C^\infty(E_\infty^*)$ *and* $n \in \mathbb{N}_0$. *Then the equality*

$$\varphi(\mathbf{B}_n(t)) - \varphi(\mathbf{B}_n(s)) = \sum_{k=-n}^\infty \int_s^t \varphi'(\mathbf{B}_n(u))(e_k) dB_k(u) + \frac{1}{2} \int_s^t \mathcal{A}\varphi(\mathbf{B}_n(u)) du$$

holds for $t \geq s \geq 0$, *which is written by*

$$d\varphi(\mathbf{B}_n(t)) = \sum_{k=-n}^\infty \varphi'(\mathbf{B}_n(t))(e_k) dB_k(t) + \frac{1}{2} \mathcal{A}\varphi(\mathbf{B}_n(t)) dt.$$

For $n \in \mathbb{N}_0$ and $p \geq 1$, we denote by $\mathbb{A}_{n,p}$ the class of all entire functions φ defined on $E_{n,-p}$ such that

$$\|\varphi\|_{\mathbb{A}_{n,p}} := \sup_{x \in E_{n,-p}} \{|\varphi(x)|e^{-\frac{1}{2}|x|^2_{n,-p}}\} < \infty.$$

Define

$$\mathbb{A}_{n,\infty} := \bigcap_{p \in \mathbb{R}} \mathbb{A}_{n,p}$$

endowed with the projective limit topology of spaces $\mathbb{A}_{n,p}, p \geq 1$ and also define

$$\mathbb{A}_{\infty} := \bigcap_{n \in \mathbb{N}_0} \mathbb{A}_{n,\infty}$$

endowed with the projective limit topology of spaces $\mathbb{A}_{n,\infty}, n \in \mathbb{N}_0$. The space $\mathbb{A}_{\infty}$ is an analytic version of the space of test white noise functionals. For any $n \in \mathbb{N}$ and $p \geq 1$ we define an $\mathbb{A}^*_{\infty}$-valued stochastic process $\{X(t); \ 0 \leq t < 1\}$ by

$$X(t)(\varphi) := \varphi(\mathbf{B}_n(t)), \ \varphi \in \mathbb{A}_{\infty}.$$

We can denote $X(t)$ by $\delta_{\mathbf{B}_n(t)}$ as an element of $\mathbb{A}^*_{\infty}$ by the following lemma:

Lemma 6.2. *For any* $t \in [0,1)$, $n \in \mathbb{N}_0, p \geq 1$ *and* $\varphi \in \mathbb{A}_{n,-p}$, *there exists a constant* $C > 0$ *such that*

$$E[|\varphi(\mathbf{B}_n(t))|] \leq C\|\varphi\|_{\mathbb{A}_{n,-p}}.$$

Proof. For any $0 \leq t < 1$, $n \in \mathbb{N}_0, p \geq 1$ and $\varphi \in \mathbb{A}_{n,-p}$, we can estimate $E[|\varphi(\mathbf{B}_n(t))|]$ as follows.

$$
\begin{aligned}
E[|\varphi(\mathbf{B}_n(t))|] &\leq E[|\varphi(\mathbf{B}_n(t))|e^{-\frac{1}{2}|\mathbf{B}_n(t)|^2_{n,-p}}e^{\frac{1}{2}|\mathbf{B}_n(t)|^2_{n,-p}}] \\
&\leq E[e^{\frac{1}{2}|\mathbf{B}_n(t)|^2_{n,-p}}] \sup_{x \in E_{n,-p}} \{|\varphi(x)|e^{-\frac{1}{2}|x|^2_{n,-p}}\} \\
&= \exp\left(-\frac{1}{2}\sum_{j=-n}^{\infty} \log(1 - t\ell_j^{-2p})\right) \|\varphi\|_{\mathbb{A}_{n,-p}}.
\end{aligned}
$$

Since $\lim_{j \to \infty} \frac{-\log(1-t\ell_j^{-2p})}{t\ell_j^{-2p}} = 1$ and $\sum_{j=-n}^{\infty} \ell_j^{-2p} < \infty$, we have

$$-\sum_{j=-n}^{\infty} \log(1 - t\ell_j^{-2p}) < \infty.$$

Thus taking $C = \exp\left(-\frac{1}{2}\sum_{j=-n}^{\infty}\log(1 - t\ell_j^{-2p})\right)$, we obtain the assertion. $\square$

For any $y \in E^*_{\infty}$, the differential operator D_y on $\mathbb{A}_{\infty}$ is defined by

$$D_y\varphi(x) := \varphi'(x)(y), \ \varphi \in \mathbb{A}_{\infty}.$$

For any $n \in \mathbb{N}_0, p \geq 1, x \in E_{n,-p}$ with $x \perp e_j$ for $j \geq -n$ and $\beta \in \mathbb{R}$, we have an estimation of $|D_{e_j}\varphi^{(k)}(x + \beta e_j)|e^{-\frac{1}{2}|x+\beta e_j|^2_{n,-p}}$:

$$|D_{e_j}\varphi(x + \beta e_j)|e^{-\frac{1}{2}|x+\beta e_j|^2_{n,-p}}$$

$$= \left| \frac{1}{2\pi i} \int_{|\alpha|=r} \frac{\varphi(x + (\alpha + \beta)e_j)}{\alpha^2} d\alpha \right| e^{-\frac{1}{2}|x+\beta e_j|^2_{n,-p}}$$

$$\leq \frac{1}{2\pi} \int_{|\alpha|=r} \left| \frac{\varphi(x + (\alpha + \beta)e_j)}{\alpha^2} \right| d\alpha e^{-\frac{1}{2}|x+\beta e_j|^2_{n,-p}}$$

$$= \frac{1}{2\pi} \int_{|\alpha|=r} \left| \frac{\varphi(x + (\alpha + \beta)e_j)}{\alpha^2} \right| e^{-\frac{1}{2}|x+(\alpha+\beta)e_j|^2_{n,-p}} e^{\frac{1}{2}|\alpha e_j|^2_{n,-p}} d\alpha$$

$$\leq \frac{1}{2\pi r^2} \int_{|\alpha|=r} |\varphi(x + (\alpha + \beta)e_j)| e^{-\frac{1}{2}|x+(\alpha+\beta)e_j|^2_{n,-p}} d\alpha e^{\frac{1}{2}r^2|e_j|^2_{n,-p}}$$

$$\leq \frac{1}{2\pi r^2} e^{\frac{1}{2}r^2|e_j|^2_{n,-p}} \int_{|\alpha|=r} \sup_{x \in E_{n,-p}} \{|\varphi(x)|e^{-\frac{1}{2}|x|^2_{n,-p}}\} d\alpha.$$

Taking $r = \ell_j^p$ for $j \geq -n$, we have the following:

Lemma 6.3. *Let $n \in \mathbb{N}_0$. Then for any $j \geq -n$, D_{e_j} is a continuous linear operator from $\mathbb{A}_\infty$ into itself. More precisely, the inequality*

$$\|D_{e_j}\varphi\|_{\mathbb{A}_{n,p}} \leq \sqrt{e}\ell_j^{-p} \|\varphi\|_{\mathbb{A}_{n,p}}$$

holds for any $p \geq 1$ and $\varphi \in \mathbb{A}_{n,p}$.

Lemma 6.4. *For any $n \in \mathbb{N}_0$, $p \geq 1, \varphi \in \mathbb{A}_{n,p}$ and $X \in (\mathbb{A}_{n,p})^*$, we have*

$$\sum_{k=-n}^{\infty} X(D_{e_k}\varphi)e_k \in E_{n,-p}.$$

Proof. For any $n \in \mathbb{N}_0$, $p \geq 1, \varphi \in \mathbb{A}_{n,p}$ and $X \in (\mathbb{A}_{n,p})^*$, we have the estimation:

$$\left| \sum_{k=-n}^{\infty} X(D_{e_k}\varphi)e_k \right|^2_{n,-p} = \sum_{j=-n}^{\infty} \ell_j^{-2p} \left| \left\langle \sum_{k=-n}^{\infty} X(D_{e_k}\varphi)e_k, e_j \right\rangle_* \right|^2$$

$$= \sum_{j=-n}^{\infty} \ell_j^{-2p} \left| X(D_{e_j}\varphi) \right|^2$$

$$\leq \sum_{j=-n}^{\infty} \ell_j^{-2p} |X|^2_{(\mathbb{A}_{n,p})^*} |D_{e_j}\varphi|^2_{\mathbb{A}_{n,p}}.$$

Since $|D_{e_j}\varphi|_{\mathbb{A}_{n,p}} \leq \sqrt{e}\ell_j^{-p} \|\varphi\|_{\mathbb{A}_{n,p}}$ by Lemma 6.3, we obtain

$$\left| \sum_{k=-n}^{\infty} X(D_{e_k}\varphi)e_k \right|^2_{n,-p} \leq \sum_{j=-\infty}^{\infty} \ell_j^{-4p} |X|^2_{(\mathbb{A}_{n,p})^*} \|\varphi\|^2_{\mathbb{A}_{n,p}} < \infty.$$

$\square$

Lemma 6.5. *For any* $n \in \mathbb{N}_0$, $p \geq 1$ *and* $\varphi \in \mathbb{A}_{n,p}$, *we have*

$$\mathcal{A}_n \varphi \in \mathbb{A}_{n,p}.$$

Proof. Since

$$\mathcal{A}_n = \sum_{k=-n}^{\infty} D_{e_k}^2$$

for any $n \in \mathbb{N}_0$ and Lemma 6.3, we obtain

$$\|\mathcal{A}_n \varphi\|_{\mathbb{A}_{n,p}} \leq \sum_{k=-n}^{\infty} \|D_{e_k}^2 \varphi\|_{\mathbb{A}_{n,p}} \leq e \sum_{k=-n}^{\infty} \ell_k^{-2p} \|\varphi\|_{\mathbb{A}_{n,p}} < \infty$$

for any $n \in \mathbb{N}_0$ and $p \geq 1$. This implies the assertion. $\square$

By Lemmas 6.4 and 6.5, for any $n \in \mathbb{N}_0$, operators ∇_n^* and $\mathcal{A}_n^*$ on $\mathbb{A}_\infty^*$ are defined by

$$\nabla_n^* X(\varphi) := \sum_{k=-n}^{\infty} X(D_{e_k} \varphi) e_k, \ X \in \mathbb{A}_\infty^*,$$

and

$$\mathcal{A}_n^* X(\varphi) := X(\mathcal{A}_n \varphi), \ X \in \mathbb{A}_\infty^*,$$

respectively. Then $X(t)$ satisfies a stochastic differential equation

$$dX(t) = \nabla_n^* X(t) d\mathbf{B}_n(t) + \frac{1}{2} \mathcal{A}_n^* X(t) dt.$$

Remark 6.6. As in Example 5.3, $\mathbb{A}_\infty$ includes functionals of polynomials of $x \in E_\infty^*$.

7. A Stochastic Differential Equation Induced by a Sum of Delta Distributions Centered at Brownian Motions

For $N, n \in \mathbb{N}_0$ and $p \geq 1$, we denote by $\mathbb{A}_{n,p}^{N,\star}$ the class of all entire functions φ defined on $E_{n,-p}$ such that

$$\|\varphi\|_{\mathbb{A}_{n,p}^{N,\star}} := \sup_{x \in E_{n,-p}} \left\{ |\varphi(x)| \left(1 + |x|_{n,-p}^2\right)^{-N} \right\} < \infty.$$

Then, for $N, n \in \mathbb{N}_0$ and $p \geq 1$, we note that $\mathbb{A}_{n,p}^{N,\star} \subset \mathbb{A}_{n,p}$.

Define $\mathbb{A}_{n,\infty}^\star := \bigcap_{N=0}^{\infty} \bigcap_{p \geq 1} \mathbb{A}_{n,p}^{N,\star}$ endowed with the projective limit topology of spaces $\mathbb{A}_{n,p}^{N,\star}, N \in \mathbb{N}_0, p \geq 1$ and also define $\mathbb{A}_\infty^\star := \bigcap_{n \in \mathbb{N}_0} \mathbb{A}_{n,\infty}^\star$ endowed with the projective limit topology of spaces $\mathbb{A}_{n,\infty}, n \in \mathbb{N}_0$.

Lemma 7.1. *Let* $t \geq 0$ *and* $\varphi \in \mathbb{A}_\infty^\star$. *Then for any* $m \in \mathbb{N}_0$ *and* $p \geq 1$, *the following estimate holds:*

$$E[|Y(t)(\varphi)|] \leq (1+t) \sum_{n=-\infty}^{\infty} \ell_n^{-2p} \|\varphi\|_{\mathbb{A}_{m,p}^{1,\star}}. \tag{7.1}$$

For any $m \in \mathbb{N}$ and $p \geq 1$, we can consider an $(\mathbb{A}_\infty^\star)^*$-valued stochastic process $\{Y(t);\ t \geq 0\}$ by

$$Y(t)(\varphi) := \sum_{n=-m}^{\infty} \varphi(B_n(t)e_n),\ \varphi \in \mathbb{A}_\infty^\star$$

in view of Lemma 7.1. We can denote $Y(t)$ by $\sum_{k=-m}^{\infty} \delta_{B_k(t)e_k}$ as an element of $(\mathbb{A}_\infty^\star)^*$.

For any $y \in E_\infty^*$, we can consider the differential operator D_y restricted on $\mathbb{A}_\infty^\star$ as

$$D_y\varphi(x) := \varphi'(x)(y),\ \varphi \in \mathbb{A}_\infty^\star.$$

For any $m \in \mathbb{N}_0, p \geq 1, x \in E_{m,-p}$ with $x \perp e_j$ for $j \geq -m$ and $\beta \in \mathbb{R}$, we have an estimation of $|D_{e_j}\varphi(x + \beta e_j)| \left(1 + |x + \beta e_j|_{m,-p}^2\right)^{-N}$:

$$|D_{e_j}\varphi(x + \beta e_j)| \left(1 + |x + \beta e_j|_{m,-p}^2\right)^{-N}$$

$$= \left|\frac{1}{2\pi i} \int_{|\alpha|=r} \frac{\varphi(x + (\alpha + \beta)e_j)}{\alpha^2} d\alpha\right| \left(1 + |x + \beta e_j|_{m,-p}^2\right)^{-N}$$

$$\leq \frac{1}{2\pi} \int_{|\alpha|=r} \left|\frac{\varphi(x + (\alpha + \beta)e_j)}{\alpha^2}\right| d\alpha \left(1 + |x + \beta e_j|_{m,-p}^2\right)^{-N}$$

$$= \frac{1}{2\pi} \int_{|\alpha|=r} \left|\frac{\varphi(x + (\alpha + \beta)e_j)}{\alpha^2}\right| \left(1 + |x + (\alpha + \beta)e_j|_{m,-p}^2 - |\alpha e_j|_{m,-p}^2\right)^{-N} d\alpha$$

$$|D_{e_j}\varphi(x + \beta e_j)| \left(1 + |x + \beta e_j|_{m,-p}^2\right)^{-N}$$

$$\leq \frac{2^N}{2\pi r^2} \int_{|\alpha|=r} |\varphi(x + (\alpha + \beta)e_j)| \left(1 + |x + (\alpha + \beta)e_j|_{m,-p}^2\right)^{-N} d\alpha \cdot$$

$$\leq \frac{2^N}{2\pi r^2} \int_{|\alpha|=r} \sup_{x \in E_{m,-p}} \{|\varphi(x)| \left(1 + |x|_{m,-p}^2\right)^{-N}\} d\alpha.$$

Taking $r = \frac{1}{2}\ell_j^p$, we have the following:

Lemma 7.2. *Let $m, j \in \mathbb{N}_0$. Then for any $j \geq -m$, D_{e_j} is a continuous linear operator from $\mathbb{A}_\infty^\star$ into itself. More precisely, the inequality*

$$\|D_{e_j}\varphi\|_{\mathbb{A}_{m,p}^{N,\star}} \leq \frac{2^{N-1}}{\pi} \ell_j^{-p} \|\varphi\|_{\mathbb{A}_{m,p}^{N,\star}}$$

holds for any $N \in \mathbb{N}_0$, $p \in \mathbb{R}$ and $\varphi \in \mathbb{A}_{m,p}^{N,\star}$.

Similarly, we also have the following facts by Lemma 6.4.

Lemma 7.3. *For any $N, m \in \mathbb{N}_0$, $p \geq 1, \varphi \in \mathbb{A}_{m,p}^{N,\star}$ and $Y \in (\mathbb{A}_{m,p}^{N,\star})^*$, we have*

$$\sum_{k=-m}^{\infty} Y(D_{e_k}\varphi)e_k \in E_{m,-p}.$$

Lemma 7.4. *For any $N, n \in \mathbb{N}_0$, $p \geq 1$ and $\varphi \in \mathbb{A}_{n,p}^{N,\star}$, we have*

$$\mathcal{A}_n\varphi \in \mathbb{A}_{n,p}^{N,\star}.$$

Proof. Since $\mathcal{A}_n = \sum_{k=-n}^{\infty} D_{e_k}^2$ for any $n \in \mathbb{N}_0$, from Lemma 7.2, we obtain

$$
\begin{aligned}
\|\mathcal{A}_n\varphi\|_{\mathbb{A}_{n,p}^{N,\star}} &\leq \sum_{k=-n}^{\infty} \|D_{e_k}^2\varphi\|_{\mathbb{A}_{n,p}^{N,\star}} \\
&\leq \left(\frac{2^{N-1}}{\pi}\right)^2 \sum_{k=-n}^{\infty} \ell_k^{-2p} \|\varphi\|_{\mathbb{A}_{n,p}^{N,\star}} < \infty
\end{aligned}
$$

for any $n, N \in \mathbb{N}_0$ and $p \geq 1$. This implies the assertion. $\qquad\square$

By Lemmas 7.3 and 7.4, for any $n \in \mathbb{N}_0$, the operators ∇_n^* and $\mathcal{A}_n^*$ can be defined on $\mathbb{A}_\infty^*$ by

$$
\nabla_n^* Y(\varphi) := \sum_{k=-n}^{\infty} Y(D_{e_k}\varphi)e_k, \ Y \in (\mathbb{A}_\infty^\star)^*,
$$

and

$$
\mathcal{A}_n^* Y(\varphi) := Y(\mathcal{A}_n\varphi), \ Y \in (\mathbb{A}_\infty^\star)^*.
$$

Then $Y(t)$ also satisfies the stochastic differential equation

$$
dY(t) = \nabla_n^* Y(t)d\mathbf{B}_n(t) + \frac{1}{2}\mathcal{A}_n^* Y(t)dt.
$$

8. Concluding Remarks

Since for any $n \in \mathbb{N}_0$, $\varphi \in \mathbb{A}_\infty$ and $m \in \mathbb{N}$, the equalities

$$
d\varphi(\mathbf{B}_n(t)^m) = \varphi'(\mathbf{B}_n(t)^m)(d(\mathbf{B}_n(t)^m)) + \frac{1}{2}\varphi''(\mathbf{B}_n(t)^m)(d(\mathbf{B}_n(t)^m), d(\mathbf{B}_n(t)^m))
$$

$$
\tag{8.1}
$$

and

$$
\begin{aligned}
d(\mathbf{B}_n(t)^m) &= m\mathbf{B}_n(t)^{m-1}d\mathbf{B}_n(t) + \frac{m(m-1)}{2}\mathbf{B}_n(t)^{m-2}(d\mathbf{B}_n(t))^2 \\
&= m\mathbf{B}_n(t)^{m-1}d\mathbf{B}_n(t) + \frac{m(m-1)}{2}\mathbf{B}_n(t)^{m-2}\sum_{k=-n}^{\infty} e_{2k}dt
\end{aligned}
$$

hold, we have the following:

Theorem 8.1. (cf. [12]) *For any $n \in \mathbb{N}_0$, $\varphi \in \mathbb{A}_\infty$ and $m \in \mathbb{N}$, the equality*

$$
\begin{aligned}
d\varphi(\mathbf{B}_n(t)^m) &= m\sum_{k=-n}^{\infty} \varphi'(\mathbf{B}_n(t)^m)(\mathbf{B}_n(t)^{m-1}e_k)dB_k(t) \\
&\quad + \frac{m(m-1)}{2}\sum_{k=-n}^{\infty} \varphi'(\mathbf{B}_n(t)^m)(\mathbf{B}_n(t)^{m-2}e_{2k})dt \\
&\quad + \frac{m^2}{2}\sum_{k=-n}^{\infty} \varphi''(\mathbf{B}_n(t)^m)(\mathbf{B}_n(t)^{m-1}e_k, \mathbf{B}_n(t)^{m-1}e_k)dt
\end{aligned}
$$

holds.

For $x \in E_\infty^*$, we introduce operator $\mathcal{A}_{n,x}$ defined on $\mathbb{A}_\infty$ by

$$\mathcal{A}_{n,x}\varphi := \sum_{k=-n}^{\infty} D_{xe_k}^2 \varphi, \ \varphi \in \mathbb{A}_\infty.$$

For $x \in E_\infty^*$, we also introduce operators $\nabla_{n,x}^*$ and $\mathcal{A}_{n,x}^*$ defined on $\mathbb{A}_\infty^*$ by

$$\nabla_{n,x}^* X(\varphi) := \sum_{k=-n}^{\infty} X(D_{xe_k}\varphi)e_k, \ \varphi \in \mathbb{A}_\infty,$$

and

$$\mathcal{A}_{n,x}^* X(\varphi) := X(\mathcal{A}_{n,x}\varphi), \ \varphi \in \mathbb{A}_\infty.$$

For any $n \in \mathbb{N}_0$, $t \geq 0$ and $m \in \mathbb{N}_0$, we set

$$X_m(t)(\varphi) := \varphi(\mathbf{B}_n(t)^m), \ \varphi \in \mathbb{A}_\infty.$$

If $\{X_m(t); \ 0 \leq t < 1\}$ is an $\mathbb{A}_\infty^*$-valued stochastic process, then it satisfies the stochastic differential equation by Theorem 8.1:

$$\begin{aligned} dX_m(t) = {}& m\nabla_{n,\mathbf{B}_n(t)^{m-1}}^* X_m(t)d\mathbf{B}_n(t) \\ & + \left(\frac{m(m-1)}{2} D_{\mathbf{B}_n(t)^{m-2}\delta_n(0)}^* + \frac{m^2}{2} \mathcal{A}_{n,\mathbf{B}_n(t)^{m-1}}^* \right) X_m(t)dt. \end{aligned} \quad (8.2)$$

We note that the equation (8.2) includes the random differential operators in $\nabla_{n,\mathbf{B}_n(t)^{m-1}}$ and $\mathcal{A}_{n,\mathbf{B}_n(t)^{m-1}}$, for $m \geq 2$.

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