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A NOTE ON PERFECT ORDER SUBSETS CONJECTURE

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Abstract: In this paper, we review the Perfect Order Conjecture posed by Carrie E Finch and Lenny Jones and. This structured review will be helpful for research.

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INTRODUCTION

1. PERFECT ORDER SUBSETS CONJECTURE

Let G be a finite group and x be an element of G. The order subset of G determined by x is defined as the set of all elements in G with the same order as x. A finite group is said to have perfect order subsets (POS) if the number of elements of each given order can divide the order of the group. The study of such groups was initiated by Finch and Jones [2002]. They demonstrated several methods for the construction of finite abelian groups having perfect order subsets and also established a curious connection between such groups and Fermat numbers. The article concluded with the observation that the symmetric group S_3 has perfect order subsets, the only known example of a minimal POS group that contains a noncyclic Sylow p-subgroup of odd order is $(Z_2)^{11} \times Z_3 \times Z_5 \times (Z_{11})^2 \times Z_{23} \times Z_{89}$ and provided the following open questions.

Question 1.1. Are there nonabelian groups other than S_3 that have perfect order subsets?

Question 1.2. Are there only finitely many minimal POS groups that contain non-cyclic Sylow p-subgroups of odd order?

Question 1.3. If G has perfect order subsets and some odd prime p divides $|G|$, then is it true that $|G|$ is divisible by 3?

Finch and Jones [2003, 2004] have extended their study for nonabelian groups and they have obtained some interesting properties for such groups. Also, they proved that there are infinitely many nonabelian groups with perfect order subsets. The authors conjectured that for $n \geq 4$, A_n and S_n do not have perfect order subsets. In addition, all values of q are determined such that the special linear group, $SL_2(q)$, has perfect order subsets. In contrast, each of the simple groups $PSL_2(q)$ possesses an order subset that is not perfect; and so Finch-Jones posed the following conjectures:

Question 1.4. If G has perfect order subsets and $|G|$ is not a power of 2, then 3 divides .

Question 1.5. For $n \geq 4, A_n$ and S_n do not have perfect order subsets.

Question 1.6. If *G* has perfect order subsets and *G* is nonabelian, then *G* is not simple.

Many authors furnished different examples and counter examples for each of these conjectures. Foote and Reist [2013] verified the Perfect Order Subset Conjecture for Simple Groups for all but one family of finite simple groups. Specifically, for each nonabelian finite simple group G there is some N such that the cardinality of the nonempty subset of all elements of order N in *G* does not divide the order of *G*, unless *G* is an orthogonal group of plus type in dimension 4*n*, for some $n \geq 2$.

Theorem 1.1. (Foote and Reist [2013]) If G is any finite simple group other than $O_{2n}^{+}(q)$ with n even and ≥ 4, then the POS Conjecture is true for *G*.

Libera and Tlucek [2003] considered a few examples of finite non-abelian group with two generators dihedral, quaternion, semi-dihedral, and quasi-dihedral groups and detemined that only the dihedral groups can be POS groups. In his paper, Das [2009] studied some of the properties of arbitrary POS-groups, and constructed a couple of new families of nonabelian POS-groups. Also, he established the facts that there are infinitely many nonabelian POS-groups other than the symmetric group S_3, and that if a POS-group has its order divisible by an odd prime then it is not necessary that 3 divides the order of the group. It was proved that the alternating group $A_n, n \geq 3$ is not a POS-group.

Given a positive integer n, let C_n denote the cyclic group of order n. The couple families of nonabelian POS-groups constructed in the following theorem serve as counter examples to the first and the third question posed by Finch and Jones [2002].

Theorem 1.2. (Das [2009]) Let p p be a Fermat's prime. Let α, β be two positive integers such that $2^{\alpha} \geq p-1$. Then there exists a homomorphism $\theta: C_{2^{\alpha}} \rightarrow Aut(C_{p^{\beta}})$ such that the semidirect product $C_{2^{\alpha}} \times_{\theta} Aut(C_{p^{\beta}})$ is a nonabelian POS group.

Das settled the Conjecture of Finch and Jones [2003] regarding An using the following theorem.

Theorem 1.3. (Das [2009]) For $n \geq 3$, the alternating group A_n An is not a POS group.

Tuan and Hai [2010] proved that for any $n \geq 4$, the symmetric group S_n is not POS-group. Taun and Hai answered the Conjecture by Finch and Jones [2003] using the conclusions proved by Das. Shen [2010] studied the structure of POS groups of order $2m$ with $(2.m)=1$, and confirmed a conjecture of Das's. Jones and Toppin [2011] provided a new proof to one of these questions and evidence to support answers to the other two questions. Ford et al. [2012] proved that if G is abelian, then such a group has order divisible by 3 except in the case $G = Z/(2^k Z)$.

Al-Hasanat et al. [2014] derived the order classes of dihedral groups by deducing a formula. Applying this formula, GAP software was used to find the order classes of a given dihedral group. The special cases of dihedral groups, namely, the groups of order $2(p^m)$ with $p=2$ or 3 will give nonabelian groups other than S_3 that have perfect order subsets. Benesh [2016] answered two conjectures by providing the counter example of $Z_4 \times Z_{3.5^n}, n \geq 1$ with the inversion action.

Ford et al. [2012] used sieve techniques to prove a result which implies that Z_2 is the only minimal POS having order not divisible by 3. They conjecture that Z_2 and $(Z_2)^2 \times Z_3$ are the only such groups.

Question 1.7. If G is a minimal POS group, and 5 and 7 do not divide $|G|$, then G is isomorphic to either or Z_2 or $(Z_2)^2 \times Z_3$.

Fulkerson [2015] proved that there are only three known minimal abelian POS groups having order not divisible by 5 and shown that they are the only such groups G for which $|G| < 6.62 \times 10^{336}$. Fulkerson [2016] showed that if G is a minimal abelian perfect order subset group with $|G|$ divisible by neither 5 nor 7 and with G isomorphic to neither Z_2 or $(Z_2)^2 \times Z_3$, then $|G| > 10^{10^7}$. Shen et al. [2013] studied the structure of POS-groups with some cyclic Sylow subgroups and posed the following questions.

Question 1.8. Let G be a POS-group and $p \in \pi(G)$. Then satisfies one of following conditions:

(a) G has a cylic Sylow p-subgroup or a generalized quaternion Sylow 2-group;

(b) G has a normal p-complement;

(c) G has a normal Sylow p-subgroup.

We pose a conjecture to close this note.

Question 1.9.

For all Fermat prime p, $Aut(D_p)$ is a nonabelian POS group whose order is not divisible 3.

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