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IMPULSIVE INTEGRO DIFFERENTIAL EQUATION WITH ANTIPERIODIC BOUNDARY CONDITIONS ON TIME SCALES

Krishnaveni.V, Sathiyanathan.K and Arunkumar.D

Abstract: In this paper, we have investigated the existence of solutions for system of impulsive integro-differential equations with antiperiodic boundary conditions on time scales. Existence of solutions are established via Schauder's fixed point and Schaefer's fixed point theorem for operators in a Banach space. An example is given to illustrate the effectiveness of our proposed result.

Key Words*: Integro-differential equations, antiperiodic boundary conditions, Green's function, time scale, 2010 Mathematics Subject Classification: 39A10, 34A40.*

1. INTRODUCTION

The time scale theory presents a structure where, an established result of a general time scale is applied to special cases. If $T = R$ and $T = Z$ then we have the results for differential and difference equations respectively. A great deal of work has been done since 1988 unifying and extending the theories of differential and difference equations. Some basic definitions and theorems on time scales can be found in the standard books [3,4] Impulsive differential equations serve as an essential model to learn the dynamics of processes that are subject to abrupt changes in their states. Alternatively, impulsive control theory has become a awfully main direction in the theory of impulsive differential equations, inspired by their various applications to problems arising in orbital transfer of satellite[9], ecosystems management[6], electrical engineering[12] and so on. Several literatures have been published about existence of solutions for anti periodic boundary value problems for first and second order impulsive differential equations [1, 2, 10, 11, 12], which are essential for complementing the theory of impulsive equations. In recent times, the existence results were extended to anti periodic boundary value problem for second order impulsive differential equations on time scales[13].

Impulsive Integro differential equations have also been considered by many researchers in [5, 7, 8]. Motivated by above, this study consider second order impulsive integro differential equations with antiperiodic boundary value conditions on time scales. First, we use Schauder's and Schaefer's fixed point theorem to study the existence results of the following antiperiodic boundary value problem

Department of Mathematics, SRMV College of Arts and Science, Coimbatore, Tamilnadu, India
E-Mails- krishveni83@gmail.com, sathimanu63@gmail.com, arunkumarmat@gmail.com

$$\left.\begin{aligned} -u^{\Delta\Delta}(t) = h\left(t, u(t), u^{\Delta}(t), \int_0^t g\left(t, s, u(s), u^{\Delta}(s)\right)\Delta s\right), \quad & t \in [0,T]\backslash\Omega := \{t_1, t_2, \dots, t_m\}, \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), & \\ u^{\Delta}(t_k^+) = u^{\Delta}(t_k) + J_k\left(u(t_k), u^{\Delta}(t_k)\right), \quad & k = 1,2,\dots,m, \\ u(0) = -u(\sigma(T)), \quad u^{\Delta}(0) = -u^{\Delta}(\sigma(T)). & \end{aligned}\right\} \tag{1}$$

Secondly, we employ Schaefer's fixed point theorem to investigate the following antiperiodic boundary value problem on time scales:

$$\left.\begin{aligned} u^{\Delta\Delta}(t) = h\left(t, u^{\sigma}(t), u^{\Delta}(t), \int_0^t g\left(t, s, u^{\sigma}(s), u^{\Delta}(s)\right)\Delta s\right), \quad & t \in [0,T]\backslash\Omega := \{t_1, t_2, \dots, t_m\}, \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), & \\ u^{\Delta}(t_k^+) = u^{\Delta}(t_k) + J_k\left(u(t_k), u^{\Delta}(t_k)\right), \quad & k = 1,2,\dots,m, \\ u(0) = -u(\sigma(T)), \quad u^{\Delta}(0) = -u^{\Delta}(\sigma(T)). & \end{aligned}\right\} \tag{2}$$

We assume throughout this paper that $h : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0,T] \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $J_k : \mathbb{R}^n \to \mathbb{R}^n$ are also continuous $k = 1,2,\dots,m)$.

2. PRELIMINARIES

In this section, we present some definitions of time scales, lemmas and theorems before starting our main results.

A time scale is an arbitrary nonempty closed subset of the real numbers. Throughout this paper, we will denote a time scale by the symbol T. And the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in T : s > t\}, \quad \rho(t) = \sup\{s \in T : s < t\} \tag{3}$$

respectively. The point $t \in T$ is called left dense, left scattered, right dense or right scattered if $\rho(t) = t,\ \rho(t) = t,\ \sigma(t) = t \text{ or } \sigma(t) > t$ respectively. Points that are right dense and left dense at the same time are called dense.

We denote $\sigma(\sigma(t))$ by $\sigma^2(t)$.

If T has a left scattered maximum m, define $\mathbb{T}^k := \mathbb{T} - \{m\};$ otherwise, set $\mathbb{T}^k := \mathbb{T}$. The symbols $[a,b], [a,b)$ and so on, denote time scales intervals, for example,

$$[a,b] = \{t \in T : a \leq t \leq b \tag{4}$$

where $a, b \in \mathbb{T}$ with $a < \rho(b)$.

Definition 1. A vector function $f: \mathbb{T} \to \mathbb{R}^n$ is rd-continuous provided that it is continuous at each right dense point in $\mathbb{T}$ and has a left-sided limit at each left dense point in $\mathbb{T}$. The set of rd-continuous functions $f: \mathbb{T} \to \mathbb{R}^n$
will be denoted in this paper by $C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}^n)$.

Definition 2. Assume $f: \mathbb{T} \to \mathbb{R}$ is a function and let $t \in T_k$. Then we define $f^{\Delta}(t)$ to be the number(provided it exists) with the property that given any $\epsilon > 0$ there exists a neighbourhood $\mathcal{U}$ of t (i.e).,$\mathcal{U} = \left(t - \delta, t + \delta\right) \cap \mathbb{T}$ for some $\delta > 0$ such that

$$\left|f\left(\sigma(t)\right) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s|, \ \forall s \in \mathcal{U} \tag{5}$$

We call $f^{\Delta}(t)$ the delta (or Higher) derivative of f at t.

Definition 3. A function $F: \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f: \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \text{ holds } \forall\, t \in T \tag{6}$$

Theorem 1. [existence of antiderivatives] Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by

$$F(t) = \int_{t_0}^{t} f(t)\Delta t \quad for \quad t \in \mathbb{T} \tag{7}$$

is an antiderivative of f.

We assume that for
$f(t_k^+, x, y, z) := \lim_{t \to t_k^+} f(t, x, y, z)$ *and* $f(t_k^-, x, y, z) := \lim_{t \to t_k^-} f(t, x, y, z)$
both exist with

$f(t_k^+, x, y, z) := f(t_k, x, y, z), \mathrm{k} = 1,2, \ldots, \mathrm{m}$. In order to define a solution of (1) and (2), we introduce and denote the Banach space $PC([0,T],\backslash\mathbb{R}^n)$ by

$$PC([0,T],\backslash\mathbb{R}^n) := \left\{\begin{matrix} u: [0,T] \to R^n, u \in C([0,T]\backslash\Omega, R^n), \\ \text{u is left continuous at } t = t_k, \\ \text{the right} - \text{hand limit } u(t_k^+) \text{exists} \end{matrix}\right\} \tag{8}$$

with the norm $\|u\|_{PC} := \sup_{t\in[0,T]}\|u(t)\|$ where $\|\cdot\|$ is the usual Euclidean norm and $\langle .,. \rangle$ will be the Euclidean inner product.

In a similar fashion to the above, define and denote the Banach space $PC^1([0,T],\backslash\mathbb{R}^n)$ by

$$PC^1([0,T],\backslash\mathbb{R}^n) := \left\{ \begin{matrix} u \in PC([0,T],\backslash\mathbb{R}^n), u^\Delta(t) \in C([0,T]\backslash\Omega, R^n), \\ \text{the limits } u^\Delta(t_k^-), u^\Delta(t_k^+) \text{exists with} \\ u^\Delta(t_k^-) = u^\Delta(t_k^+) \end{matrix} \right\} \quad (9)$$

with the norm $\|u\|_{PC^1} := \sup_{t\in[0,T]}\{\|u(t)\|_{PC}, \|u^\Delta(t)\|_{PC}\}$

The following fixed point theorem is our main tool to prove the existence of at least one solution to (2).

Theorem 2. (Schaefer's fixed point theorem) Let X be a Banach space and let $A: X \to X$ be a Completely continuous operator. Then either:

1. the operator equation $x = \lambda Ax$ has a solution for$\lambda = 1$. or

2. the set $S := \{x \in X, x = \lambda Ax, \lambda \in]0,1[\}$ is unbounded.

Theorem 3. Let $a \in \mathbb{T}^k, b \in \mathbb{T}$ and assume $f: \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^k$ with $t > a$. Also assume that $f^\Delta(t,.)$ is rd-continuous on $[a,\sigma(t)]$. Suppose that for each $\epsilon > 0$ there exist a neighbourhood U of t, independent of $\tau \in [a,\sigma(t)]$, such that

$| f(\sigma(t),\tau) - f(s,\tau) - f\,\Delta\,(t,\tau)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|\ \forall s \in U$

where f^Δ denotes the derivative of f with respect to the first variable. Then

1. $g(t) = \int_a^t f(t,\tau)\Delta\tau$ implies $g^\Delta(t) = \int_a^t f^\Delta(t,\tau)\Delta\tau + f(\sigma(t),t)$
2. $h(t) = \int_t^b f(t,\tau)\Delta\tau$ implies $g^\Delta(t) = \int_t^b f^\Delta(t,\tau)\Delta\tau + f(\sigma(t),t)$

Lemma 1. (Compact result) Assume that {f n } n∈N is a function sequence on J such that

1. $\{f_n\}_{n\in N}$ is uniformly bounded on J,

2. $\{f_n^\Delta\}_{n\in N}$ is uniformly bounded on J.

Then there is a subsequence of $\{f_n\}_{n\in N}$ converges uniformly on J.

Theorem 4. (Schauder's fixed point theorem) Let K be a closed convex subset of the banach space X. Suppose $f : K \to K$ and f is compact (i.e., bounded sets in K are mapped into relatively compact sets). Then, f has a fixed point in K.

Lemma 2. For any $k(t) \in PC([0,T],\mathbb{R}_n)$, $u(t)$ solves

$$\left.\begin{array}{c} -u^{\Delta\Delta}(t) = k(t),\ t \in [0,T]\backslash\Omega, \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), \\ u^{\Delta}(t_k^+) = u^{\Delta}(t_k) + J_k\left(u(t_k), u^{\Delta}(t_k)\right),\ k = 1,2,\dots,m, \\ u(0) = -u(\sigma(T)),\ u^{\Delta}(0) = -u^{\Delta}(\sigma(T)). \end{array}\right\} \quad (10)$$

if and only if $x(t)$ is the solution of integral equation

$$u(t) = \int_0^{\sigma(T)} G(t,\sigma(s))k(s)\Delta s + \sum_{k=1}^{m} H(t,t_k)I_k(u(t_k))$$

$$-\sum_{k=1}^{m} G(t,t_k)J_k\left(u(t_k), u^{\Delta}(t_k)\right), \qquad \forall\, t \in [0,\sigma^2(T)] \quad (11)$$

where

$$G(t,s) = \begin{cases} \frac{1}{2}\left[\frac{1}{2}\sigma(T) - t + s\right] & 0 \le s \le t \le \sigma^2(T), \\ \frac{1}{2}\left[\frac{1}{2}\sigma(T) + t - s\right] & 0 \le t \le s \le \sigma(T) \end{cases} \quad (12)$$

$$H(t,s) = \begin{cases} \frac{1}{2} & 0 \le s \le t \le \sigma^2(T), \\ -\frac{1}{2} & 0 \le t \le s \le \sigma(T) \end{cases} \quad (13)$$

Proof. The proof is similar to that of Lemma 2.1 in [13]

By Lemma 2, we have for every $t \in [0,\sigma(T)]$

$$\max_{(t,s)\in[0,\sigma(T)]\times[0,T]} |G(t,\sigma(s))| \le \frac{3}{4}\sigma(T) \quad (14)$$

If $t = \sigma^2(T) > \sigma(T)$, then

$$|G(\sigma^2(T),\sigma(s))| \le \frac{1}{4}[\sigma(T) + 2\sigma^2(T)] \quad (15)$$

Therefore we have the upper bounds

$$\max_{(t,s)\in[0,\sigma^2(T)]\times[0,\sigma(T)]} |G(t,\sigma(s))| \le \frac{1}{4}[\sigma(T) + 2\sigma^2(T)] := G_0$$

$$\max_{(t,s)\in[0,\sigma^2(T)]\times[0,\sigma(T)]} |H(t,s)| \le \frac{1}{2} \quad (16)$$

$$\max_{(t,s)\in[0,\sigma 2(T)]\times[0,\sigma(T)]}\left|G^{\Delta}(t,s)\right| \leq \frac{1}{2}$$

Recall that a mapping between Banach spaces is compact if it is continuous and carries bounded sets into relatively compact sets.

Lemma 3. Suppose that $h : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0,T] \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and $I_k: \mathbb{R}^n \to \mathbb{R}^n$, $J_k: \mathbb{R}^n \to \mathbb{R}^n$ are continuous. Define an operator $B : PC^1([0,\sigma^2(T)],\mathbb{R}_n) \to PC^1([0,\sigma^2(T)],\mathbb{R}_n)$ as

$$Bu(t) := \int_0^{\sigma(T)} G(t,\sigma(s))\, h\left(s,u(s),u^{\Delta}(s),\int_0^t g\left(t,s,u(s),u^{\Delta}(s)\right)\Delta s\right)\Delta s$$

$$+\sum_{k=1}^{m} H(t,t_k)I_k(u(t_k)) - \sum_{k=1}^{m} G(t,t_k)J_k\left(u(t_k),u^{\Delta}(t_k)\right), \qquad (17)$$

where $G(t,s)$ and $H(t,s)$ are as given in Lemma 2 Then B is a compact map.

Proof. we know G^{Δ} ,H^{Δ} denotes the derivatives of G,H with respect to the first variable, we have

$$(Bu)^{\Delta}(t) := -\frac{1}{2}\int_0^t h\left(s,u(s),u^{\Delta}(s),\int_0^t g\left(t,s,u(s),u^{\Delta}(s)\right)\Delta s\right)\Delta s$$

$$+\frac{1}{2}\int_t^{\sigma(T)} h\left(s,u(s),u^{\Delta}(s),\int_t^{\sigma(T)} g\left(t,s,u(s),u^{\Delta}(s)\right)\Delta s\right)\Delta s \qquad (18)$$

$$+\sum_{k=1}^{m} J_k\left(u(t_k),u^{\Delta}(t_k)\right)$$

Then the continuity of h,g,I_k,J_k implies B is a continuous map from $PC[J,\mathbb{R}_n]$ to $PC[J,\mathbb{R}_n]$. On the other hand, for any bounded subset$S \subset PC[J,\mathbb{R}_n]$. (18) implies $\{(Bu)^{\Delta}(t)|u(t) \in S\}$ is also a bounded subset of $PC[J,\mathbb{R}_n]$. Deducing in a similar way as proving Lemma 1, we have B is a compact map.

Lemma 4. $u \in PC^1[J,\mathbb{R}_n] \cap PC^2[J,\mathbb{R}_n]$ is a solution of (1) if and only if $u(t) \in PC^1[J,\mathbb{R}_n]$ is a fixed point of B.

Proof. It can be easily obtain from Lemma 2 and we omit the proof here.

3. EXISTENCE OF SOLUTIONS

In this section, we prove the existence results for (1)and (2) in presence of Schauder's fixed point theorem and Schaefer's fixed point theorem, respectively.

We first set

$$\eta_1 = \overline{\lim_{\|u\|+\|v\|+\|w\|\to\infty}}\left(\max_{t\in[0,T]}\frac{\|h(t,u,v,w\|}{\|u\|+\|v\|+\|w\|}\right),\ \eta_2 = \overline{\lim_{\|u\|+\|v\|\to\infty}}\left(\max_{t\in[0,T]}\frac{\|g(t,s,u,v\|}{\|u\|+\|v\|}\right)$$

$$\alpha_k = \overline{\lim_{\|u\|\to\infty}}\left(\frac{\|I_k(u)\|}{\|u\|}\right)(k=1,2,\dots,m), \qquad \beta_k = \overline{\lim_{\|u\|+\|v\|\to\infty}}\left(\frac{\|J_k(u,v)\|}{\|u\|+\|v\|}\right)(k=1,2,\dots,m), \tag{19}$$

Theorem 5. Suppose that $h:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $g:[0,T]\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous and $I_k:\mathbb{R}^n\to\mathbb{R}^n$, $J_k:\mathbb{R}^n\to\mathbb{R}^n$ are continuous. $(k=1,2,\cdots,m)$. If

$$\delta = max\{\delta_1, \delta_2\} < 1, \tag{20}$$

where

$$\delta_1 = \frac{1}{2}\left[\eta_1(1+\eta_2)\sigma(T)+\sum_{k=1}^{m}\beta_k\right][\sigma(T)+2\sigma^2(T)]+\frac{1}{2}\sum_{k=1}^{m}\alpha_k, \tag{21}$$

$$\delta_2 = \eta_1(1+\eta_2)\sigma(T)+\sum_{k=1}^{m}\beta_k,$$

then (1) has atleast one solution in $PC^1[J,\mathbb{R}_n]\cap PC^2[J,\mathbb{R}_n]$.

Proof. By Lemma 3 it is sufficient to show that B has atleast one solution in $PC^1[J,\mathbb{R}_n]$. First, we can choose by (20) $\eta_1' > \eta_1$, $\eta_2' > \eta_2$, $\alpha_k' > \alpha_k$, $\beta_k' > \beta_k$, $(k=1,2,\dots,m)$ such that

$$\delta_1' = \frac{1}{2}\left[\eta_1'(1+\eta_2')\sigma(T)+\sum_{k=1}^{m}\beta_k'\right][\sigma(T)+2\sigma^2(T)]+\frac{1}{2}\sum_{k=1}^{m}\alpha_k' < 1$$

$$\delta_2' = \eta_1'(1+\eta_2')\sigma(T)+\sum_{k=1}^{m}\beta_k' < 1 \tag{22}$$

By (19) we can choose a positive number N such that

$$\|h(t,u,v,w\| < \eta_1'(\|u\|+\|v\|+\|w\|), \qquad \forall\ t\in[0,\sigma(T)], \|u\|+\|v\|\ge N. \tag{23}$$

Then

$$\|h(t,u,v,w\| < \eta_1'(\|u\|+\|v\|+\|w\|)+M,\ \forall\ t\in[0,\sigma(T)], u,v,w\in\mathbb{R}^n. \tag{24}$$

where

$$M = \max_{t\in[0,\sigma(T)],\|u\|+\|v\|+\|w\|\le N}\|f(t,u,v,w)\| < \infty \tag{25}$$

Similarly, there exist positive constants $\overline{M}, M_k, \overline{M_k}$,$(k = 1,2,\cdots,m)$ such that

$$\|g(t,s,u,v\| \leq \eta_2'(\|u\| + \|v\|) + \overline{M}\ , \quad \forall\ t \in [0,\sigma(T)], \qquad u,v \in \mathbb{R}^n\ , \tag{26}$$

$$\|I_k(u\| \leq \alpha_k'\|u\| + M_k\ , \quad \forall\ u \in \mathbb{R}^n\ , \tag{27}$$

$$\|J_k(u,v\| \leq \beta_k'(\|u\| + \|v\|) + \overline{M_k}\ , \quad \forall\ u,v \in \mathbb{R}^n\ , \tag{28}$$

It then follows by (16), (17), (24)-(28) that

$$\begin{aligned}\|Bu(t)\| \leq \frac{1}{4}\sigma(T)[\sigma(T) + 2\sigma^2(T)]\big[\eta_1'(\|u(t)\| + \|u^{\Delta}(t)\| + \eta_1'(\|u(t)\| + \|u^{\Delta}(t)\|) + \overline{M}) + M\big] \\ + \frac{1}{2}\left(\sum_{k=1}^{m}\alpha_k'\|u\| + \sum_{k=1}^{m}M_k\right) \\ + \frac{1}{4}[\sigma(T) + 2\sigma^2(T)]\left[\left(\sum_{k=1}^{m}\beta_k'\|u(t)\| + \|u^{\Delta}(t)\|\right) + \sum_{k=1}^{m}\overline{M_k}\right]\end{aligned}$$

$$\leq \delta_1'\|u\|_{PC^1} + P^{(1)}\ , \quad \forall\ t \in [0,\sigma^2(T)], \tag{29}$$

where

$$P^{(1)} = \frac{1}{4}[\sigma(T) + 2\sigma^2(T)]\left[\left(\eta_1'\overline{M}\sigma(T) + M\sigma(T) + \sum_{k=1}^{m}\overline{M_k}\right) + \frac{1}{2}\left(\sum_{k=1}^{m}\overline{M_k}\right)\right] \tag{30}$$

is a constant.

Similarly, we can prove that

$$\|Bu^{\Delta}(t)\| \leq \delta_2'\|u\|_{PC^1} + P^{(2)}\ , \quad \forall\ t \in [0,\sigma^2(T)], \tag{31}$$

where

$$P^{(1)} = \frac{1}{2}\left(\eta_1'\overline{M}\sigma(T) + M\sigma(T) + \sum_{k=1}^{m}\overline{M_k}\right)$$

is a constant.

Consequently, by (29) and (31) we have

$$\|(Bu)\|_{PC^1} \leq \delta_1'\|u\|_{PC^1} + P'\ , \quad \forall\ u \in PC^1[0,\sigma^2(T),\mathbb{R}^n], \tag{32}$$

where

(33)

$\delta = max\{\delta_1, \delta_2\} < 1$, $P' = \max\{P^{(1)}, P^{(2)}\}$

Thus we can choose $l > 0$ such that

$$B(H_l) \subset H_l \, , \tag{34}$$

where $H_l = \{ u \in PC^1[0,\sigma^2(T),\mathbb{R}^n] | \|u\|_{PC^1} \leq l\}$. On the otherhand,from Lemma 4 B is a complete continuous operator. Hence by Schauder's fixed point theorem B has a fixed point in H_l . This completes the proof.

Remark

Assumption (20) is true if there hold

$$\frac{\|h(t,u,v,w\|}{\|u\| + \|v\| + \|w\|} \to 0, \qquad as \ \|u\| + \|v\| + \|w\| \to \infty$$

$$\frac{\|g(t,s,u,v\|}{\|u\| + \|v\|} \to 0 \, , \frac{\|J_k(u,v)\|}{\|u\| + \|v\|} \to 0 \, , \ as \ \ \|u\| + \|v\| \to \infty \tag{35}$$

$$\frac{\|I_k(u)\|}{\|u\|} \to 0 \, , \ as \ \ \|u\| \to \infty \ (k = 1,2, \dots, m),$$

We now go on to study the existence results for (2). We should mention that the idea used in the following theorem is initiated by Tisdell [5, 11].

Theorem 5. Suppose that $h : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0,T] \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and $I_k: \mathbb{R}^n \to \mathbb{R}^n$, $J_k: \mathbb{R}^n \to \mathbb{R}^n$ are continuous. $(k = 1,2,\cdots,m)$. If there exist nonnegative constants $\alpha_1, \alpha_2, \gamma_k, \zeta_k, C_k, D_k, E$ and $\overline{E}$ such that

$$\|h(t,u,v,w\| \leq \alpha_1[\langle u, h(t,u,v,w)\rangle + \|v\|^2 + \|w\|^2] + E \tag{36}$$

$$\forall \ (t,u,v,w) \quad \in ([0,\sigma(T)]\backslash\Omega \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$$

$$\left\| \int_0^t g(t,s,u,v) \, \Delta s \right\| \leq \left(\alpha_2[\|u\| + \|v\|] + \overline{E}\right)^{\frac{1}{2}} \quad \forall \, u,v \in \mathbb{R}^n$$

$$\|I_k(u)\| \leq \gamma_k \|u\| + C_k \, , \qquad \|J_k(u,v)\| \leq \zeta_k(\|u\| + \|v\|) + D_k \, , \ \ \forall \, u,v \in \mathbb{R}^n$$

$$\max\left\{2G_0\alpha_1\alpha_2\sigma(T)+\frac{1}{2}\sum_{k=1}^{m}\gamma_k+2G_0\sum_{k=1}^{m}\zeta_k\ , \quad \alpha_1\alpha_2\sigma(T)+2G_0\sum_{k=1}^{m}\zeta_k\right\}<1 \tag{37}$$

where $\langle\cdot\rangle$ is the Euclidean inner product. Then (2) has atleast one solution.

Proof. Consider the mapping

$$B:\ PC^1([0,\sigma^2(T)],\mathbb{R}_n)\to PC^1([0,\sigma^2(T)],\mathbb{R}_n) \tag{38}$$

$$Bu(t):=\int_0^{\sigma(T)}G(t,\sigma(s))\,h\left(s,u^\sigma(s),u^\Delta(s),\int_0^t g\left(t,s,u^{\sigma(s)}(s),u^\Delta(s)\right)\Delta s\right)\Delta s$$

$$+\sum_{k=1}^{m}H(t,t_k)I_k(u(t_k))-\sum_{k=1}^{m}G(t,t_k)J_k\left(u(t_k),u^\Delta(t_k)\right), \tag{39}$$

$$G(t,s)=\begin{cases}-\frac{1}{2}\left[\frac{1}{2}\sigma(T)-t+s\right] & 0\le s\le t\le\sigma^2(T),\\ -\frac{1}{2}\left[\frac{1}{2}\sigma(T)+t-s\right] & 0\le t\le s\le\sigma(T)\end{cases} \tag{40}$$

$$H(t,s)=G^\Delta(t,s)=\begin{cases}\frac{1}{2} & 0\le s\le t\le\sigma^2(T),\\ -\frac{1}{2} & 0\le t\le s\le\sigma(T)\end{cases} \tag{41}$$

In a similar way as we prove Lemmas 2, 3, 4 we obtain

(i) $u\in PC^1[J,\mathbb{R}_n]\cap PC^2[J,\mathbb{R}_n]$ is a solution of (2) if and only if $u(t)\in PC^1[J,\mathbb{R}_n]$ is a fixed point of B;

(ii) B is a compact operator.

Consider the equation

$$u = Bu \tag{42}$$

To show B has atleast one fixed point, we apply Schaefer's theorem by showing that all potential solutions to

$$u=\lambda Bu,\ \lambda\in(0,1) \tag{43}$$

are bounded a priori, with the bound being independent of λ. With this in mind, let $u(t)$ be a solution of (43). Note that $u(t)$ is also a solution to

$$\left.\begin{aligned} u^{\Delta\Delta}(t) = \lambda h\left(t, u^{\sigma}(t), u^{\Delta}(t), \int_0^t g\left(t, s, u^{\sigma}(s), u^{\Delta}(s)\right)\Delta s\right),\ t \neq t_k,\ k = 1,2,\dots,m, \\ u(t_k^+) = u(t_k) + \lambda I_k\big(u(t_k)\big), \\ u^{\Delta}(t_k^+) = u^{\Delta}(t_k) + \lambda J_k\left(u(t_k), u^{\Delta}(t_k)\right),\ k = 1,2,\dots,m, \\ u(0) = -u\big(\sigma(T)\big),\ u^{\Delta}(0) = -u^{\Delta}\big(\sigma(T)\big). \end{aligned}\right\} \tag{44}$$

On one hand, we see that for $\lambda \in (0,1)$

$$\lambda\left\|h\left(t, u^{\sigma}(t), u^{\Delta}(t), \int_0^t g\left(t, s, u^{\sigma}(s), u^{\Delta}(s)\right)\Delta s\right)\right\|$$

$$\leq \lambda\left\{\alpha_1\left[\langle u^{\sigma}(t), \lambda h\left(t, u^{\sigma}(t), u^{\Delta}(t), \int_0^t g\left(t, s, u^{\sigma}(s), u^{\Delta}(s)\right)\Delta s\right)\rangle + \|u^{\Delta}(t)\|^2 + \left[\alpha_2\left(\|u(t)\| + \|u^{\Delta}(t)\|\right) + \overline{E}\right]\right] + E\right\}$$

$$\leq \alpha_1\left\{\langle u^{\sigma}(t), \lambda h\left(t, u^{\sigma}(t), u^{\Delta}(t), \int_0^t g\left(t, s, u^{\sigma}(s), u^{\Delta}(s)\right)\Delta s\right)\rangle + \lambda\langle u^{\Delta}(t), u^{\Delta}(t)\rangle + \lambda\left[\alpha_2\left(\|u(t)\| + \|u^{\Delta}(t)\|\right) + \overline{E}\right]\right\} + \lambda E$$

$$\leq \alpha_1\left\{\langle u^{\sigma}(t), u^{\Delta\Delta}(t)\rangle + \langle u^{\Delta}(t), u^{\Delta}(t)\rangle + \left[\alpha_2\left(\|u(t)\| + \|u^{\Delta}(t)\|\right) + \overline{E}\right]\right\} + E$$

$$\leq \alpha_1\langle u(t), u^{\Delta}(t)\rangle^{\Delta} + \alpha_1\alpha_2\left(\|u(t)\| + \|u^{\Delta}(t)\|\right) + \alpha_1\overline{E} + E$$

$$\leq \alpha_1\langle u(t), u^{\Delta}(t)\rangle^{\Delta} + 2\alpha_1\alpha_2\|u\|_{PC^1} + \alpha_1\overline{E} + E \tag{45}$$

On the other hand, by the antiperiodic boundary condition we have

$$\int_0^{\sigma(T)} [\langle u(t), u^{\Delta}(t)\rangle)^{\Delta}]\Delta t = \langle u\big(\sigma(T)\big), u^{\Delta}(\sigma(T))\rangle - \langle u(0), u^{\Delta}(0)\rangle = 0 \tag{46}$$

It therefore follows that

$$\int_0^{\sigma(T)} \lambda \left\| h\left(t, u^\sigma(t), u^\Delta(t), \int_0^t g\left(t, s, u^\sigma(s), u^\Delta(s)\right)\Delta s\right)\right\| \Delta t \qquad (47)$$

$$\leq 2\alpha_1\alpha_2\|u\|_{PC^1}\sigma(T) + \alpha_1\bar{E}\sigma(T) + E\sigma(T)$$

Consequently,

$$\|u(t)\| = \lambda\|Bu(t)\|$$

$$= \left\| \int_0^{\sigma(T)} G(t,\sigma(s))\, h\left(s, u^\sigma(s), u^\Delta(s), \int_0^t g\left(t, s, u^{\sigma(s)}(s), u^\Delta(s)\right)\Delta s\right)\Delta s + \sum_{k=1}^m H(t,t_k)I_k(u(t_k)) - \sum_{k=1}^m G(t,t_k)J_k\left(u(t_k), u^\Delta(t_k)\right)\right\|$$

$$\leq 2G_0\alpha_1\alpha_2\|u\|_{PC^1}\sigma(T) + G_0\alpha_1\bar{E}\sigma(T) + G_0E\sigma(T) + \frac{1}{2}\sum_{k=1}^m C_k + G_0\sum_{k=1}^m D_k + \left[2G_0\alpha_1\alpha_2\sigma(T) + \frac{1}{2}\sum_{k=1}^m \gamma_k + 2G_0\sum_{k=1}^m \zeta_k\right]\|u\|_{PC^1}$$

$$u(t) \leq G_0\alpha_1\bar{E}\sigma(T) + G_0E\sigma(T) + \frac{1}{2}\sum_{k=1}^m C_k + G_0\sum_{k=1}^m D_k + \left[2G_0\alpha_1\alpha_2\sigma(T) + \frac{1}{2}\sum_{k=1}^m \gamma_k + 2G_0\sum_{k=1}^m \zeta_k\right]\|u\|_{PC^1} \qquad (48)$$

Differentiating both sides of (48), we can easily have

$$\|u^\Delta(t)\| \leq \frac{1}{2}\alpha_1\bar{E}\sigma(T) + \frac{1}{2}E\sigma(T) + \frac{1}{2}\sum_{k=1}^m D_k + \left[\alpha_1\alpha_2\sigma(T) + \sum_{k=1}^m \zeta_k\right]\|u\|_{PC^1} \qquad (49)$$

Choose

$$Q^{(1)} = \max\left\{G_0\alpha_1\bar{E}\sigma(T) + G_0E\sigma(T) + \frac{1}{2}\sum_{k=1}^m C_k + G_0\sum_{k=1}^m D_k\,, \quad \frac{1}{2}\alpha_1\bar{E}\sigma(T) + \frac{1}{2}E\sigma(T) + \frac{1}{2}\sum_{k=1}^m D_k\right\}$$

$$Q^{(2)} = \max\left\{2G_0\alpha_1\alpha_2\sigma(T) + \frac{1}{2}\sum_{k=1}^{m}\gamma_k + 2G_0\sum_{k=1}^{m}\zeta_k\ , \qquad \alpha_1\alpha_2\sigma(T) + \sum_{k=1}^{m}\zeta_k\right\} \tag{50}$$

Thus we have by (37) that

$$\|u\|_{PC^1} \leq \frac{Q^{(1)}}{1-Q^{(2)}} \tag{51}$$

Then the proof is completed.

4. EXAMPLES

Example 4.1

Consider the antiperiodic value problem on

$$\mathbb{T} = \bigcup_{k=0}^{\infty}([3k+1\,, 3k+2] \cup \{3k+2\}), \tag{52}$$

$$\left.\begin{aligned} u^{\Delta\Delta}(t) = u(\sigma(t)) + u(t)\left(u^{\Delta}(t)\right)^2 + \int_0^t ts\left(u^{\Delta}(t) + u(t)\right)(1-2s)^{\frac{1}{2}}\Delta s + \cos t, \qquad t \in [0,1], t \neq 1, \\ u(1^+) = u(1) + \frac{1}{8}u(1) + 4, \\ u^{\Delta}(1^+) = u^{\Delta}(1) - \frac{1}{8}u(1) + 10, \\ u(0) = -u(3), \qquad u^{\Delta}(0) = -u^{\Delta}(3) \end{aligned}\right\} (53)$$

We claim (53) has atleast one solution.

Proof:

Let $T = 3$ and

$f(t,u,v,w) = u + uv^2 + \int_0^t ts(u+v)(1-2s)^{\frac{1}{2}}\Delta s + \cos t$ In Theorem 6 Choosing $E = 4$,

we have for $(t,u,v,w) \in [0,1] \cup \{2,3\} \times \mathbb{R}^3$ that

(54)

$$f(t,u,v,w) = |u| + |u|v^2 + \int_0^1 s|u+v|(1-2s)^{\frac{1}{2}}\Delta s + 1$$

$$\langle u, f(t,u,v,w)\rangle + v^2 + w^2 = u^2 + u^2v^2 + u\cos t + \int_0^t ts(u^2+uv)(1-2s)^{\frac{1}{2}}\Delta s + v^2 + w^2 \quad (55)$$

We need the following inequalities,

$|u|^2 \geq |u| - 1$, $|u|^2v^2 + v^2 + w^2 + |u|\cos t \geq |u|v^2 - 2$, $u \geq 100.$

Since $\min_{v\geq 0}\{v^2 - 2v\} \geq -1$, we have $u^2v^2 + v^2 - |u|v^2 = v^2(u^2 - |u| + 1) > 0.$ Thus, for $\alpha_1 = 1$, $\alpha_2 = \frac{1}{100}$, $E = 4$ and $\overline{E} = 0$,

$$\alpha_1[\langle u, f(t,u,v,w)\rangle + v^2 + w^2] + E$$
$$= \alpha_1\left[u^2 + u^2v^2 + u\cos t + \int_0^t ts(u^2+uv)(1-2s)^{\frac{1}{2}}\Delta s + v^2 + w^2\right]$$
$$\geq |f(t,u,v,w)|, (t,u,v,w) \in [0,1] \cup \{2,3\} \times \mathbb{R}^3 \quad (56)$$

Moreover,

$$G_0 = \frac{9}{4} \text{ and } 2G_0\alpha_1\alpha_2\sigma(T) + \frac{1}{2}\sum_{k=1}^{m}\gamma_k + 2G_0\sum_{k=1}^{m}\zeta_k, \alpha_1\alpha_2\sigma(T) + \sum_{k=1}^{m}\zeta_k = \frac{19}{25} < 1$$

Then the conclusion follows from Theorem 6.

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