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HYPERGRAPH ON REGULAR SEMIGROUPS

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Abstract: *Some basic concepts related to hypergraphs arising from semigroups are introduced here. The relationship between regular semigroups and hypergraphs are studied. Finite regular semigroups and its hypergraph transversals are characterized.*

Key words: *Hypergraph; Semigroup; Regular semigroup; Hypergraph transversal; Inverse transversal.*

1. INTRODUCTION

The notation and terminology of Berge [1] and Howie [2] are followed. Let $X = \{x_1, x_2,..., x_n\}$ be a finite set. A hypergraph on X is a family $H = (E_1, E_2,\ldots, E_m)$ of subsets of X such that

1. $E_i \neq \emptyset$, $i = 1, 2,\ldots, m$,
2. $\bigcup_{i=1}^{m} E_i = X$.

The elements $x_1, x_2,\ldots, x_n$ of X are called vertices and the sets $E_1, E_2,\ldots, E_m$ are the edges of the hypergraph. A simple hypergraph is a hypergraph $H = (E_1, E_2,\ldots, E_m)$ such that $E_i \subset E_j \Rightarrow j$. A simple graph is a simple hypergraph each of whose edges has cardinality 2. A multigraph is a hypergraph in which each edge has cardinality ≤ 2. A hypergraph H may be drawn as a set of points representing the vertices. The edge E_j is represented by a continuous curve joining the two elements if $|E_j| = 2$, by a loop if $|E_j| = 1$ and by a simple closed curve enclosing the elements if $|E_j| \geq 3$. Let $H = (E_1, E_2,\ldots, E_m)$ be a hypergraph on a set X. For a set $J \subset \{1, 2, 3,\ldots, m\}$, we call the family $H' = \{E_j : j \in J\}$ the partial hypergraph generated by the set J. The set of vertices of H' is a nonempty subset $\cup_{j\in J} E_j$ of X. Two vertices u and v are adjacent in H if their exits an edge of H that contains both u and v; nonadjacent if they are not adjacent. Let $H = (E_1, E_2,\ldots, E_m)$ be a hypergraph on a set X. A set $T \subset X$ is a transversal of H if it meets all the edges, that is to say: $T \cap E_i \neq \emptyset$ $(i = 1,2,,\ldots,m)$. The family of minimal transversals of H constitutes a simple hypergraph on X called the transversal hypergraph of H, and denoted by $Tr\ H$. If the hypergraph is a simple graph G, a set S is stable if it contains no edge,

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that is, if its complement $X - S$ meets all the edges of G. Thus, $Tr\ G = \{X - S : S$ is a maximal stable set of $G\}$. By a groupoid (S, μ) we shall mean a non-empty set S on which a binary operation μ is defined. That is to say, we have a mapping $\mu: S \times S \to S$. We shall say that (S, μ) is a semigroup if μ is associative, i.e. if $(\forall\, x, y, z \in S)\ ((x, y)\mu, z) = (x,(y, z)\mu)\mu$. If $(S, .)$ is a semigroup, then a non-empty subset T of S is called a subsemigroup of S if it is closed with respect to multiplication, i.e. if $(\forall\, x, y \in T)\ xy \in T$. Let S be a semigroup. An element $x \in S$ is called regular if there exists $y \in S$ such that $x = xyx$. A semigroup is called regular if every element of S is regular. Let $a \in S$. An element $b \in S$ is called an inverse of a if $a = aba$ and $b = bab$. Denote $V(x) = \{y \in S : y$ is an inverse of $x\}$ and $W(x) = \{y \in S : x = xyx\}$. Then it follows that $V(x) \subseteq W(x)$. The following lemma is trivial.

Lemma 1.1 If S is a regular semigroup, then $W(x) \neq \emptyset$ for any x.

By definition we have $W(x) \subseteq S$ and $V(x) \subseteq S$ and

$$\cup_{x \in S} W(x) \subseteq S$$

and

$$\cup_{x \in S} V(x) \subseteq S.$$

This leads to the following lemma.

Lemma 1.2 If S is a finite regular semigroup, then

$$\cup_{x \in S} W(x) \cup_{x \in S} V(x) = S.$$

Proof. If $x \in S$ then there exists at least one $y \in S$ such that $x = xyx$.

Therefore,

$$(yxy)x(yxy) = yxy$$

and

$$x(yxy)x = xyx = x$$

so that $x \in V(yxy)$.

Hence $x \in W(yxy)$. Therefore

$$x \in \cup_{x \in S} W(x)$$

so that

$$\cup_{x \in S} W(x) = S$$

So also

$$x \in \cup_{x \in S} V(x)$$

and

$$\cup_{x \in S} V(x) = S.$$

In view of Lemma 1.1 and Lemma 1.2, we have the following theorem.

Theorem 1.1 If S is a finite regular semigroup then, $\{W(x): x \in S\}$ is a hypergraph on S.

2. INVERSE TRANSVERSAL

A semigroup S is called an inverse semigroup if every a in S possesses a unique inverse, i.e. if there exists a unique element a^{-1} in S such that $aa^{-1} a = a$, $a^{-1} aa^{-1} = a^{-1}$. Such a semigroup is certainly regular, but not every regular semigroup is an inverse semigroup: a rectangular band, in which every element is an inverse of every other element, is an obvious example. Let S be a semigroup. A subsemigroup T of S is called a transversal if it meets $W(x)$ for all $x \in S$. An inverse semigroup T' of S is called an inverse transversal if it meets $V(x)$ for all $x \in S$.

Theorem 2.1 Every transversal T of a regular semigroup S is a transversal of the hypergraph on T.

Proof. Since T is a transversal o the semigroup S, it follows that T is a subsemigroup of S and

$$T \cap V(x) \neq \emptyset \text{ for all } x \in S$$

Let $E_x = E \cap V(x)$. Then $E_x \neq \emptyset$.

Also

$$\cup_{x \in S} (T \cap W(x)) = T \cap (\cup_{x \in S} W(x)) = T \cap S = T$$

Similarly,

$$\cup_{x \in S} (T \cap V(x)) = T \cap (\cup_{x \in S} V(x)) = T \cap S = T.$$

In either case

$$\cup_{x \in S} E_x = T.$$

Hence $H = \{E_x : x \in S\}$ is a hypergraph on T and $T \cap E_x \neq \emptyset$.

That is, T meets all edges of the hypergraph H. So T is a hypergraph transversal. This completes the proof.

Theorem 2.2 Every inverse transversal T' of a regular semigroup S is an inverse transversal of the hypergraph on T'.

We leave the proof since it is similar to theorem 2.1.

We illustrate the above concepts by taking the following example.

Example 2.1 1 Consider Z_{15} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14}. Then Z_{15} is a regular semigroup under multiplication modulo 15 follows from the multiplication table.

$\times_{15}$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	0	2	4	6	8	10	12	14	1	3	5	7	9	11	13
3	0	3	6	9	12	0	3	6	9	12	0	3	6	9	12
4	0	4	8	12	1	5	9	13	2	6	10	14	3	7	11
5	0	5	10	0	5	10	0	5	10	0	5	10	0	5	10
6	0	6	12	3	9	0	6	12	3	9	0	6	12	3	9
7	0	7	14	6	13	5	12	4	11	3	10	2	9	1	8
8	0	8	1	9	2	10	3	11	4	12	5	13	6	14	7
9	0	9	3	12	6	0	9	3	12	6	0	9	3	12	6
10	0	10	5	0	10	5	0	10	5	0	10	5	0	10	5
11	0	11	7	3	14	10	6	2	13	9	5	1	12	8	4
12	0	12	9	6	3	0	12	9	6	3	0	12	9	6	3
13	0	13	11	9	7	5	3	1	14	12	10	8	6	4	2
14	0	14	13	12	11	10	9	8	7	6	5	4	3	2	1

Here

$W(0) = \{0\}$, $W(1) = \{1\}$, $W(2) = \{8\}$, $W(3) = \{2,7,12\}$, $W(4) = \{4\}$,
$W(5) = \{2,5,8,11,14\}$, $W(6) = \{1,6,11\}$, $W(7) = \{13\}$, $W(8) = \{2\}$, $W(9) = \{4,9,14\}$,
$W(10) = \{1,4,7,10,13\}$, $W(11) = \{11\}$, $W(12) = \{3,8,13\}$, $W(13) = \{7\}$,
$W(14) = \{14\}$.

Then it follows that $W(x) \neq$ and $\cup\{W(x): x \in S\} = Z_{15}$.

Hence $H = \{W(x): x \in S\}$ is a hypergraph on Z_{15}. Considering $W = \cup \{W(x): W(x) = 1\}$ we see that $W = \{0, 1, 2, 4, 7, 8, 11, 13, 14\}$ is a subsemigroup of Z_{15} and W intersects all $W(x)$ so that W is a hypergraph transversal of the hypergraph on Z_{15} and is a semigroup transversal.

References

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[2] J. M. Howie, An Introduction to Semigroup Theory, University of St Andrews, Scot lane, 1976, Academic Press.