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Functional Equations Related To Spatial Filtering In Image Enhancement

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Abstract : In this paper, we discuss the solutions of the functional equations arising from arithmetic mean and harmonic mean and we illustrate how these functional equations can be used to remove noise in an image by filtering techniques.

Keywords : Functional equation; additive functional equation; quadratic functional equation; cubic functional equation; reciprocal functional equation.

1. INTRODUCTION AND PRELIMINARIES

A Hungarian Mathematician J. Aczel [1], an excellent specialist in functional equations, defines the functional equation as follows:

"Functional equations are equations in which both sides are terms constructed from the finite number of unknown functions and a finite number of independent variables".

Nowadays, the field of functional equations is an ever-growing branch of mathematics with far-reaching applications. The theory of functional equations is relatively new and it contributes to the development of strong tools in contemporary mathematics. Functional equations also comprise a traditional branch of Mathematics offering wide scope for algebraic, analytic, order theoretic and topological considerations. Conversely many mathematical ideas in different fields have become essential to the foundation of functional equations. It is increasingly used to investigate problems in other fields such as Mathematical analysis, Combinatorics, Biology, Behavioral and Social sciences, and Engineering.

Solving a functional equation means to find all functions that satisfy the functional equation. Solutions to functional equations have become important tools in an increasing number of problems in social sciences such as Economics and Psychology. Many new applied problems and theories have inspired and encouraged specialists on functional equations to develop new approaches and new methods.

Applications of functional equations to characterizing various probability laws and statistics can be found in Ramachandran and Lau [14], and Rao and Shanbhag [15]. Castillo, Cobo, Gutierrez and Pruneda [5] introduced the functional networks using functional equations. These functional networks have found many applications like the neural networks. Information measures play an important role in information theory and also in coding theory. Various functional equations are used for characterizing information measures.

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Functional equations occur practically everywhere. Their influence and applications are felt in every field, and all fields benefit from their contact, use and technique. The growth and development used to be influenced by their spectacular application in several areas and not only in mathematics but also in other disciplines. Applications can be found in a wide variety of fields such as Classical mechanics, Decision theory, Dynamic programming, Fuzzy set theory, Game theory, Geometry, Group Theory, Inequalities, Inner product space, Measure Theory, Mechanics, Multivalued logic, Polynomials, Cluster analysis, Stochastic process, Physics, Astronomy, Reproducing scoring system, taxation, Population ethics, Applied Science, Computer graphics, Artificial intelligence, Neural networks, Digital image processing and many other fields. Functional equations are being used with vigor in ever-increasing numbers to investigate problems in the above mentioned areas and other fields.

Let us evoke definition of some basic functional equations, and their solutions and properties which are required for our main results.

Definition 1.1.

[1] A function $g : \mathbb{R} \to \mathbb{R}$ is called additive if g satisfies

$$g(x+y) = g(x) + g(y) \tag{1.1}$$

for all $x, y \in \mathbb{R}$.

The additive function $g(x) = cx$ is a solution of the equation (1.1), where c is a constant.

Definition 1.2.

([1], [11]) A function $g : \mathbb{R} \to \mathbb{R}$ is called quadratic if g satisfies

$$g(x+y) + (x-y) = 2g(x) + 2g(y) \tag{1.2}$$

for all $x, y \in \mathbb{R}$.

The quadratic function $g(x) = kx^2$ is a solution of the equation (1.2), where k is a constant. The functional equation (1.2) is related to a symmetric bi-additive function ([1], [11]). A function $f : \mathrm{X} \to \mathrm{Y}$ between vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function $\mathrm{B} : \mathrm{X} \times \mathrm{X} \to \mathrm{Y}$ such that $f(x) = \mathrm{B}(x, x)$ for all $x \in \mathrm{X}$, where the function is given by

$$\mathrm{B}(x, y) = \frac{1}{4}[f(x+y) - f(x-y)], \text{ for all } x, y \in \mathrm{X}.$$

Definition 1.3.

[9] A function $g : \mathbb{R} \to \mathbb{R}$ is called cubic if g satisfies

$$g(2x+y) + g(2x-y) = 2g(x+y) + 2g(x-y) + 12g(x) \tag{1.3}$$

for all $x, y \in \mathbb{R}$. The cubic function $g(x) = cx^3$, is a solution of the equation (1.3), where c is a constant.

The motivation for studying cubic functional equations came from the fact that recently polynomial equations have found applications in approximate checking, self-testing and self-correcting of computer programs that compute polynomials.

Theorem 1.4.

[9] *A function $f : X \to Y$ between vector spaces satisfies the functional equation (1.3) if and only if there exists a function $B : X \times X \times X \to Y$ such that $f(x) = B(x, x, x)$ for all $x \in X$ and B is symmetric for each fixed variable and is additive for fixed two variables.*

K.W. Jun and H.M. Kim ([9],[10]) proved that a function g satisfying (1.3) also satisfies

$$f(x+y+z)+f(x+y-z)+2f(x)+2f(y) = 2f(x+y)+f(x+z)+f(x-z)+f(y+z)+f(y-z) \quad (1.4)$$

for all $x, y \in \mathbb{R}$.

H. Azadi Kenary [4] showed that the functional equation

$$g(x+2y)+g(x-2y) = 4[g(x+y)+g(x-y)-6g(x)] \quad (1.5)$$

is also a cubic functional equation.

K. Ravi and B.V. Senthil Kumar [16] investigated the general solution of a mapping $r:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying the reciprocal functional equation

$$r(x+y) = \frac{r(x)\,r(y)}{r(x)+r(y)}. \quad (1.6)$$

The reciprocal function $r(x)=\frac{c}{x}$ is the solution of the functional equation (1.6), where c is a constant.

Definition 1.5.

[16] A mapping $r:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is called reciprocal if r satisfies the functional equation (1.6).

The functional equation (1.6) has an application in the well-known "Reciprocal formula" to find the equivalent resistance of an electric circuit consisting of two resistors connected in parallel and also the equation (1.6) holds good in a geometric construction discussed in [17].

Consider a function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfying the square functional equation

$$f(x+y) = \frac{1}{4}[f(x+y, y+t)+f(x+t, y-t)+f(x-y, y+t)+f(x-t, y-t)]$$

for all $x, y, t \in \mathbb{N}$. The equation (1.7) gives the value of f at the centre of any square, the sides of which are parallel to the coordinate axes, equals the mean value of f over the four vertices of the square. The general solution of the equation (1.7) is

$$f(x, y) = B(x, y)+A_1(x)+A_2(y)+\alpha,$$

where B is an arbitrary bi-additive map, A_1 and A_2 are arbitrary additive maps, and α is an arbitrary constant [2].

The functional equation similar to (1.7) arises in the problem of solving 2-dimensional Laplace equation

$$f_{xx}+f_{yy} = 0. \quad (1.8)$$

Motivated by equation (1.7), we introduce the following two variable reciprocal functional equations of the form

$$f(x, y) = \frac{1}{4}[f(x+t, y)+f(x-t, y)+f(x, y, +t)+f(x, y-t)] \quad (1.9)$$

and
$$f(x, y) = \frac{f_1(x, y, t)}{f_2(x, y, t)} \quad (1.10)$$

where $f_1(x, y, t) = 4f(x+t, y+t)\,f(x+t, y-t)\,f(x-t, y+t)f(x-t, y-t)$

and $f_2(x, y, t) = f(x+t, y+t)\,f(x+t, y-t)\,f(x-t, y+t)+f(x+t, y+t)\,f(x+t, y-t)\,f(x-t, y-t)$
$+f(x+t, y+t)\,f(x-t, y+t)\,f(x-t, y-t)+f(x+t, y-t)\,f(x-t, y+t)\,f(x-t, y-t)\neq 0.$

P.K. Sahoo and L. Szekelyhidi ([18],[19]) obtained the general solution of the functional equation

$$f(x+t, y+t)+f(x-t, y)+f(x, y-t) = f(x-t, y-t)+f(x, y+t)+f(x+t, y) \quad (1.11)$$

for all $x, y, t \in G$, where G denotes a 2-divisible abelian group. The functional equation (1.11) arises while characterizing quadratic polynomials in two variables and also connected to a problem in spatial filtering of digital images. The functional equation (1.11) can be used as a filter to retain certain information contained in the original image. Other type of functional equations related to digital filtering are available in [12].

In this paper, we find the solution of the functional equations (1.7) and (1.10). We also illustrate the functional equations (1.7) and (1.10) can be related to spatial filtering in image enhancement.

The paper is organised as follows. In Section 2, we present basic concepts of image enhancement. In Section 3, we prove equation (1.7) is equivalent to (1.9). In Section 4, we find the solution of the functional equation (1.7). In Section 5, we obtain the general solution of the functional equation (1.10) and in Section 6, we illustrate the application of the functional equations (1.7) and (1.10) in image enhancement. In Section 7, we present the conclusion of this paper.

2. BASIC CONCEPTS OF IMAGE ENHANCEMENT

In this section, we present some basic concepts related to filtering techniques used in digital image processing.

Many functional equations occur in various fields of Applied Science, such as Mathematical Physics, Statistics, Economics, Astronomy, Engineering, Technology and Computer. Particularly, visualizing using computer is a comparatively new and rapid growing field which mostly helps computers with artificial sensory perception. In other words, the computer vision deals with image understanding. Pre-processing is a fundamental task in every image understanding problem. Among others, pre-processing involves filtering. Spatial filters are used for deblurring, smoothing, sharpening and enhancing of images. Thus filtering is a technique for transforming or improving the quality of an image.

The primary objective of the image enhancement is to adjust the digital image so that the resultant image is more suitable than the original image for a specific application. There are many image enhancement techniques. We focus on the techniques which are based on the position manipulation of image pixels.

An image is defined as a two dimensional function $f(x, y)$, where x and y are spatial (plane) coordinates and the amplitude of f at any pair of coordinates (x, y) is called the intensity or grey level of pixel value of the image [21]. Data sets collected by image sensor are generally contaminated by noise. An efficient filtering method is necessary for enhancing the quality of images by removing noise. The noise removal in the image is still a challenging problem for researcher because noise removal introduces artifacts and causes blurring of the image. A few types of noise sources which corrupt images are Gaussian Noise, Poisson Noise, Salt & Pepper Noise, Speckle Noise, Erlang (Gamma) Noise, Exponential Noise, Uniform Noise. For more information about different types of noises, one can refer ([3], [6], [7], [8], [13], [20], [21]). For our experimental purpose, we deal with Gaussian noise and Salt & Pepper noise.

2.1. Gaussian Noise

Gaussian noise is statistical noise that has a probability density function of the normal distribution (also known as Gaussian distribution). In other words, the values that the noise can take on are Gaussian distributed. It is most commonly used as additive white noise to yield additive white Gaussian noise.

2.2. Salt & Pepper Noise

It represents itself as randomly occurring white and black pixels. This type of noise is also caused by errors in data transmission and is a special case of data drop out noise when in some single pixels are set alternatively to zero or to the maximum value, giving the image a salt and pepper like appearance.

3. RELATION BETWEEN EQUATION (1.7) AND EQUATION (1.9)

In this section, we prove that the functional equations (1.7) and (1.9) are equivalent.

Theorem 3.1.

A mapping $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (1.7) if and only if f satisfies the functional equation (1.9). Hence the solution of (1.7) and (1.9) are same.

Proof. Let satisfy equation (1.7). Replacing (x, y) by $(x+t, x+t)$ in (1.7), we find

$$f(x+t, y+t) = \frac{1}{4}[f(x+2t, y+2t) + f(x+2t, y) + f(x, y) + 2t + f(x, y)] \quad (3.1)$$

for all $x, y, t \in \mathbb{R}$. Now, substituting (x, y) as $(x-t, y-t)$ in (1.7), we obtain

$$f(x-t, y-t) = \frac{1}{4}[f(x, y) + f(x, y-2t) + f(x-2t, y) + f(x-2t, y-2t)] \quad (3.2)$$

for all $x, y, t \in \mathbb{R}$. Again, taking $(x+t, y-t)$ instead of (x, y)in (1.7), we have

$$f(x+t, y-t) = \frac{1}{4}[f(x+2t, y) + f(x+2t, y-2t) + f(x, y) + f(x, y-2t)] \quad (3.3)$$

for all $x, y, t \in \mathbb{R}$. Now, replacing (x, y) by $(x-t, y+t)$ in (1.7), one finds

$$f(x-t, y+t) = \frac{1}{4}[f(x, y+2t) + f(x, y) + f(x-2t, y+2t) + f(x-2t, y)] \quad (3.4)$$

for all $x, y, t \in \mathbb{R}$. Adding equations (3.1), (3.2), (3.3) and (3.4), we arrive

$$f(x+t, y+t) + f(x-t, y-t) + f(x+t, y-t) + f(x-t, y+t)$$
$$= f(x, y) + \frac{1}{2}[f(x+2t, y) + f(x, y+2t) + f(x, y-2t) + f(x-2t, y)]$$
$$+\frac{1}{4}[f(x+2t, y+2t) + f(x-2t, y-2t) + f(x+2t, y-2t) + f(x-2t, y+2t)] \quad (3.5)$$

for all $x, y, t \in \mathbb{R}$. Now, replacing t by $2t$ in (1.7), we obtain

$$f(x, y) = \frac{1}{4}[f(x+2t, y+2t) + f(x+2t, y-2t) + f(x-2t, y+2t) + f(x-2t, y-2t)] \quad (3.6)$$

for all $x, y, t \in \mathbb{R}$. Using (1.7) and (3.6) in (3.5), we arrive (1.9).

On the other hand, let f satisfy equation (1.9). Replacing $y+t$ by in (1.9), we have

$$f(x, y+t) = \frac{1}{4}[f(x+t, y+t) + f(x-t, y+t) + f(x, y+2t) + f(x, y)] \quad (3.7)$$

for all $x, y, t \in \mathbb{R}$. Substituting y as y in (1.9), we obtain

$$f(x, y-t) = \frac{1}{4}[f(x+t, y-t) + f(x-t, y-t) + f(x, y) + f(x, y-2t)] \quad (3.8)$$

for all $x, y, t \in \mathbb{R}$. Again taking $x+t$ instead of x in (1.9), one finds

$$f(x+t, y) = \frac{1}{4}[f(x+2t, y) + f(x, y) + f(x+t, y+t) + f(x+t, y-t)] \quad (3.9)$$

for all $x, y, t \in \mathbb{R}$. Now, replacing x by $x-t$ in (1.9), one obtains

$$f(x-t, y) = \frac{1}{4}[f(x, y) + f(x-2t, y) + f(x-t, y+t) + f(x-t, y-t)] \quad (3.10)$$

for all $x, y, t \in \mathbb{R}$. Summing equations (3.7), (3.8), (3.9) and (3.1), we arrive

$$f(x, y+t) + f(x, y-t) + f(x+t, y) + f(x-t, y)$$
$$= f(x, y) + \frac{1}{2}[f(x+t, y+t) + f(x-t, y+t) + f(x-t, y-t) + f(x+t, y-t)]$$
$$+\frac{1}{4}[f(x, y+2t) + f(x, y-2t) + f(x+2t, y) + f(x-2t, y)] \quad (3.11)$$

for all $x, y, t \in \mathbb{R}$. Now, replacing t by $2t$ in (1.9), we have

$$f(x, y) = \frac{1}{4}[f(x+2t, y) + f(x-2t, y) + f(x, y+t) + f(x, y-2t)] \tag{3.12}$$

for all $x, y, t \in \mathbb{R}$. Using (1.9) and (3.12) in (3.11), we arrive (1.7), which completes the proof.

4. SOLUTION OF THE FUNCTIONAL EQUATION (1.7)

In this section, we obtain the general solution of the functional equation (1.7).

Lemma 4.1.

If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a mapping given by*

$$f(x, y) = \alpha A(x) + \beta A(y)$$

where $A : \mathbb{R} \to \mathbb{R}$ *is an additive mapping satisfying (1.1), and and are constants. Then f satisfies the equation (1.7).*

Proof. $f(x+t, y+t) + f(x-t, y-t) + f(x+t, y-t) + f(x-t, y+t)$

$$= \alpha A(x+t) + \beta A(y+t) + \alpha A(x-t) + \beta A(y-t) + \alpha A(x+t) + \beta A(y-t) + \alpha A(x-t) + \beta A(y+t)$$

$$= 2\alpha A(x+t) + 2\alpha A(x-t) + 2\beta A(y+t) + \beta A(y-t)$$

$$= 4\alpha A(x) + 4\beta A(y)$$

$$= 4f(x, y)$$

for all $x, y, t \in \mathbb{R}$, which completes the proof of the lemma.

Lemma 4.2.

If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a mapping given by*

$$f(x, y) = \frac{\gamma}{4}[Q(x+y) - Q(x-y)]$$

where $Q : \mathbb{R} \to \mathbb{R}$ *is a quadratic mapping satisfying (1.2), and* γ *is a constant. Then f satisfies the equation (1.7).*

Proof. $f(x+t, y+t)\ f(x-t, y-t) + f(x+t, y-t) + f(x-t, y+t)$

$$= \frac{\gamma}{4}[Q(x+y+2t) - Q(x-y)] + \frac{\gamma}{4}[Q(x+y-2t) - Q(x-y)] + \frac{\gamma}{4}[Q(x+y) - Q(x-y-2t)] + \frac{\gamma}{4}[Q(x+y) - Q(x-y+2t)]$$

$$= +\frac{\gamma}{4}[Q(x+y) - Q(x-y)] + \frac{\gamma}{4}[Q(x+y+2t) - Q(x+y-2t)] - \frac{\gamma}{4}[Q(x-y+2t) + Q(x-y-2t)]$$

$$= 2f(x, y) + \frac{\gamma}{4}[2Q(x+y) + 2Q(t)] - \frac{\gamma}{4}[2Q(x-y) + 2Q(2t)]$$

$$= 4f(x, y)$$

for all $x, y, t \in \mathbb{R}$, which completes the proof of the lemma.

Lemma 4.3.

Let $C:\mathbb{R}\to\mathbb{R}$ *be a cubic mapping satisfying (1.4) and (1.5). If* $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is a mapping given by*

$$f(x, y) = \delta[2C(x) + 2C(y) - C(x + y)]$$

for all $x, y, t \in \mathbb{R}$, where δ is a constant. Then satisfies (1.7).

Proof. $f(x+t, y+t)\ f(x-t, y-t) + f(x+t, y-t) + f(x-t, y+t)$

$$\begin{aligned}
&= 2\delta C(x + t) + 2\delta C(y + t) - 2\delta C(x + y + 2t) + 2\delta C(x - t) + 2\delta C(y - t)\\
&\quad - \delta C(x + y - 2t) + 2\delta C(x - t) + 2\delta C(y - t) - \delta C(x + y)\\
&= 4\delta C(x + t) + 4\delta C(x + t) - 4\delta C(y + t) + 4\delta C(y - t)\\
&\quad - \delta C(x + y + 2t) - \delta C(x + y - 2t) - 2\delta C(x + y)\\
&= 4\delta C(x + t) + 4\delta C(x - t) - 4\delta C(y + t) + 4\delta C(y - t)\\
&\quad - 2\delta C(x + y) - [4\delta C(x + y + t) + 4\delta C(x + y - t) - 6\delta C(x + y)]\\
&= 4\delta C(x + t) + 4\delta C(x - t) - 4\delta C(y + t) + 4\delta C(y - t) + 4\delta C(x + y)\\
&\quad - [8\delta C(x + y) - 4\delta C(x + t) + 4\delta C(x - t) + 4\delta C(y + t) + 4\delta C(y - t)] - 8\delta C(x) - 8\delta C(yt)\\
&= 8\delta C(x) + 8\delta C(y) - 4\delta C(x + y)\\
&= 4f(x + y)
\end{aligned}$$

for all $x, y, t \in \mathbb{R}$, which completes the proof of the lemma.

Theorem 4.4.

Let $A:\mathbb{R}\to\mathbb{R}$ *be an additive mapping satisfying (1.1),* $Q:\mathbb{R}\to\mathbb{R}$ *be a quadratic mapping satisfying (1.2) and* $C:\mathbb{R}\to\mathbb{R}$ *be a cubic mapping satisfying (1.4) and (1.5). If* $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is a mapping given by*

$$f(x, y) = \alpha A(x) + \beta A(y) + \frac{\gamma}{4}[Q(x + y) - Q(x - y)] + \delta[2C(x) + 2C(y) - C(x + y)]$$

for all $x, y \in \mathbb{R}$, where $\alpha, \beta, \gamma, \delta$ are constants. Then f satisfies (1.7).

Proof. The proof of this theorem follows immediately from Lemmas 4.1, 4.2 and 4.3.

5. GENERAL SOLUTION OF THE FUNCTIONAL EQUATION (1.10)

In this section, we achieve the general solution of the functional equation (1.10).

Lemma 5.1.

Let $r:\mathbb{R}\{0\}\to\mathbb{R}$ *be a function satisfying the functional equation (1.6). Then r is an odd function.*

Proof. Replacing (x, y) by (x, x) in (1.6), we obtain

$$r(2x) = \frac{1}{2}r(x) \tag{5.1}$$

for all $x \in \mathbb{R}\setminus\{0\}$. Similarly, we arrive at $r(3x) = \frac{1}{3}r(x)$, for all $x \in \mathbb{R}\setminus\{0\}$. Now, replacing by in (1.6) and using (5.1), we have

$$r(x + 2y) = \frac{r(x)\,r(y)}{2r(x) + r(y)} \tag{5.2}$$

for all $x \in \mathbb{R}\setminus\{0\}$. Substituting $(x, y) = (x, -2y)$ in (1.6) and using (5.1), we obtain

$$r(x - 2y) = \frac{r(x)\,r(-y)}{2r(x) + r(-y)} \tag{5.3}$$

for all $x \in \mathbb{R} \setminus \{0\}$. Equation (5.2) divided by equation (5.3) yields,

$$\frac{r(x+2y)}{r(x-2y)} = \frac{r(y)[2r(x)+r(-y)}{r(-y)[2r(x)+r(y)]}$$

for all $x \in \mathbb{R} \setminus \{0\}$. Now, replacing (x, y) by $(x, -x)$ in (5.4) to get

$$\frac{r(-x)}{r(3x)} = \frac{r(-x)[2r(x)+r(x)]}{r(x)[2r(x)+r(-x)]}$$

which on further simplification yields $r(-x) = -r(x)$, for all $x \in \mathbb{R} \setminus \{0\}$. Hence is an odd function.

Theorem 5.2.

A mapping $f : \mathbb{R} \setminus \{0\} \times \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ *satisfies (1.10) if and only if there exist two reciprocal mappings* $r_1, r_2 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *such that*

$$f(x, y) = \frac{r_1(x)\, r_2(y)}{r_1(x) + r_2(y)'}$$

for all $x, y \in \mathbb{R} \setminus \{0\}$.

Proof. Assume that f is a solution of (1.10). Define $h_1(x) = f(x, 0)$, $h_2(y) = f(0, y)$, for all $x, y \in \mathbb{R} \setminus \{0\}$. It is easy to verify that h_1, h_2 are reciprocal functions. Let $h_1(x) = r_1(x)$ and $h_2(x) = r_2(x)$, $x \in \mathbb{R} \setminus \{0\}$. Hence for all $x, y \in \mathbb{R} \setminus \{0\}$.

Conversely, assume that there exist two reciprocal mappings $r_1, r_2 : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ such that $f(x, y) = \dfrac{r_1(x)\, r_2(y)}{r_1(x) + r_2(y)}$, for all $x, y \in \mathbb{R} \setminus \{0\}$. Hence

$$\begin{aligned}
\frac{f_1(x, y, t)}{f_2(x, y, t)} &= \frac{4}{\frac{1}{f(x+t, y+t)} + \frac{1}{f(x+t, y-t)} + \frac{1}{f(x-t, y+t)} + \frac{1}{f(x-t, y-t)}} \\
&= \frac{4}{\frac{r_1(x+t)+r_2(y+t)}{r_1(x+t)\, r_2(y+t)} + \frac{r_1(x+t)+r_2(y-t)}{r_1(x+t)\, r_2(y-t)} + \frac{r_1(x-t)+r_2(y+t)}{r_1(x-t)\, r_2(y+t)} + \frac{r_1(x-t)+r_2(y-t)}{r_1(x-t)\, r_2(y-t)}} \\
&= \frac{4}{\frac{1}{r_1(x+t)} + \frac{1}{r_1(y+t)} + \frac{1}{r_1(x+t)} + \frac{1}{r_1(y-t)} + \frac{1}{r_1(x-t)} + \frac{1}{r_1(y+t)} + \frac{1}{r_1(x-t)} + \frac{1}{r_1(y-t)}} \\
&= \frac{4}{\frac{1}{r_1(x+t)} + \frac{1}{r_1(y+t)} + \frac{1}{r_1(x+t)} + \frac{1}{r_1(y-t)} + \frac{1}{r_1(x-t)} + \frac{1}{r_1(y+t)} + \frac{1}{r_1(x-t)} + \frac{1}{r_1(y-t)}} \\
&= \frac{2}{\frac{1}{r_1(x+t)} + \frac{1}{r_2(y+t)} + \frac{1}{r_1(x+t)} + \frac{1}{r_2(y-t)}} \\
&= \frac{1}{\frac{1}{r_1(x)} + \frac{1}{r_2(y)}} \\
&= \frac{r_1(x) r_2(y)}{r_1(x) + r_2(y)} = f(x, y),
\end{aligned}$$

for all $x, y \in \mathbb{R} \setminus \{0\}$.

6. APPLICATION OF THE FUNCTIONAL EQUATIONS (1.7) AND (1.10) IN IMAGE ENHANCEMENT

In this section, we illustrate the application of the functional equations (1.7) and (1.10) in image enhancement.

In image enhancement technique, we introduce the functional equations (1.7) and (1.10) to filter noisy images. The functional equation (1.7) computes the arithmetic mean of the diagonal pixels of a 3×3 window and replaces the resulting value at the centre pixel. The Figure 6.1 portrays the performance of the functional equations (1.7) and (1.10) to improve the quality of an image with Gaussian noise.

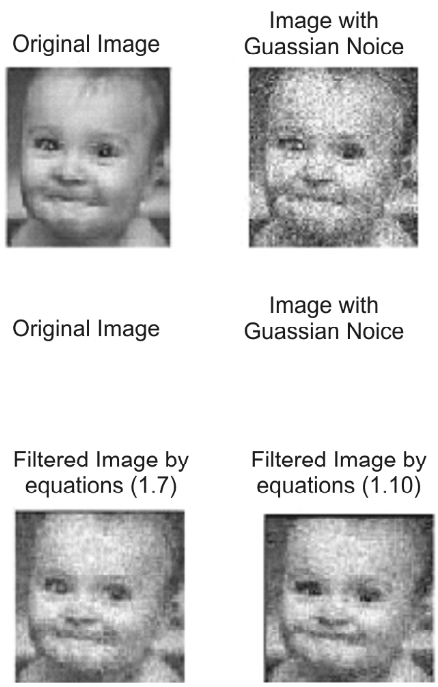

Fig. 6.1.

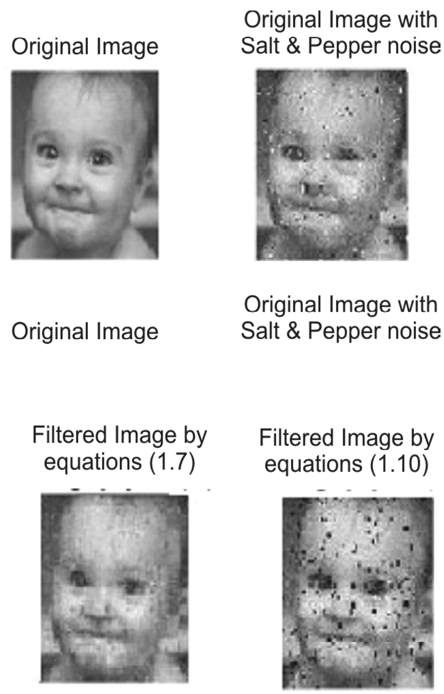

Fig. 6.1.

The functional equation (1.10) computes the harmonic mean of the diagonal pixels of a window and replaces the resulting value at the centre pixel. The Figure 6.2 shows the performance of the functional equations (1.7) and (1.10) to improve the quality of an image with Salt & Pepper noise.

7. CONCLUSION

1. In Section 1, we have introduced two new functional equations (1.7) and (1.10) and studied their general solutions in Sections 5 and 5 respectively.
2. In [2], the solution of the functional equation (1.7) is obtained as sum of additive function, bi-additive function and a constant. But in Section 4, we have improved the solution of equation (1.7) and proved that its improved solution is sum of additive, quadratic and cubic functions.
3. The solutions of the functional equations (1.7) and (1.10) represent the position of the pixels in the image.
4. The functional equation (1.7) works well for filtering Salt & Pepper noise from corrupted image whereas the functional equation (1.10) works well for Gaussian noise as show in Section 6. Moreover, the functional equation (1.10) establishes the concept that the harmonic mean filter is more suitable for salt noise removal but it fails to remove the pepper noise (Refer Figure 6.2).
5. The equation (1.11) dealt in [18] is used as a filter to retain certain basic information of an image whereas the proposed functional equations (1.7) and (1.10) can be used both for retaining basic information of an image and enhancing the quality of image by filtering process.

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