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η-RICCI SOLITONS ON LORENTZIAN α-SASAKIAN MANIFOLDS

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Abstract

The present paper deals with the study of η-Ricci solitons on Lorentzian α-Sasakian manifolds satisfying certain curvature conditions like Ricci-semisymmetry, $S \cdot R = 0$, Einstein-semisymmetry, partially Ricci-pseudosymmetry, projectively Ricci-semisymmetry and projectively Ricci-pseudosymmetry. We prove the existence of η-Ricci solitons on Lorentzian α-Sasakian manifold ($^M n$, φ, ξ, η, g) if the Ricci curvature satisfies one of the previous conditions. Further, we show that (M^n, g) is Einstein.

Key words*: Lorentzian α-Sasakian manifold, η-Ricci solitons, Ricci-semisymmetric manifold, Projective curvature tensor.*

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1. INTRODUCTION

From the last few years, the Ricci and other geometric flows [1] have been an interesting research topic in the field of Mathematics as well as Physics. The concept of Ricci flow was first introduced by Hamilton [7] in 1982. This concept was developed to answer Thurston's geometric conjecture which says that each closed three manifolds admits a geometric decomposition. Hamilton [8] also classified all compact manifolds with positive curvature operator in dimension four. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2\,Ricg,$$

on a compact Riemannian manifold M with Riemannian metric g and $Ricg$ is Ricci curvature tensor, t is time. A self-similar solution to the Ricci flow is called a Ricci soliton [9] if it moves only by a one parameter family of diffeomorphism and scaling. Therefore A Ricci soliton is a natural generalization of an Einstein metric (i.e The Ricci tensor S is a constant multiple of g) and is defined on a Riemannian manifold (M, g) by

$$(L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.1}$$

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where S is the Ricci tensor associated to g. In the context of paracontact geometry, the study of η-Ricci solitons were first considered by Blaga [2] and he gave a sufficient condition for the existence of an η-Ricci soliton on a para-Kenmotsu manifold. Later the same author treated this notion for Lorentzian para-Sasakian manifold [3]. Recently, Prakasha and Hadimani [10] have shown the non-existence of certain geometric conditions on para-Sasakian η-Ricci solitons.

Motivated by the above ideas, in this paper we plan to study η-Ricci solitons on Lorentzian α-Sasakian manifolds admitting some geometric conditions. The paper is organized as follows: In section 2, we review some preliminary results. In section 3, we give the existence of η-Ricci solitons on Lorentzian α-Sasakian manifolds in our settings. In section 4, we study existence of η-Ricci solitons on Lorentzian α-Sasakian manifold satisfying Ricci-semisymmetry condition. In sections 5, 6 and 7, we consider α-Ricci solitons on Lorentzian α-Sasakian manifold satisfying $S \cdot R = 0$, Einstein semisymmetry and partially Ricci-pseudosymmetry conditions respectively. Sections 8 and 9 contain the study of η-Ricci solitons on projectively Ricci-semisymmetry and projectively Ricci-pseudosymmetry Lorentzian α-Sasakian manifold.

2. PRELIMINARIES

A differential manifold of dimension n is called a Lorentzian _-Sasakian manifold if it admits a (1, 1)-tensor field φ, a contravariant vector field ξ, a covariant vector field η and the Lorentzian metric g which satisfy [11]

$$\varphi\xi = 0,\ \eta(\xi) = -1,\ \eta(\varphi X) = 0,\ g(X, \xi) = \eta(X), \tag{2.1}$$

$$\varphi^2 X = X + \eta(X)\xi, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\,\eta(Y) \tag{2.3}$$

for all $X, Y \in TM$.

Also Lorentzian -Sasakian manifold satisfy [11]

$$\nabla_X \xi = -\alpha\varphi X, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\varphi X, Y), \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and $\alpha \in R$.

A Lorentzian α-Sasakian manifold (M^n, g) is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\,\eta(Y) \tag{2.6}$$

for any vector field X, Y, where a, b are smooth functions on M.

Further, on a Lorentzian α-Sasakian manifold M the following relations hold [11]:

$$\eta(R(X, Y) Z) = \alpha^2[g(Y, Z)X - g(X, Z)Y], \tag{2.7}$$

$$R(\xi, X)Y = \alpha^2 [g(X, Y)\xi - \eta (Y) X], \tag{2.8}$$

$$R(X, Y)\xi = \alpha^2 [\eta(Y) X - \eta(X)Y], \tag{2.9}$$

$$S(X, \xi) = (n - 1) \alpha^2 \eta(X), \tag{2.10}$$

$$Q\eta = (n - 1) \alpha^2 \xi, \tag{2.11}$$

$$S(\xi, \xi) = -(n - 1) \alpha^2, \tag{2.12}$$

$$(\nabla_X\varphi) (Y) = \alpha [g(X, Y) - \eta(X)\eta (Y), \tag{2.13}$$

for any vector fields X, Y, $Z \in TM$. Here R denotes Riemannian curvature tensor and S is the Ricci tensor defined by $S(X, Y) = g(QX, Y)$, where Q is the Ricci operator.

We use the following definitions for our results:

Definition 2.1. An n-dimensional Lorentzian α-Sasakian manifold (M^n, g) is called Ricci-semisymmetric if $R \cdot S = 0$.

Definition 2.2. An n-dimensional Lorentzian α-Sasakian manifold (M^n, g) is called projectively Ricci-semisymmetric if $P(\_,X) \cdot S = 0$.

Definition 2.3. An n-dimensional Lorentzian _-Sasakian manifold (M^n, g) is called Einstein-semisymmetric if $R \cdot E = 0$. Where is the Einstein tensor defined by

$$E(X,Y) = S(X,Y) - \frac{r}{n} g(X,Y), \tag{2.14}$$

Where S is the Ricci tensor and r is the scalar curvature.

Definition 2.4. An n-dimensional Lorentzian α-Sasakian manifold (M^n, g) is called partially Riccipseudosymmetric if and only if the relation

$$R \cdot S = f(p)\, Q(g, S), \tag{2.15}$$

holds on the set $A = \{x \in M : Q(g, S) \neq 0 \text{ at } x\}$. where $f \in C^\infty(M)$ for $p \in A$. $R \cdot S$, $Q(g, S)$ and $(X \Lambda_g Y)$ are respectively defined as

$$(R(X, Y) \cdot S (U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V), \tag{2.16}$$

$$Q(g, S) = (X(\Lambda_g Y) \cdot S) (U, V), \tag{2.17}$$

$$(X(\Lambda_g Y) = g(Y, Z) X - g(X, Z)Y, \tag{2.18}$$

for all X, Y, U and $V \in TM$.

Definition 2.5. An -dimensional Lorentzian α-Sasakian manifold $(M^n, g))$ is called projectively Ricci-pseudosymmetric if the tensors $P \cdot S$ and $Q(g, S)$ are linearly dependent. This is equivalent to

$$P \cdot S = LSQ\,(g, S), \tag{2.19}$$

holding on the set $U_S = \{x \in M : P \neq 0 \text{ at } x\}$. where L_S is some function on U_S.

3. η-RICCI SOLITONS ON LORENTZIAN α-SASAKIAN MANIFOLDS

Let $(M^n, \varphi, \xi, n, g)$ be an n-dimensional Lorentzian α-Sasakian manifold and (g, ξ, λ, μ) be an η-Ricci soliton on (M^n, g). Then the relation (1.2) implies that

$$(L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\,\eta\,(Y) = 0.$$

i.e

$$2S(X, Y) = -(L_V g)(X, Y) - 2\lambda g(X, Y) - 2\mu\eta\,(X)\,\eta\,(Y). \tag{3.1}$$

On a Lorentzian -Sasakian manifold M and from (2.4), we have

$$(L_V g)(X, Y) = -2\alpha g\,(\varphi X, \varphi Y). \tag{3.2}$$

By substituting (3.2) in (3.1), we get

$$S(X, Y) = -\alpha g\,(\varphi X, \varphi Y) - \lambda g(X, Y) - \mu\eta(X)\,\eta(Y). \tag{3.3}$$

Putting $Y = \xi$ in (3.3), we have

$$S(X, \xi) = (\mu - \lambda)\,\eta(X) \tag{3.4}$$

But, in [5] the authors proved that on a n-dimensional Lorentzian α-Sasakian manifold $(M^n, \varphi, \xi, \eta, g)$, the Ricci tensor field satisfies

$$S(X, \xi) = (n - 1)\alpha^2\eta\,(X). \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$(\mu - \lambda) = (n - 1)\alpha^2. \tag{3.6}$$

4. η-RICCI SOLITONS ON RICCI-SEMISYMMETRIC LORENTZIAN α-SASAKIAN MANIFOLD

Let us consider Ricci-semisymmetric Lorentzian α-Sasakian manifold. Then from definition 2.1, we have

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)\,Z) = 0 \tag{4.1}$$

for any $X, Y, Z \in TM$.

Let a Ricci-semisymmetric Lorentzian α-Sasakian manifold (M^n, g) admits an η-Ricci soliton (g, ξ, λ, μ) then relation (3.3) holds. By making use of (3.3) in (4.1), we get

$$\alpha[R(\xi, X)\,Y, \varphi Z) + g(R(\xi, X)\,Z, \_Y)] + \alpha[g(R(\xi, X)\,Y, Z) + g\,(Y, R(\xi, X)\,Z]$$
$$+ \mu[\eta\,(R(\xi, X)\,Y)\,\eta\,(Z) + \eta(Y)\,\eta\,(R(\xi, X)\,Z)] = 0 \tag{4.2}$$

In view of (2.8), (4.2) reduces to

$$\alpha g(\eta(Y)Z + \eta(Z)\,Y, \varphi X) + \mu[\eta(Y)g(X, Z) + \eta(Z)g(X, Y) + 2\eta(X)\,\eta(Y)\,\eta(Z)] = 0 \tag{4.3}$$

Setting $Z = \xi$ in (4.3), we get

$$\alpha g(\varphi X, Y) + \mu g(\varphi X, \varphi Y) = 0 \tag{4.4}$$

Replacing Y by φY in (4.4) and making use of (2.1), we obtain

$$\alpha g(\varphi X, \varphi Y) + \mu g(\varphi X, Y) = 0. \tag{4.5}$$

Adding (4.4) and (4.5), we get

$$(\lambda + \mu)\,[g(\varphi X, Y) + g\,(\varphi X, \varphi Y)] = 0. \tag{4.6}$$

It follows from above equation that $\mu = -\alpha$. Also from the relation (3.6), we get $\lambda = -(n-1)\,\alpha^2 - \alpha$.

Therefore, we can state the following:

Theorem 4.1. If (M^n, g) is a Ricci-semisymmetric Lorentzian α-Sasakian manifold and (g, ξ, λ, μ) be an η-Ricci soliton on (M^n, g) then $\mu = -\alpha$ and $\lambda = -(n-1)\alpha^2 - \alpha$.

By virtue of (3.6) and (4.4), equation (3.3) becomes:

$$S(X, Y) = (n-1)\,\alpha^2\,g(X, Y) \tag{4.7}$$

Hence, we state the following corollary:

Corollary 4.1. If (M^n, g) is a Ricci-semisymmetric Lorentzian α-Sasakian manifold and (g, ξ, λ, μ) is an η-Ricci soliton on (M^n, g) then (M^n, g) is Einstein manifold.

5. η-RICCI SOLITONS ON LORENTZIAN α-SASAKIAN MANIFOLD SATISFYING $S \cdot R = 0$

Using the following equations

$$(S(\xi, X) \cdot R)(Y, Z)W = ((\xi \wedge_S X) \cdot R)(Y, Z)W$$

$$= (\xi \wedge_S X)R(Y, Z)W + R((\xi \wedge_S X)Y, Z)W + R(Y, (\xi \wedge_S X)Z)W + R(Y, Z)(\xi \wedge_S X)W \tag{5.1}$$

where the endomorphism $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y. \tag{5.2}$$

Using (5.2) in (5.1), we have

$$S(\xi, X) \cdot R)(Y, Z)W = S(X, R(Y, Z)W)\_ - S(\xi, R(Y, Z)W)X$$

$$+ S(X, Y)R(\xi, Z)W - S(\xi,Y) R(X,Z)W + S(X, Z)R(Y, \xi)W$$

$$- S(\xi, Z)R(Y,X)W + S(X,W)R(Y,Z)\xi - S(\xi, W)R(Y,Z)X. \qquad (5.3)$$

Let (M^n, g) be a Lorentzian α-Sasakian manifold satisfies $S(\xi, X) \cdot R = 0$ then above equation reduces to

$$S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W - S(\xi, Y)R(X, Z)W$$

$$+ S(X, Z)\mathrm{R}(\mathrm{Y}, \xi)W - S(\xi, Z)R(Y, X)W + S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X = 0. \qquad (5.4)$$

Taking inner product with ξ, the relation (5.4) becomes:

$$S(X, R(Y, Z)W) - S(\xi, R(Y, Z)W)\_(X) + S(X, Y)\eta(R(\xi, Z)W)$$

$$- S(\xi, Y)\eta(R(X, Z)W) + S(X, Z)\eta(R(Y, \xi)W) - S(\xi, Z)\eta(R(Y, X)W)$$

$$+ S(X, W)\eta(R(Y, Z)\xi) - S(\xi, W)\, \eta(R(Y, Z)X) = 0. \qquad (5.5)$$

Replacing the expression of S from (3.3), we get

$$2(\lambda - \mu)\alpha^2 [g(Z, W)\eta(X)\eta(Y) - g(Y, W)\eta(X)\eta(Z)] + \lambda\alpha^2 [g(X, Y)g(Z, W)$$

$$- g(X, Z)g(Y, W)] + \mu\alpha^2 [g(X, Y)\eta(Z)\,\eta(W) - g(X, Z)\,\eta(Y)\,\eta(W)$$

$$+ 2g(Z, W)\eta(X)\eta(Y) - 2g(Y, W)\eta(X)\eta(Z)] - g(\alpha\varphi X + \lambda X, R(Y, Z)W)$$

$$+ \alpha^3 g(\varphi X, Y)[g(Z, W) + \eta(Z)\eta(W)] - \alpha^3 g(\varphi X, Z)[g(Y, W) + \eta(Y)\eta(W)]. (5.6)$$

For $Z = \xi$ and $W = \xi$ in (5.6), we have

$$\alpha g(\varphi X, Y) + \mu[g(X, Y) + \eta(\mathrm{X})?(\mathrm{Y})] = 0. \qquad (5.7)$$

As in the proof of Theorem 4.1 we then $\mu = -\alpha$ and $\lambda = -(n-1)\alpha^2 - \alpha$.

Therefore, we have the following:

Theorem 5.2. If (M^n, g) is a Lorentzian α-Sasakian manifold satisfying $S \cdot R = 0$ and (g, ξ, λ, μ) be an η-Ricci soliton on (Mn, g) then $\mu = -\alpha$ and $\lambda = -(n-1)\alpha^2 - \alpha$.

Again, from the relations (3.3) and (3.6) and (5.7), we get

$$S(X, Y) = (n - 1)\, \alpha^2\, g(X, Y). \qquad (5.8)$$

Hence we state the following corollary:

Corollary 5.2. If (M^n, g) is a Lorentzian α-Sasakian manifold satisfying $S \cdot R = 0$ and (g, ξ, λ, μ) is an η-Ricci soliton on (M^n, g) then (M^n, g) is Einstein manifold.

6. η-RICCI SOLITONS ON EINSTEIN-SEMISYMMETRIC LORENTZIAN α-SASAKIAN MANIFOLD

Let (M^n, g) be an Einstein-semisymmetric Lorentzian α-Sasakian manifold. Then by definition 2.2, we have

$$E(R(\xi, X)Y, Z) + E(Y, R(\xi, X)Z) = 0. \tag{6.1}$$

Plugging (2.14) in (6.1), we get

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) - \frac{r}{n}\ [g(R(\xi, X)Y, Z) + g(Y, R(\xi, X)Z)] = 0. \tag{6.2}$$

Let an Einstein-semisymmetric Lorentzian α-Sasakian manifold (M^n, g) admits an η-Ricci soliton (g, ξ, λ, μ). Then by virtue of (3.3), it follows that

$$\alpha[g(\varphi Z, R(\xi, X)Y) + g(\varphi Y, R(\xi, X)Z)] + \left(\lambda + \frac{r}{n}\right)[g(R(\xi, X)Y, Z) + g(Y, R(\xi, X)Z)]$$
$$+ \mu[\eta(R(\xi, X)Y)\,\eta(Z) + \eta(Y)\eta(R(\xi, X)] = 0. \tag{6.3}$$

In view of (2.8), (6.3) reduces to

$$\alpha[g(X, \varphi Z)\eta(Y) + g(X, \varphi Y)\eta(Z)] + \mu[g(X, Y)\eta(Z) + g(X, Z)\eta(Y)$$
$$+ 2\eta(X)\eta(Y)\eta(Z)] = 0. \tag{6.4}$$

Setting $Z = \xi$ in (6.4), we get

$$\alpha g(\varphi X, Y) + \mu[g(X, Y) + \eta(X)\ \eta(Y)] = 0 \tag{6.5}$$

Again as in the proof of Theorem 4.1 we obtain $\mu = -\alpha$ and $\lambda = -(n-1)\alpha^2 - \alpha$.

Hence, we leads to the following:

Theorem 6.3. If (M^n, g) is an Einstein-semisymmetric Lorentzian α-Sasakian manifold and (g, ξ, λ, μ) be an η-Ricci soliton on (M^n, g) then $\mu = -\alpha$ and $\lambda = -(n-1)\alpha^2 - \alpha$.

From the relations (3.3) and (3.6) and (6.5), we get

$$S(X, Y) = (n-1)\ \alpha^2\ g(X, Y) \tag{6.6}$$

This leads to the following corollary:

Corollary 6.3. If (M^n, g) is an Einstein-semisymmetric Lorentzian α-Sasakian manifold and (g, ξ, λ, μ) is an η-Ricci soliton on (M^n, g) then (M^n, g) is Einstein manifold.

7. η-RICCI SOLITONS ON PARTIALLY RICCI-PSEUDOSYMMETRIC LORENTZIAN α-SASAKIAN MANIFOLD

Let (M^n, g) be a partially Ricci-pseudosymmetric Lorentzian α-Sasakian manifold. Then by definition 2.4, we have

$$(R(\xi, X) \cdot S)(Y, Z) = f(p)[((\xi \wedge_g X) \cdot S)(Y, Z)]. \tag{7.1}$$

From equations (2.16) and (2.18), it follows that

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = f(p)[S((\xi \Lambda_g X)Y, Z) + S(Y, (\xi \xi_g X)Z)] \quad (7.2)$$

Let a partially Ricci-pseudosymmetric Lorentzian α-Sasakian manifold (M^n, g) admits a η-Ricci soliton (g, ξ, λ, μ) then the relation (3.3) holds on (M^n, g). By using (3.3) in (7.2), we get

$$\alpha[g(R(\xi, X)Y, \varphi Z) + g(\varphi Y, R(\xi, X)Z)] + \lambda[g(R(\xi, X)Y, Z) + g(Y, R(\xi, X)Z)] + \mu[\eta(R(\xi, X)Y)\,\eta(Z) + \eta(Y)\eta(R(\xi, X)Z)]$$

$$= f(p)[g(X, Y)\,S(\xi, Z) - \eta(Y)\,S(X, Z) + g(X, Z)S(Y, \xi) - \eta(Z)S(X, Y)] \quad (7.3)$$

Using (2.8) and (2.10) in (7.3), we obtain

$$[f(p) - \alpha^2][\alpha \cdot g(\varphi(Y)Z + \eta(Z)Y, \varphi X) + \mu\{g(X, Y)\,\eta(Z) + g(X, Z)\,\eta(Y) + 2\eta(X)\,\eta(Y)\,\eta(Z)\}] = 0. \quad (7.4)$$

Taking $Z = \xi$ in (7.4) and using (2.1), we have

$$[f(p) - \alpha^2][\alpha g(\varphi X, Y) + \mu\{g(X, Y) + \eta(X)\eta(Y)\}] = 0. \quad (7.5)$$

This can be hold only if either

$$f(p) = \alpha^2 \text{ or } \alpha g(\varphi X, Y) + \mu g(\varphi X, \varphi Y) = 0. \quad (7.6)$$

Suppose $f(p) \neq \alpha^2$ then from the above equation, we have

$$\alpha g(\varphi X, Y) + \mu g(\varphi X, \varphi Y) = 0. \quad (7.7)$$

Again as in the proof of Theorem 4.1, we get the following:

Theorem 7.4. If (M^n, g) is a partially Ricci-pseudosymmetric Lorentzian α-Sasakian manifold and (g, ξ, λ, μ) be an η-Ricci soliton on (M^n, g) then $\mu = -\alpha$ and $\lambda = -(n-1)\alpha^2 - \alpha$ provided $f(p) \neq \alpha^2$.

From the relations (3.3) and (3.6) and (7.6), we get

$$S(X, Y) = (n-1)\,\alpha^2\, g(X, Y) \quad (7.8)$$

This leads to the following corollary:

Corollary 6.3. If (M^n, g) is a partially Ricci-pseudosymmetric Lorentzian α-Sasakian manifold and (g, ξ, λ, μ) is an η-Ricci soliton on (M^n, g) then (M^n, g) is Einstein manifold.

8. η-RICCI SOLITONS ON PROJECTIVELY RICCI-SEMISYMMETRIC LORENTZIAN α-SASAKIAN MANIFOLD

In this section we consider a projectively Ricci-semisymmetric Lorentzian α-Sasakian manifold (M^n, g) which admits an η-Ricci soliton (g, ξ, λ, μ). Then the condition $P(X, Y) \cdot S = 0$ implies that

$$S(P(\xi, X)Y, Z) + S(Y, P(\xi, X)Z) = 0. \tag{8.1}$$

for any vector fields $X, Y, Z \in TM$ and P denotes Projective curvature tensor defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{8.2}$$

In view of (3.3), (8.1) reduces to

$$\alpha[g(P(\xi, X)Y, \varphi Z) + g(\varphi Y, P(\xi, X)Z)] + \lambda[g(P(\xi,X)Y, Z) + g(Y, P(\xi, X)Z)]$$
$$+ \mu[\eta(P(\xi, X)Y)\,\eta(Z) + \eta(Y)\eta(P(\xi, X)Z)] = 0. \tag{8.3}$$

Now using (2.8) and (3.3) in (8.2), we obtain

$$\begin{aligned} P(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X] + \frac{1}{n-1}[\alpha g(\varphi X,Y)\xi + \lambda g(X,Y)\xi \\ + \mu\eta(X)\eta(Y)\xi + (\mu - \alpha)\eta(Y)X]. \end{aligned} \tag{8.4}$$

and

$$\eta(P(\xi,X)Y) = \left[-\alpha^2 - \frac{\lambda}{n-1}\right][g(X,Y) + \eta(X)\eta(Y)] - \frac{\alpha}{n-1}g(\varphi X,Y). \tag{8.5}$$

Taking account of (8.4) and (8.5) in (8.3), we get

$$\alpha[g(\varphi X, Y)\eta(Z) + g(\varphi X, Z)\,\varphi(Y)]$$
$$+ \mu[g(X, Y)\,\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)] = 0. \tag{8.6}$$

For $Z = \xi$ in (8.6), we have

$$\alpha g(\varphi X, Y) + \mu[g(X, Y) + \eta(X)\,\eta(Y)] = 0. \tag{8.7}$$

Or equivalently

$$\alpha g(\varphi X, Y) + \mu g(\varphi X, \varphi Y) = 0. \tag{8.8}$$

Therefore again as in the proof of Theorem 4.1, we get the following:

Theorem 7.4. If (M^n, g) is a projectively Ricci-semisymmetric Lorentzian α-Sasakian manifold and (g, ξ, λ, μ) be an η-Ricci soliton on (M^n, g) then $\mu = -\alpha$ and $\lambda = -(n-1)\alpha^2 - \alpha$.

From the relations (3.3) and (3.6) and (8.7), we get

$$S(X, Y) = (n-1)\,\alpha^2\, g(X, Y). \tag{8.9}$$

Therefore we have the following corollary:

Corollary 6.3. If (M^n, g) is a projectively Ricci-semisymmetric Lorentzian α-Sasakian manifold and (g, ξ, λ, μ) is an η-Ricci soliton on (M^n, g) then (M^n, g) is Einstein manifold.

9. η-RICCI SOLITONS ON PROJECTIVELY RICCI-PSEUDOSYMMETRIC LORENTZIAN α-SASAKIAN MANIFOLD

Let (M^n, g) be a projectively Ricci-pseudosymmetric Lorentzian α-Sasakian manifold. Then by definition 2.5, we have

$$(P(\xi, X)\cdot S)(Y, Z) = L_P\,[((\xi \wedge_g X)\cdot S)(Y, Z). \tag{9.1}$$

The above equation can be written as

$$S(P(\xi, X)Y, Z) + S(Y, P(\xi, X)Z) = L_P\,[S((\xi \wedge_g X)Y, Z) + S(Y, (\xi \wedge_g X)Z)]. \tag{9.2}$$

Let a projectively Ricci-pseudosymmetric Lorentzian α-Sasakian manifold (Mn, g) admits an η-Ricci soliton (g, ξ, λ, μ) then the relation (3.3) holds on (M^n, g). Therefore using (3.3) and (2.18) in (9.2), we get

$$\alpha[g(P(\xi, X)Y, \varphi Z) + g(\varphi Y, P(\xi, X)Z)] + \lambda[g(P(\xi, X)Y, Z) + g(Y, P(\xi, X)Z) + \mu[\eta(P(\xi, X)\,Y)\,\eta\,(Z) + \eta(Y)\eta(P(\xi, X)Z)]$$

$$= L_P\,[g(X, Y)\,S(\xi, Z) - \eta(Y)\,S(X, Z) + g(X, Z)S(\xi, Y) - \eta(Z)S(X,Y)]. \tag{9.3}$$

Applying (8.4) and (2.10) in (9.3), we obtain

$$[L_P - \alpha^2]\,[\alpha\{g(X, \varphi Z)\,\eta(Y) + g(X, \varphi Y)\,\eta(Z)\} + \mu\{g(X, Y)\,\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)\}] = 0. \tag{9.4}$$

Putting $Z = \xi$ in (9.4) and using (2.1), we get

$$[L_P - \alpha^2]\,[\alpha g(\varphi X, Y) + \mu\{g(X, Y) + \eta(X)\eta(Y)\}] = 0. \tag{9.5}$$

This can be hold only if either

$$L_P = \alpha^2 \text{ or } \alpha g(\varphi X, Y) + \mu g(\varphi X, \varphi Y) = 0. \tag{9.6}$$

Suppose $L_P \neq \alpha^2$ then equation (9.6) becomes

$$\alpha g(\varphi X, Y) + \mu g(\varphi X, \varphi Y) = 0. \tag{9.7}$$

Again as in the proof of Theorem 4.1 we conclude the following:

Theorem 9.6. If (M^n, g) is a projectively Ricci-pseudosymmetric Lorentzian α-Sasakian manifold and $(g, \varphi, \lambda, \mu)$ be an η-Ricci soliton on (M^n, g) then $\mu = -\alpha$ and $\lambda = -(n-1)\,\alpha^2 - \alpha$ provided $L_P \neq \alpha^2$.

From the relations (3.3) and (3.6) and (9.6), we get

$$S(X, Y) = (n-1)\,\alpha^2\,g(X, Y). \tag{9.8}$$

Therefore, we can state the following corollary:

Corollary 6.3. If (M^n, g) is a projectively Ricci-pseudosymmetric Lorentzian α-Sasakian manifold and (g, ξ, λ, μ) is an η-Ricci soliton on (M^n, g) then (M^n, g) is Einstein manifold.

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