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ANALYSIS OF AN $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ RETRIAL QUEUEING SYSTEM WITH PRIORITY SERVICES, ABANDONED CUSTOMERS, WORKING BREAKDOWN, REPAIR AND RANDOMIZED VACATION POLICY

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Abstract: The topic under study is a single server retrial queue consisting of two classes of batch arrivals with non-preemptive priority service, abandoned customers, working breakdown, repair and randomized vacation policy. Two independent compound Poisson processes are encountered where the service times are generally distributed for two classes of customers. At a breakdown instant during busy period, the system continues to provide auxiliary service for the current customer, who is in the service station. The customer who is being served could be left out by the server for certain probabilities, in case of a new arrival for a low priority service. When the system becomes empty, the server goes for vacation of random length. If there are no customers waiting at the completion epoch of vacation, the server either takes another vacation or remains idle. We use the established norm which is the corresponding steady state results for time dependent probability generating functions are obtained. Along with that, the expected waiting time for the expected number of customers in the high and low priority queues are computed. Numerical results along with the graphical presentations are shown elaborately.

Keyword:batch arrival;retrial queue; priority services; working breakdown; randomized vacation policy.

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1. INTRODUCTION

The retrial queues is a major area of research for its applications such as computer systems and telecommunication networks, telephone switching systems, etc. Kapyrin [10] first proposed the $M/G/1$ retrial queue with non-exponential retrial time. Gomez-Corral [6] has investigated a retrial queueing system with general retrial times under constant retrial policy. A single server retrial queuing system with general retrial times and Bernoulli vacation have been extensively discussed by Krishna Kumar and Arivudainambi [11]. Gautam Choudhury and Jau-Chuan Ke [5] discussed an $M^X/G/1$ queue with retrial time follows general distribution breakdown, delay, repair and Bernoulli vacation schedule.

Artalejo et al.[1] studied stationary analysis of a retrial queue with preemptive repeated attempts. Bong and Dong [2] have been studied an $M_1, M_2/G/1/K$ retrial queue with priority service in which two types of arrival were blocked high priority arrival may allow to join the orbit. The $M_1, M_2/G_1, G_2/1$ retrial queueing system with priority service, breakdown and repair has been investigated by Wang [17] and Madhu Jain [14] discussed the double orbit finite capacity retrial queues with priority service, breakdown and repair.

The idea of working breakdown is not the same as the working vacations. This idea was introduced by Kalidass and Kasturi [9]. More recently, Dong and Yang [4] discussed an $M/M/1$ queue with optional service, working breakdown and repair. Ke and Chang [8] examined the $M/G/1$ retrial queue with balking, feedback and modified vacation policy. Ke et al.[7] studied an $M^x/G/1$ queueing system with a randomized vacation policy and at most J vacations. Shan and Yunfei [16] developed the variant working vacation policy for an $M^x/G/1$ queueing system, where the server operates a randomized vacation policy and at most J working vacations.

Besides, there are many papers dealing with customers abandoned. Krishna Kumar et al.[12] considered an $M/G/1$ retrial queue with additional service phase and possible preemptive resume service discipline. Recently, Shan Gao [15] studied an $M/G/1$ retrial queue with abandoned customers and multi-optional vacations. Krishnamoorthy [13] presented an $M_1,M_2/PH/1$ retrial queueing system with preemptive priority service, orbit search and abandonment customers in which the retrial attempted failed then customers abandonment the system with certain probabilities.

In this paper, we study priority retrial queue with abandoned customers, working breakdown and repair under general randomized vacation policy. If the system becomes empty, the server takes vacation of random length, and in case of server for no customers in the system the server comes back from the vacation, it can either take another vacation or stay idle with certain probability. Otherwise, service starts immediately. Here we assume that arriving two types of customers, namely high and low priority (retrial) customer under FCFS discipline. The high priority customers who found the server busy, formed a queue with infinite capacity. If a low priority customer found the server busy, he joined the retrial group (orbit). Whenever, the system is subjected to breakdowns; Once the system breaks down, the server will not stop the service immediately, the service continues only to the current customer at a slower rate such kind of service is known as a working breakdown service (WB) which follows general distribution. If the server is occupied with low priority customer, the arriving new low priority customer may interrupt the customer in service and pushed them out of the service area and he gets the service immediately with some probability.

The paper is arranged as follows. The mathematical notations, definitions are mentioned in section 2, equations defining the model and the time dependent solutions are obtained in section 3. The steady state results are derived in section 4. Some important system performance measures and stochastic decomposition are given in section 5 and 6 respectively. Some particular cases are mentioned in section 7 and in section 8, some numerical results and their graphical representations are presented.

2. MATHEMATICAL DESCRIPTION

1. High priority and low priority customers arrive in batches of variable size in a compound Poisson process. Let $\lambda_h c_i dt$ (i = 1, 2, 3, ...) and $\lambda_l c_j$ dt (j = 1, 2, 3, ...) be the first order probability that a batch of i and j customers arrives during a short interval of time (t, $t + dt$), where $0 \le c_i \le 1$, $\Sigma_{i=1}^{\infty} c_i = 1$, $0 \le c_j \le 1$, $\Sigma_{i=1}^{\infty} c_j = 1$, and $\lambda_h > 0$, $\lambda_l > 0$ are the mean arrival rate for high and low priority customers entering into the system. Note that low priority customers will be served only when there are no high priority customers in the queue.
2. The retrial customers are the customers with low priority. A new batch of low priority customers who find the server idle begins to be one of the customers served immediately and remaining customer joins the orbit in order to seek service again after random length of time. A low priority customer in the orbit always returns to the orbit when he finds the server busy on his retrial attempt.
3. For each customer under high and low priority service provided by a single service channel on a 'First In - First Out' service basis.
4. A new low priority customer who finds the server idle, the service is started immediately. Otherwise, he either with probability $q(0 \le q \le 1)$ remove the customers being served out of the service area and begins his service immediately or with probability $\overline{q}$ (= 1 – q) joins the orbit.
5. The system may break down at any point of time during busy period and breakdowns are assumed to follow Poisson process with breakdown rate $\alpha > 0$. However, the server works slower than the regular service rate for the current customer, after which it will be repaired.
6. If server detects the system empty after a service completion, it takes a vacation with random length of time. At the end, if server finds that there are no customers in the system, the server will take another vacation with probability p or remains idle with probability (1– p).
7. All stochastic processes involved are mutually independent.

2.1 Definitions

For an arbitrary time , the stochastic behaviour of the system can be described mathematically by a Markov process $\{\aleph_1(t), \aleph_2(t), Y(t)\}; t \geq 0$, where $\aleph_1(t)$ is the number of customers in the high priority queue, $\aleph_1(t)$ is the number of customers in the orbit, Y(t) = 0,1,2,3,4,5,6 denotes the server status, depending on the server being idle, busy with high priority, low priority, working breakdown service for high and low priority, under repair and on vacation respectively. If $Y(t)=0$, $M^0(t)$ denotes the elapsed time for retrial; if $Y(t) = 1$, $B_1^0(t)$ denotes the elapsed time for high priority service; if $Y(t)=2$, $B_2^0(t)$ denotes the elapsed time for low priority service; if $Y(t)=3$, $W_1^0(t)$ denotes the elapsed time for working breakdown high priority service; if , $Y(t)=4$, $W_2^0(t)$ denotes the elapsed time for working breakdown low priority service; if $Y(t) = 5$, $R^0(t)$ denotes the elapsed time for repair; if $Y(t)= 6$, $V^0(t)$ denote the elapsed time for vacation.

The high and low priority service time, working breakdown service time, vacation time, repair time follows general (arbitrary) distribution and the notions for their cumulative distribution function (CDF), the probability density function (pdf) and the Laplace transform (LT) are given in table 1.

Table 1: Notations

Server state	CDF	pdf	LT	Hazard rate
Retrial	$A(t)$	$a(t)$	$\bar{M}(s)$	$\beta(k)$
High priority service for regular period	$B_1(t)$	$b_1(t)$	$\bar{B}_1(s)$	$\mu_1(k)$
Low priority service for regular period	$B_2(t)$	$b_2(t)$	$\bar{B}_2(s)$	$\mu_2(k)$
High priority service for WB period	$W_1(t)$	$w_1(t)$	$\bar{W}_1(s)$	$\omega_1(k)$
Low priority service for WB period	$W_2(t)$	$w_2(t)$	$\bar{W}_2(s)$	$\omega_2(k)$
Vacation	$V(t)$	$v(t)$	$\bar{V}(s)$	$\gamma(k)$
Repair	$R(t)$	$r(t)$	$\bar{R}(s)$	$\eta(k)$

Next, we define the probability and probability densities $I_{0,0}(t) = Pr\{\aleph_1(t) = 0, \aleph_2(t) = 0, Y(t) = 0\}$ probability densitities

$I_{0,n}(t,\kappa)d\kappa = Pr\{\aleph_1(t) = 0, \aleph_2(t) = n, Y(t) = 0; \kappa \leq M^0(t) < \kappa + d\kappa\}, n \geq 1,$

$$P_{m,n}^{(1)}(t,\kappa)d\kappa = Pr\{\aleph_1(t)=m, \aleph_2(t)=n, Y(t)=1; \kappa \le B_1^0(t) < \kappa + d\kappa\}, n \ge 0,$$
$$P_{m,n}^{(2)}(t,\kappa)d\kappa = Pr\{\aleph_1(t)=m, \aleph_2(t)=n, Y(t)=2; \kappa \le B_2^0(t) < \kappa + d\kappa\}, n \ge 0,$$
$$Q_{m,n}^{(1)}(t,\kappa)d\kappa = Pr\{\aleph_1(t)=m, \aleph_2(t)=n, Y(t)=3; \kappa \le W_1^0(t) < \kappa + d\kappa\}, n \ge 0,$$
$$Q_{m,n}^{(2)}(t,\kappa)d\kappa = Pr\{\aleph_1(t)=m, \aleph_2(t)=n, Y(t)=4; \kappa \le W_2^0(t) < \kappa + d\kappa\}, n \ge 0,$$
$$R_{m,n}(t,\kappa)d\kappa = Pr\{\aleph_1(t)=m, \aleph_2(t)=n, Y(t)=5; \kappa \le R^0(t) < \kappa + d\kappa\}, n \ge 0,$$
$$V_{m,n}(t,\kappa)d\kappa = Pr\{\aleph_1(t)=m, \aleph_2(t)=n, Y(t)=6; \kappa \le V^0(t) < \kappa + d\kappa\}, n \ge 0,$$

for $\kappa \ge 0$, $t \ge 0$ and $m \ge 0$.

3. EQUATION GOVERNING THE SYSTEM

We construct a set of Kolmogorov forward equations using supplementary variables technique as follows:

$$\frac{d}{dt} I_{0,0}(t) = -(\lambda_h + \lambda_l) I_{0,0}(t) + (1-p)\int_0^\infty V_{0,0}(t,\kappa)\gamma(\kappa)d\kappa, \tag{1}$$

$$\frac{\partial}{\partial t} I_{0,n}(t,\kappa) + \frac{\partial}{\partial \kappa} I_{0,n}(t,\kappa) = -(\lambda_h + \lambda_l + \beta(\kappa)) I_{0,n}(t,\kappa) \tag{2}$$

$$\frac{\partial}{\partial t} P_{m,n}^{(1)}(t,\kappa) + \frac{\partial}{\partial \kappa} P_{m,n}^{(1)}(t,\kappa) = -(\lambda_h + \lambda_l + \alpha + \mu_1(\kappa)) P_{m,n}^{(1)}(t,\kappa)$$
$$+(1-\delta_{m0})\lambda_h \sum_{i=1}^{m} c_i P_{m-i,n}^{(1)}(t,\kappa) + (1-\delta_{n0})\lambda_l \sum_{j=1}^{n} c_j P_{m,n-j}^{(1)}(t,\kappa), \tag{3}$$

$$\frac{\partial}{\partial t} P_{m,n}^{(2)}(t,\kappa) + \frac{\partial}{\partial \kappa} P_{m,n}^{(2)}(t,\kappa) = -(\lambda_h + \lambda_l + \alpha + \mu_2(\kappa)) P_{m,n}^{(2)}(t,\kappa)$$
$$+(1-\delta_{m0})\lambda_h \sum_{i=1}^{m} c_i P_{m-i,n}^{(2)}(t,\kappa) + (1-\delta_{n0})\bar{q}\lambda_l \sum_{j=1}^{n} c_j P_{m,n-j}^{(2)}(t,\kappa) \tag{4}$$

$$\frac{\partial}{\partial t} Q_{m,n}^{(1)}(t,\kappa) + \frac{\partial}{\partial \kappa} Q_{m,n}^{(1)}(t,\kappa) = -(\lambda_h + \lambda_l + \omega_1(\kappa)) Q_{m,n}^{(1)}(t,\kappa)$$
$$+(1-\delta_{m0})\lambda_h \sum_{i=1}^{m} c_i Q_{m-i,n}^{(1)}(t,\kappa) + (1-\delta_{n0})\lambda_l \sum_{j=1}^{n} c_j Q_{m,n-j}^{(1)}(t,\kappa), \tag{5}$$

$$\frac{\partial}{\partial t} Q_{m,n}^{(2)}(t,\kappa) + \frac{\partial}{\partial \kappa} Q_{m,n}^{(2)}(t,\kappa) = -(\lambda_h + \lambda_l + \omega_2(\kappa)) Q_{m,n}^{(2)}(t,\kappa)$$
$$+(1-\delta_{m0})\lambda_h \sum_{i=1}^{m} c_i Q_{m-i,n}^{(1)}(t,\kappa) + (1-\delta_{n0})\lambda_l \sum_{j=1}^{n} c_j Q_{m,n-j}^{(2)}(t,\kappa), \tag{6}$$

$$\frac{\partial}{\partial t} R_{m,n}(t,\kappa) + \frac{\partial}{\partial \kappa} R_{m,n}(t,\kappa) = -(\lambda_h + \lambda_l + \eta(\kappa)) R_{m,n}(t,\kappa)$$
$$+(1-\delta_{m0})\lambda_h \sum_{i=1}^{m} c_i R_{m-i,n}(t,\kappa) + (1-\delta_{n0})\lambda_l \sum_{j=1}^{n} c_j R_{m,n-j}(t,\kappa), \tag{7}$$

$$\frac{\partial}{\partial t}V_{m,n}(t,\kappa)+\frac{\partial}{\partial \kappa}V_{m,n}(t,\kappa)=-(\lambda_h+\lambda_l+\gamma(\kappa))V_{m,n}(t,\kappa) +(1-\delta_{m0})\lambda_h\sum_{i=1}^{m}c_iV_{m-i,n}(t,\kappa)+(1-\delta_{n0})\lambda_l\sum_{j=1}^{n}c_jV_{m,n-j}(t,\kappa). \quad (8)$$

To solve the equations (2)-(8) for the boundary conditions at

$$I_{0,n}(t,0)=\int_0^\infty P_{0,n}^{(1)}(t,\kappa)\mu_1(\kappa)d\kappa+\int_0^\infty P_{0,n}^{(2)}(t,\kappa)\mu_2(\kappa)d\kappa+\int_0^\infty R_{0,n}(t,\kappa)\eta(\kappa)d\kappa +\int_0^\infty V_{0,n}(t,\kappa)\gamma(\kappa)d\kappa, \quad (9)$$

$$P_{m,n}^{(1)}(t,0)=\lambda_hc_{m+1}I_{0,0}(t)+\int_0^\infty P_{m+1,n}^{(1)}(t,\kappa)\mu_1(\kappa)d\kappa+\int_0^\infty P_{m+1,n}^{(2)}(t,\kappa)\mu_2(\kappa)d\kappa +\int_0^\infty R_{m+1,n}(t,\kappa)\eta(\kappa)d\kappa+\int_0^\infty V_{m+1,n}(t,\kappa)\gamma(\kappa)d\kappa, \quad (10)$$

$$P_{0,n}^{(2)}(t,0)=\lambda_lc_{n+1}I_{0,0}(t)+\sum_{j=1}^{n}\lambda_lc_j\int_0^\infty I_{0,n+1-j}(t,\kappa)d\kappa+\int_0^\infty I_{0,n+1}(t,\kappa)\beta(\kappa)d\kappa +\lambda_lq\int_0^\infty P_{0,n-1}^{(2)}(t,\kappa)d\kappa, \quad (11)$$

$$Q_{m,n}^{(i)}(t,0)=\alpha\int_0^\infty P_{m,n}^{(i)}(t,\kappa)d\kappa; i=1,2, \quad (12)$$

$$R_{m,n}(t,0)=\int_0^\infty Q_{m,n}^{(1)}(t,\kappa)\omega_1(\kappa)d\kappa+\int_0^\infty Q_{m,n}^{(2)}(t,\kappa)\omega_2(\kappa)d\kappa, \quad (13)$$

$$V_{0,0}(t,0)=\int_0^\infty P_{0,0}^{(1)}(t,\kappa)\mu_1(\kappa)d\kappa+\int_0^\infty P_{0,0}^{(2)}(t,\kappa)\mu_2(\kappa)d\kappa+\int_0^\infty R_{0,0}(t,\kappa)\eta(\kappa)d\kappa +p\int_0^\infty V_{0,0}(t,\kappa)\gamma(\kappa)d\kappa, \quad (14)$$

$$V_{m,n}(t,0)=0; m\geq 1, n\geq 1. \quad (15)$$

The initial conditions are,

$$P_{m,n}^{(i)}(0)=Q_{m,n}^{(i)}(0)=V_{m,n}(0)=R_{m,n}(0)=0; i=1,2m,n\geq 0 \quad I_{0,n}(0)=0; n\geq 1 \text{and} I_{0,0}(0)=1. \quad (16)$$

The Probability Generating Function(PGF) of this model:

$$I(t,\kappa,z_l)=\sum_{n=1}^{\infty}z_l^nI(t,\kappa);\quad A(t,\kappa,z_h,z_l)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}z_h^mz_l^nA_{m,n}(t,\kappa);$$

$$A(t,\kappa,z_h) = \sum_{m=0}^{\infty} z_h^m A_m(t,\kappa); \qquad A(t,\kappa,z_l) = \sum_{n=0}^{\infty} z_l^n A_n(t,\kappa);$$

where $A = P^{(i)}, Q^{(i)}, V, R$; = 1, 2, which are convergent inside the circle given by $|z_h| \le 1$, $|z_i| \le 1$. By taking Laplace transforms from equation (1) to (15) and solving those equations, we get,

$$\bar{I}_0(s,\kappa,z_l) = \bar{I}_0(s,0,z_l)e^{-(s+\lambda_h+\lambda_l)\kappa-\int_0^\kappa \beta(t)dt}, \tag{17}$$
$$\bar{P}^{(1)}(s,\kappa,z_h,z_l) = \bar{P}^{(1)}(s,0,z_h,z_l)e^{-\phi(s,z_h,z_l)\kappa-\int_0^\kappa \mu_1(t)dt}, \tag{18}$$
$$\bar{P}^{(2)}(s,\kappa,z_h,z_l) = \bar{P}^{(2)}(s,0,z_h,z_l)e^{-\phi_1(s,z_h,z_l)\kappa-\int_0^\kappa \mu_2(t)dt}, \tag{19}$$
$$\bar{Q}^{(1)}(s,\kappa,z_h,z_l) = \bar{Q}^{(1)}(s,0,z_h,z_l)e^{-\phi_2(s,z_h,z_l)\kappa-\int_0^\kappa \omega_1(t)dt}, \tag{20}$$
$$\bar{Q}^{(2)}(s,\kappa,z_h,z_l) = \bar{Q}^{(2)}(s,0,z_h,z_l)e^{-\phi_2(s,z_h,z_l)\kappa-\int_0^\kappa \omega_2(t)dt}, \tag{21}$$
$$\bar{R}(s,\kappa,z_h,z_l) = \bar{R}(s,0,z_h,z_l)e^{-\phi_2(s,z_h,z_l)\kappa-\int_0^\kappa \eta(t)dt}, \tag{22}$$
$$\bar{V}(s,\kappa,z_h,z_l) = \bar{V}(s,0,z_h,z_l)e^{-\phi_2(s,z_h,z_l)\kappa-\int_0^\kappa \gamma(t)dt}, \tag{23}$$
$$\bar{P}_0^{(1)}(s,\kappa,z_l) = \bar{P}_0^{(1)}(s,0,z_l)e^{-\psi(s,z_l)\kappa-\int_0^\kappa \mu_1(t)dt}, \tag{24}$$
$$\bar{P}_0^{(2)}(s,\kappa,z_l) = \bar{P}_0^{(2)}(s,0,z_l)e^{-\psi_1(s,z_l)\kappa-\int_0^\kappa \mu_2(t)dt}, \tag{25}$$
$$\bar{Q}_0^{(1)}(s,\kappa,z_l) = \bar{Q}_0^{(1)}(s,0,z_l)e^{-\psi_2(s,z_l)\kappa-\int_0^\kappa \omega_1(t)dt}, \tag{26}$$
$$\bar{Q}_0^{(2)}(s,\kappa,z_l) = \bar{Q}_0^{(2)}(s,0,z_l)e^{-\psi_2(s,z_l)\kappa-\int_0^\kappa \omega_2(t)dt}, \tag{27}$$
$$\bar{R}_0(s,\kappa,z_l) = \bar{R}_0(s,0,z_l)e^{-\psi_{2(s,z)}\kappa-\int_0^\kappa \eta(t)dt}, \tag{28}$$
$$\bar{V}_0(s,\kappa,z_l) = \bar{V}_0(s,0,z_l)e^{-\psi_2(s,z_l)\kappa-\int_0^\kappa \gamma(t)dt}, \tag{29}$$
$$\bar{V}_{0,0}(s,\kappa) = \bar{V}_{0,0}(s,0)e^{-(s+\lambda_h+\lambda_l)\kappa-\int_0^\kappa \gamma(t)dt}, \tag{30}$$

where,

$$\phi(s,z_h,z_l) = s + \lambda_h(1 - C(z_h)) + \lambda_l(1 - C(z_l)) + \alpha,$$
$$\phi_1(s,z_h,z_l) = s + \lambda_h(1 - C(z_h)) + \lambda_l(1 - C(z_l)) + \lambda_l q C(z_l) + \alpha,$$
$$\phi_2(s,z_h,z_l) = s + \lambda_h(1 - C(z_h)) + \lambda_l(1 - C(z_l)),$$

$$\psi(s,z_l) = s + \lambda_h + \lambda_l(1 - C(z_l)) + \alpha,$$
$$\psi_1(s,z_l) = s + \lambda_h + \lambda_l(1 - C(z_l)) + \lambda_l q C(z_l) + \alpha,$$
$$\psi_2(s,z_l) = s + \lambda_h + \lambda_l(1 - C(z_l)).$$

However, by definition

$$\bar{P}^{(2)}(s,0,z_h,z_l) = \bar{P}_0^{(2)}(s,0,z_l),$$
$$\bar{V}(s,0,z_h,z_l) = \bar{V}_0(s,0,z_l) = \bar{V}_{0,0}(s,0).$$

using equations (17)-(30) into (9), (10) and (11), we get

$$\begin{aligned}\bar{I}_0(s,0,z_l) &= \overline{P}_0^{(1)}(s,0,z_l)\{\overline{B}_1(\psi(s,z_l))+\alpha[\frac{1-\overline{B}_1(\psi(s,z_l))}{\psi(s,z_l)}]\overline{W}_1(\psi_2(s,z_l))\overline{R}(\psi_2(s,z_l))\}\\ &+\overline{P}_0^{(2)}(s,0,z_l)\{\overline{B}_2(\psi_1(s,z_l))+\alpha[\frac{1-\overline{B}_2(\psi_1(s,z_l))}{\psi_1(s,z_l)}]\overline{W}_1(\psi_2(s,z_l))\overline{R}(\psi_2(s,z_l))\}\\ &-\{[1-\overline{V}(\psi_2(s,z_l))]\overline{V}_{0,0}(s,0)+(s+\lambda_h+\lambda_l)\bar{I}_{0,0}(s)-1\}, \qquad (31)\end{aligned}$$

$$\begin{aligned}&\overline{P}^{(1)}(s,0,z_h,z_l)\{z_h-\overline{B}_1(\phi(s,z_h,z_l))-\alpha[\frac{1-\overline{B}_1(\phi(s,z_h,z_l))}{\phi(s,z_h,z_l)}]\overline{W}_1(\phi_2(s,z_h,z_l))\overline{R}(\phi_2(s,z_h,z_l))\}\\ &=\lambda_h C(z_h)\bar{I}_0(s,0,z_l)[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]+\overline{V}_{0,0}(s,0)\{\overline{V}(\phi_2(s,z_h,z_l))-\overline{V}(\psi_2(s,z_l))\}\\ &+\overline{P}_0^{(2)}(s,0,z_l)\{\overline{B}_2(\phi_1(s,z_h,z_l))+\alpha[\frac{1-\overline{B}_2(\phi_1(s,z_h,z_l))}{\phi_1(s,z_h,z_l)}]\overline{W}_1(\phi_2(s,z_h,z_l))\overline{R}(\phi_2(s,z_h,z_l))\\ &-\overline{B}_2(\psi_1(s,z_l))-\alpha[\frac{1-\overline{B}_2(\psi_1(s,z_l))}{\psi_1(s,z_l)}]\overline{W}_1(\psi_2(s,z_l))\overline{R}(\psi_2(s,z_l))\}-\overline{P}_0^{(1)}(s,0,z_l)\\ &\times\{\overline{B}_1(\psi(s,z_l))+\alpha[\tfrac{1-\overline{B}_1(\psi(s,z_l))}{\psi(s,z_l)}]\overline{W}_1(\psi_2(s,z_l))\overline{R}(\psi_2(s,z_l))\}, \qquad (32)\end{aligned}$$

$$\begin{aligned}\overline{P}_0^{(2)}(s,0,z_l)\{z_l-z_l^2\lambda_l q[\frac{1-\overline{B}_2(\psi_1(s,z_l))}{\psi_1(s,z_l)}]\} &= \lambda_l C(z_l)\bar{I}_{0,0}(s)+\bar{I}_0(s,0,z_l)\\ &\times\{\overline{M}(s+\lambda_h+\lambda_l)+\lambda_l C(z_l)[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]\}. \qquad (33)\end{aligned}$$

we have to solve equations (31), (32) and (33). Letting $z_h = g(z_l)$ in (32) we get,

$$\begin{aligned}&\overline{P}_0^{(1)}(s,0,z_l)\{\overline{B}_1(\psi(s,z_l))+\alpha[\frac{1-\overline{B}_1(\psi(s,z_l))}{\psi(s,z_l)}]\overline{W}_1(\psi_2(s,z_l))\overline{R}(\psi_2(s,z_l))\}\\ &=\lambda_h C[g(z_l)]\bar{I}_0(s,0,z_l)[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]+\overline{V}_{0,0}(s,0)\{\overline{V}(\sigma_2(s,z_l))-\overline{V}(\psi_2(s,z_l))\}\\ &+\overline{P}_0^{(2)}(s,0,z_l)\{\overline{B}_2(\sigma_1(s,z_l))+\alpha[\frac{1-\overline{B}_2(\sigma_1(s,z_l))}{\sigma_1(s,z_l)}]\overline{W}_1(\sigma_2(s,z_l))\overline{R}(\sigma_2(s,z_l))\\ &-\overline{B}_2(\psi_1(s,z_l))-\alpha[\tfrac{1-\overline{B}_2(\psi_1(s,z_l))}{\psi_1(s,z_l)}]\overline{W}_1(\psi_2(s,z_l))\overline{R}(\psi_2(s,z_l))\}, \qquad (34)\end{aligned}$$

substitute equation (34) into (31), we get

$$\bar{I}_0(s,0,z_l)=\frac{\left\{\begin{array}{l}1-(s+\lambda_h+\lambda_l)\bar{I}_{0,0}(s)+\overline{V}_{0,0}(s,0)[\overline{V}(\sigma_2(s,z_l))-1]\\ +\overline{P}_0^{(2)}(s,0,z_l)\{\overline{B}_2(\sigma_1(s,z_l))+\alpha[\frac{1-\overline{B}_2(\sigma_1(s,z_l))}{\sigma_1(s,z_l)}]\\ \times\overline{W}_2(\sigma_2(s,z_l))\overline{R}(\sigma_2(s,z_l))\}\end{array}\right\}}{\{1-\lambda_h C[g(z_l)][\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]\}}, \qquad (35)$$

substitute equation (35) into (33), we get

$$\overline{P}_0^{(2)}(s,0,z_l) = \frac{\left\{\begin{array}{l} \lambda_l C(z_l)\overline{I}_{0,0}(s)\{1-\lambda_h C[g(z_l)][\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]\} \\ +\{1-(s+\lambda_h+\lambda_l)\overline{I}_{0,0}(s)+\overline{V}_{0,0}(s,0)[\overline{V}(\sigma_2(s,z_l))-1]\} \\ \times\{\overline{M}(s+\lambda_h+\lambda_l)+\lambda_l C(z_l)[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]\} \end{array}\right\}}{\left\{\begin{array}{l} \{z_l - z_l^2\lambda_l q[\frac{1-\overline{B}_2(\psi_1(s,z_l))}{\psi_1(s,z_l)}]\}\{1-\lambda_h C[g(z_l)][\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]\} \\ -\{\overline{B}_2(\sigma_1(s,z_l))+\alpha[\frac{1-\overline{B}_2(\sigma_1(s,z_l))}{\sigma_1(s,z_l)}]\overline{W}_2(\sigma_2(s,z_l))\overline{R}(\sigma_2(s,z_l))\} \\ \times\{\overline{M}(s+\lambda_h+\lambda_l)+\lambda_l C(z_l)[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]\} \end{array}\right\}} \quad (36)$$

using equation (36) in (35), we get

$$\overline{I}_0(s,0,z_l) = \frac{\left\{\begin{array}{l} \{\quad 1-(s+\lambda_h+\lambda_l)\overline{I}_{0,0}(s)+\overline{V}_{0,0}(s,0)[\overline{V}(\sigma_2(s,z_l))-1]\} \\ \times\{z_l - z_l^2\lambda_l q[\frac{1-\overline{B}_2(\psi_1(s,z_l))}{\psi_1(s,z_l)}]\}+\lambda_l C(z_l)\overline{I}_{0,0}(s) \\ \times\{\overline{B}_2(\sigma_1(s,z_l))+\alpha[\frac{1-\overline{B}_2(\sigma_1(s,z_l))}{\sigma_1(s,z_l)}]\overline{W}_2(\sigma_2(s,z_l))\overline{R}(\sigma_2(s,z_l))\} \end{array}\right\}}{\left\{\begin{array}{l} \{z_l - z_l^2\lambda_l q[\frac{1-\overline{B}_2(\psi_1(s,z_l))}{\psi_1(s,z_l)}]\}\{1-\lambda_h C[g(z_l)][\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]\} \\ -\{\overline{B}_2(\sigma_1(s,z_l))+\alpha[\frac{1-\overline{B}_2(\sigma_1(s,z_l))}{\sigma_1(s,z_l)}]\overline{W}_2(\sigma_2(s,z_l))\overline{R}(\sigma_2(s,z_l))\} \\ \times\{\overline{M}(s+\lambda_h+\lambda_l)+\lambda_l C(z_l)[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]\} \end{array}\right\}} \quad (37)$$

finally substitute equations (37), (36) and (34) in (32), we get,

$$\overline{P}^{(1)}(s,0,z_h,z_l) = \frac{\left\{\begin{array}{l} \lambda_h\{C(z_h)-C[g(z_l)]\}\overline{I}_0(s,0,z_l)[\frac{1-\overline{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}]+\overline{V}_{0,0}(s,0) \\ \times\{\overline{V}(\phi_2(s,z_h,z_l))-\overline{V}(\sigma_2(s,z_l))\}+\overline{P}_0^{(2)}(s,0,z_l)\{\overline{B}_2(\phi_1(s,z_h,z_l)) \\ +\alpha[\frac{1-\overline{B}_2(\phi_1(s,z_h,z_l))}{\phi_1(s,z_h,z_l)}]\overline{W}_2(\phi_2(s,z_h,z_l))\overline{R}(\phi_2(s,z_h,z_l)) \\ -\overline{B}_2(\sigma_1(s,z_l))-\alpha[\frac{1-\overline{B}_2(\sigma_1(s,z_l))}{\sigma_1(s,z_l)}]\overline{W}_2(\sigma_2(s,z_l))\overline{R}(\sigma_2(s,z_l))\} \end{array}\right\}}{\left\{\begin{array}{l} z_h-\overline{B}_1(\phi(s,z_h,z_l))+\alpha[\frac{1-\overline{B}_1(\phi(s,z_h,z_l))}{\phi(s,z_h,z_l)}]\quad \overline{W}_1(\phi_2(s,z_h,z_l)) \\ \times\overline{R}(\phi_2(s,z_h,z_l)) \end{array}\right\}}. \quad (38)$$

Theorem 3.1 *The probability generating function for the Laplace transforms of the number of customers in the high and low priority queue while the system is in idle, regular service, working breakdown service, repair and vacation are given by*

$$\bar{I}_0(s,z_l) = \bar{I}_0(s,0,z_l)[\frac{1-\bar{M}(s+\lambda_h+\lambda_l)}{s+\lambda_h+\lambda_l}], \tag{39}$$

$$\bar{P}^{(1)}(s,z_h,z_l) = \bar{P}^{(1)}(s,0,z_h,z_l)[\frac{1-\bar{B}_1(\phi(s,z_h,z_l))}{\phi(s,z_h,z_l)}], \tag{40}$$

$$\bar{P}^{(2)}(s,z_h,z_l) = \bar{P}_0^{(2)}(s,0,z_l)[\frac{1-\bar{B}_2(\phi_1(s,z_h,z_l))}{\phi_1(s,z_h,z_l)}], \tag{41}$$

$$\bar{Q}^{(1)}(s,z_h,z_l) = \alpha\bar{P}^{(1)}(s,0,z_h,z_l)[\frac{1-\bar{B}_1(\phi(s,z_h,z_l))}{\phi(s,z_h,z_l)}][\frac{1-\bar{W}_1(\phi_2(s,z_h,z_l))}{\phi_2(s,z_h,z_l)}], \tag{42}$$

$$\bar{Q}^{(2)}(s,z_h,z_l) = \alpha\bar{P}_0^{(2)}(s,0,z_l)[\frac{1-\bar{B}_2(\phi_1(s,z_h,z_l))}{\phi_1(s,z_h,z_l)}][\frac{1-\bar{W}_2(\phi_2(s,z_h,z_l))}{\phi_2(s,z_h,z_l)}], \tag{43}$$

$$\begin{aligned}\bar{R}(s,z_h,z_l) = {} & \alpha\bar{P}^{(1)}(s,0,z_h,z_l)\bar{W}_1(\phi_2(s,z_h,z_l))[\frac{1-\bar{B}_1(\phi(s,z_h,z_l))}{\phi(s,z_h,z_l)}][\frac{1-\bar{R}(\phi_2(s,z_h,z_l))}{\phi_2(s,z_h,z_l)}] \\ & +\alpha\bar{P}_0^{(2)}(s,0,z_l)\bar{W}_2(\phi_2(s,z_h,z_l))[\frac{1-\bar{B}_2(\phi_1(s,z_h,z_l))}{\phi_1(s,z_h,z_l)}][\frac{1-\bar{R}(\phi_2(s,z_h,z_l))}{\phi_2(s,z_h,z_l)}],\end{aligned} \tag{44}$$

$$\bar{V}(s,z_h,z_l) = [\frac{(s+\lambda_h+\lambda_l)\bar{I}_{0,0}(s)-1}{(1-p)\bar{V}(s+\lambda_h+\lambda_l)}][\frac{1-\bar{V}(\phi_2(s,z_h,z_l))}{\phi_2(s,z_h,z_l)}]. \tag{45}$$

4. STEADY STATE ANALYSIS: LIMITING BEHAVIOUR

Now, we study the steady state probability distribution for our queueing model. By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t),$$

The normalizing condition

$$\sum_{i=1}^{2}\{P^{(i)}(1,1) + Q^{(i)}(1,1)\} + R(1,1) + V(1,1) + I_0(1) + I_{0,0} = 1$$

The probability generating function of the queue size irrespective of the state of the system.

$$W_q(z_h,z_l) = I_{0,0}\frac{Nr(z_h,z_l)}{Dr(z_h,z_l)}, \tag{46}$$

where,

$$Nr(z_h,z_l) = N_1(z_l)S_1(z_h,z_l) + N_2(z_l)S_2(z_h,z_l) + [\frac{(\lambda_h+\lambda_l)}{(1-p)\bar{V}(\lambda_h+\lambda_l)}]D(z_l)S_3(z_h,z_l),$$

$$\begin{aligned}Dr(z_h,z_l) = {} & D(z_l)\phi_1(z_h,z_l)\phi_2(z_h,z_l)\sigma_1(z_l)\{\phi(z_h,z_l)[z_h - \bar{B}_1(\phi(z_h,z_l))] \\ & -\alpha[1-\bar{B}_1(\phi(z_h,z_l))]\bar{W}_1(\phi_2(z_h,z_l))\bar{R}(\phi_2(z_h,z_l))\},\end{aligned}$$

$$S_1(z_h,z_l) = \phi_1(z_h,z_l)\sigma_1(z_l)[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\{\lambda_h\{C(z_h)-C[g(z_l)]\}[1-\overline{B}_1(\phi(z_h,z_l))]$$
$$\times\{\phi_2(z_h,z_l)+\alpha[1-\overline{W}_1(\phi_2(z_h,z_l))\overline{R}(\phi_2(z_h,z_l))]\}+\phi_2(z_h,z_l)\{\phi(z_h,z_l)$$
$$\times[z_h-\overline{B}_1(\phi(z_h,z_l))]-\alpha[1-\overline{B}_1(\phi(z_h,z_l))]\overline{W}_1(\phi_2(z_h,z_l))\overline{R}(\phi_2(z_h,z_l))\}\},$$

$$S_2(z_h,z_l) = \{\{\phi_1(z_h,z_l)\sigma_1(z_l)[\overline{B}_2(\phi_1(z_h,z_l))-\overline{B}_2(\sigma_1(z_l))]+\alpha\sigma_1(z_l)[1-\overline{B}_2(\phi_1(z_h,z_l))]$$
$$\times\overline{W}_2(\phi_2(z_h,z_l))\overline{R}(\phi_2(z_h,z_l))-\alpha\phi_1(z_h,z_l)[1-\overline{B}_2(\sigma_1(z_l))]\overline{W}_2(\sigma_2(z_l))\overline{R}(\sigma_2(z_l))\}$$
$$\times[1-\overline{B}_1(\phi(z_h,z_l))]\{\phi_2(z_h,z_l)+\alpha[1-\overline{W}_1(\phi_2(z_h,z_l))\overline{R}(\phi_2(z_h,z_l))]\}$$
$$+\sigma_1(z_l)[1-\overline{B}_2(\phi_1(z_h,z_l))]\{\phi_2(z_h,z_l)+\alpha[1-\overline{W}_2(\phi_2(z_h,z_l))\overline{R}(\phi_2(z_h,z_l))]\}$$
$$\times\{\phi(z_h,z_l)[z_h-\overline{B}_1(\phi(z_h,z_l))]-\alpha[1-\overline{B}_1(\alpha)]\overline{W}_1(\phi_2(z_h,z_l))\overline{R}(\phi_2(z_h,z_l))\}\},$$

$$S_3(z_h,z_l) = \phi_1(z_h,z_l)\sigma_1(z_l)\{\{\overline{V}(\phi_2(z_h,z_l))-\overline{V}(\sigma_2(z_l))\}[1-\overline{B}_1(\phi(z_h,z_l))]\{\phi_2(z_h,z_l)$$
$$+\alpha[1-\overline{W}_1(\phi_2(z_h,z_l))\overline{R}(\phi_2(z_h,z_l))]\}+[1-\overline{V}(\phi_2(z_h,z_l))]\{\phi(z_h,z_l)$$
$$\times[z_h-\overline{B}_1(\phi(z_h,z_l))]-\alpha[1-\overline{B}_1(\alpha)]\overline{W}_1(\phi_2(z_h,z_l))\overline{R}(\phi_2(z_h,z_l))\}\},$$

$$N_1(z_l) = (\lambda_h+\lambda_l)\sigma_1(z_l)\{\frac{[\overline{V}(\sigma_2(z_l))-1]}{(1-p)\overline{V}(\lambda_h+\lambda_l)}-1\}\{z_l\psi_1(z)-z_l^2\lambda_l q[1-\overline{B}_2(\psi_1(z))]\}$$
$$+\lambda_l C(z_l)\psi_1(z)\{\sigma_1(z_l)\overline{B}_2(\sigma_1(z_l))+\alpha[1-\overline{B}_2(\sigma_1(z_l))]\overline{W}_2(\sigma_2(z_l))\overline{R}(\sigma_2(z_l))\},$$

$$N_2(z_l) = \psi_1(z)\sigma_1(z_l)\{\lambda_l C(z_l)\{1-\lambda_h C[g(z_l)][\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\}+(\lambda_h+\lambda_l)$$
$$\times\{\frac{[\overline{V}(\sigma_2(z_l))-1]}{(1-p)\overline{V}(\lambda_h+\lambda_l)}-1\}\{\overline{M}(\lambda_h+\lambda_l)+\lambda_l C(z_l)[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\}\},$$

$$D(z_l) = \sigma_1(z_l)\{z_l\psi_1(z)-z_l^2\lambda_l q[1-\overline{B}_2(\psi_1(z))]\}\{1-\lambda_h C[g(z_l)][\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\}$$
$$-\psi_1(z)\{\sigma_1(z_l)\overline{B}_2(\sigma_1(z_l))+\alpha[1-\overline{W}_2(\sigma_1(z_l))]\overline{W}_2(\sigma_2(z_l))\overline{R}(\sigma_2(z_l))\}$$
$$\times\{\overline{M}(\lambda_h+\lambda_l)+\lambda_l C(z_l)[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\}.$$

Now using the normalizing condition, we get

$$I_{0,0} = \frac{D(1)\quad(\lambda_l q+\alpha)^2\{\phi_2'(1)\{\alpha+\phi'(1)[1-\overline{B}_1(\alpha)](1+\alpha[E(W_1)+E(R)])\}\}}{T} \tag{47}$$

where,

$$T = N_1(1)(\lambda_l q+\alpha)^2\phi_2'(1)[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\{[1-\overline{B}_1(\alpha)](1+\alpha[E(W_1)+E(R)])$$
$$\times\{\lambda_h E(X)(1-E(X_1))+\phi'(1)\}+\alpha\}+N_2(1)\{\{-\lambda_l q(\lambda_l q+\alpha)(\sigma_1'(1)-\phi_1'(1))$$
$$\times\overline{B}_2(\lambda_l q+\alpha)+\alpha(\sigma_1'(1)-\phi_1'(1))[1-\overline{B}_2(\lambda_l q+\alpha)]+\alpha(\lambda_l q+\alpha)(\sigma_2'(1)-\phi_2'(1))$$
$$\times[1-\overline{B}_2(\lambda_l q+\alpha)][E(W_2)+E(R)]\}\{\phi_2'(1)[1-\overline{B}_1(\alpha)](1+\alpha[E(W_1)+E(R)])\}$$
$$+\{(\lambda_l q+\alpha)[1-\overline{B}_2(\lambda_l q+\alpha)]\phi_2'(1)(1+\alpha[E(W_2)+E(R)])\}\{\alpha+\phi'(1)[1-\overline{B}_1(\alpha)]$$
$$\times(1+\alpha[E(W_1)+E(R)])\}\}+D(1)(\lambda_l q+\alpha)^2\{\alpha(1+\phi_2'(1)E(V))[1-\overline{B}_1(\alpha)]$$
$$\times(1+\alpha[E(W_1)+E(R)])\{\phi'(1)+[\frac{\lambda_h+\lambda_l}{(1-p)\overline{V}(\lambda_h+\lambda_l)}]\phi_2'(1)E(V)(\sigma_2'(1)-\phi_2'(1)$$
$$+\phi'(1))\}\}$$

5. PERFORMANCE MEASURES

Theorem 5.1 *Whenever, system is in steady state condition, then*

1. The probability that the server is idle and system is empty is $I_{0,0}(t)$ which is given by (47).

2. The probability that the server is idle but system is non-empty is

$$I_{NE} = I_{0,0}\frac{D(1) + N_1(1)[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]}{D(1)}$$

3. The probability that the server is busy on regular service is P=K

where,

$$K = \frac{\left\{\begin{array}{l} (\lambda_l q+\alpha)^2[1-\overline{B}_1(\alpha)]\{\lambda_h(1-E(X_1))E(X)N_1(1)[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}] \\ -[\frac{\lambda_h+\lambda_l}{(1-p)\overline{V}(\lambda_h+\lambda_l)}]D(1)(\phi_2'(1)-\sigma_2'(1))E(V)\}+N_2(1)\{(\lambda_l q+\alpha) \\ \times[1-\overline{B}_2(\lambda_l q+\alpha)]\{\alpha+\phi_2'(1)(1+\alpha[E(W_1)+E(R)])\}+[1-\overline{B}_1(\alpha)] \\ \times\{\lambda_l q(\lambda_l q+\alpha)(\phi_1'(1)-\sigma_1'(1))\overline{B}_2'(\lambda_l q+\alpha)-\alpha[1-\overline{B}_2(\lambda_l q+\alpha)] \\ \times\{(\phi_1'(1)-\sigma_1'(1))+(\lambda_l q+\alpha)(\phi_2'(1)-\sigma_2'(1))[E(W_4)+E(R)]\}\}\} \end{array}\right\}}{\{(\lambda_l q+\alpha)^2 D(1)\{\alpha+\phi_2'(1)[1-\overline{B}_1(\alpha)](1+\alpha[E(W_1)+E(R)])\}\}}$$

4. The probability that the server is busy on working breakdown is $Q=\alpha KE(W_1)$.

5. The probability that the server is under repair is $R=\alpha KE(R)$

6. The probability that the server is on vacation is

$$V = [\frac{\lambda_h+\lambda_l}{(1-p)\overline{V}(\lambda_h+\lambda_l)}]E(V)$$

Proof: Note that

$$I_{NE} = \lim_{z_l\to 1} I_0(z_l)$$

$$P = \lim_{z_h\to 1 z_l\to 1}[P^{(1)}(z_h,z_l)+P^{(2)}(z_h,z_l)]$$

$$Q = \lim_{z_h\to 1 z_l\to 1}[Q^{(1)}(z_h,z_l)+Q^{(2)}(z_h,z_l)]$$

$$R = \lim_{z_h\to 1 z_l\to 1} R(z_h z_l)$$

5.1 The Expected Queue Length

The expected high priority queue size is

$$L_{q_1} = \frac{d}{dz_h} W_{q_1}(z_h, 1)|_{z_h=1} \tag{48}$$

and the expected orbit size is

$$L_{q_2} = \frac{d}{dz_l} W_{q_2}(1, z_l)|_{z_l=1} \tag{49}$$

then

$$L_{q_1} = \frac{DR''(1)NR'''(1) - DR'''(1)NR''(1)}{3(DR''(1))^2}$$

$$L_{q_2} = \frac{dr''(1)nr'''(1) - dr'''(1)nr''(1)}{3(dr''(1))^2}$$

where,

$$NR''(1) = I_{0,0}\{N_1(1)S_1''(1) + N_2(1)S_2''(1) + [\frac{\lambda_h + \lambda_l}{(1-p)\overline{V}(\lambda_h + \lambda_l)}]D(1)S_3''(1)\}$$

$$NR'''(1) = I_{0,0}\{N_1(1)S_1'''(1) + N_2(1)S_2'''(1) + [\frac{\lambda_h + \lambda_l}{(1-p)\overline{V}(\lambda_h + \lambda_l)}]D(1)S_3'''(1)\}$$

$$DR''(1) = 2D(1)(\lambda_l q + \alpha)^2\bar{\phi}_2'(1)\{\alpha + \bar{\phi}'(1)[1 - \overline{B}_1(\alpha)](1 + \alpha[E(W_1) + E(R)])\}$$

$$\begin{aligned} DR'''(1) = {} & 3D(1)(\lambda_l q + \alpha)\{(2\bar{\phi}_1'(1)\bar{\phi}_2'(1) + (\lambda_l q + \alpha)\bar{\phi}_2''(1))\{\alpha + \bar{\phi}'(1)[1 - \overline{B}_1(\alpha)] \\ & \times (1 + \alpha[E(W_1) + E(R)])\} + (\lambda_l q + \alpha)\bar{\phi}_2'(1)\{2\bar{\phi}'(1) + \{\bar{\phi}''(1)[1 - \overline{B}_1(\alpha)] \\ & -2(\bar{\phi}'(1))^2\overline{B}_1'(\alpha)\}(1 + \alpha[E(W_1) + E(R)]) - \alpha[1 - \overline{B}_1(\alpha)](\bar{\phi}'(1))^2 \\ & \times (E(W_1^2) + E(R^2) + 2E(W_1)E(R))\}\} \end{aligned}$$

$$\begin{aligned} S_1''(1) = {} & 2(\lambda_l q + \alpha)^2[\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\{\alpha\bar{\phi}_2'(1) + \{\lambda_h E(X) + \bar{\phi}'(1)\}\bar{\phi}_2'(1)[1 - \overline{B}_1(\alpha)] \\ & \times (1 + \alpha[E(W_1) + E(R)])\} \end{aligned}$$

$$\begin{aligned} S_1'''(1) = {} & 3(\lambda_l q + \alpha)\bar{\phi}_1'(1)[\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\{\bar{\phi}_2'(1)(2\alpha + 2\lambda_h E(X) + 2\bar{\phi}'(1) + \bar{\phi}_2''(1) + \bar{\phi}_1''(1) \\ & +2\bar{\phi}'(1))[1 - \overline{B}_1(\alpha)](1 + \alpha[E(W_1) + E(R)]) + \{(\lambda_h E(X^2)\bar{\phi}_2'(1) + \lambda_h E(X)\bar{\phi}_2''(1)) \\ & \times [1 - \overline{B}_1(\alpha)] - 2\lambda_h E(X)\bar{\phi}'(1)\bar{\phi}_2'(1)\overline{B}_1'(\alpha)\}(1 + \alpha[E(W_1) + E(R)]) - \alpha(\lambda_h E(X) \\ & +\bar{\phi}_2'(1))\bar{\phi}_2'(1)[1 - \overline{B}_1(\alpha)](E(W_1^2) + E(R^2) + 2E(W_1)E(R)) + \alpha\bar{\phi}_2''(1) + \bar{\phi}_1'(1)\bar{\phi}_2'(1)\} \end{aligned}$$

$$\begin{aligned} S_2''(1) = {} & 2\{\lambda_l q(\lambda_l q + \alpha)\bar{\phi}_1'(1)\overline{B}_2'(\lambda_l q + \alpha) + \alpha[1 - \overline{B}_2(\lambda_l q + \alpha)]\{\bar{\phi}_1'(1) - (\lambda_l q + \alpha)\bar{\phi}_2'(1) \\ & \times (E(W_2) + E(R))\}\}\{\bar{\phi}_2'(1)[1 - \overline{B}_1(\alpha)](1 + \alpha[E(W_1) + E(R)])\} + 2\{(\lambda_l q + \alpha) \\ & \times \bar{\phi}_2'(1)[1 - \overline{B}_2(\lambda_l q + \alpha)](1 + \alpha[E(W_2) + E(R)])\}\{\alpha + \bar{\phi}_2'[1 - \overline{B}_1(\alpha)] \\ & \times (1 + \alpha[E(W_1) + E(R)])\} \end{aligned}$$

$$S_2'''(1) = 3\{(\lambda_l q+\alpha)\{2(\bar{\phi}_1'(1))^2\overline{B}_2'(\lambda_l q+\alpha)(1+\alpha[E(W_2)+E(R)])+\lambda_l q\bar{\phi}_1''(1)$$
$$\times\overline{B}_2(\lambda_l q+\alpha)+\lambda_l q(\bar{\phi}_1'(1))^2\overline{B}_2''(\lambda_l q+\alpha)-\alpha(\lambda_l q+\alpha)\bar{\phi}_2''(1)[1-\overline{B}_2(\lambda_l q+\alpha)]$$
$$+\alpha(\bar{\phi}_2')^2[1-\overline{B}_2(\lambda_l q+\alpha)](E(W_2^2)+E(R^2)+2E(W_2)E(R))\}$$
$$\times\{\bar{\phi}_2'(1)[1-\overline{B}_1(\alpha)](1+\alpha[E(W_1)+E(R)])\}+\{\bar{\phi}_2''(1)[1-\overline{B}_1(\alpha)]$$
$$\times(1+\alpha[E(W_1)+E(R)])-2(\bar{\phi}_2'(1))^2\overline{B}_1'(\alpha)(1+\alpha[E(W_1)+E(R)])$$
$$-\alpha(\bar{\phi}_2'(1))^2[1-\overline{B}_1(\alpha)](E(W_1^2)+E(R^2)+2E(W_1)E(R))\}$$
$$\times\{\lambda_l q(\lambda_l q+\alpha)\bar{\phi}_1'(1)\overline{B}_2(\lambda_l q+\alpha)+\alpha\bar{\phi}_1'(1)[1-\overline{B}_2(\lambda_l q+\alpha)]$$
$$-\alpha(\lambda_l q+\alpha)\bar{\phi}_2'(1)[1-\overline{B}_2(\lambda_l q+\alpha)](E(W_2)+E(R))\}+(\lambda_l q+\alpha)$$
$$\times\{\bar{\phi}_2''(1)[1-\overline{B}_2(\lambda_l q+\alpha)](1+\alpha[E(W_2)+E(R)])-2\bar{\phi}_1'(1)\bar{\phi}_2'(1)\overline{B}_2'(\lambda_l q+\alpha)$$
$$\times(1+\alpha[E(W_2)+E(R)])-\alpha(\bar{\phi}_2'(1))^2[1-\overline{B}_2(\lambda_l q+\alpha)](E(W_2^2)+E(R^2)$$
$$+2E(W_2)E(R))\}\{\alpha+\bar{\phi}'(1)[1-\overline{B}_1(\alpha)](1+\alpha[E(W_1)+E(R)])\}$$
$$+\{2\bar{\phi}'(1)+\bar{\phi}''(1)[1-\overline{B}_1(\alpha)](1+\alpha[E(W_1)+E(R)])-2(\bar{\phi}'(1))^2\overline{B}_1'(\alpha)$$
$$\times(1+\alpha[E(W_1)+E(R)])-\alpha(\bar{\phi}_2'(1))^2[1-\overline{B}_1(\alpha)](E(W_1^2)+E(R^2)$$
$$+2E(W_1)E(R))\}\{(\lambda_l q+\alpha)\bar{\phi}_2'(1)[1-\overline{B}_2(\lambda_l q+\alpha)](1+\alpha[E(W_2)+E(R)])\}\}$$

$$S_3''(1) = 2\alpha(\lambda_l q+\alpha)^2\bar{\phi}_2'(1)E(V)$$
$$S_3'''(1) = 3(\lambda_l q+\alpha)\{2\alpha\bar{\phi}_1'(1)\bar{\phi}_2'(1)E(V)+\alpha(\lambda_l q+\alpha)\{\bar{\phi}_2''(1)E(V)-(\phi_2'(1))^2E(V^2)\}$$
$$+(\lambda_l q+\alpha)(\bar{\phi}'(1))^2E(V)\}$$
$$nr''(1) = I_{0,0}\{N_1(1)s_1''(1)+N_2(1)s_2''(1)+[\frac{\lambda_h+\lambda_l}{(1-p)\overline{V}(\lambda_h+\lambda_l)}]D(1)s_3''(1)\}$$
$$nr'''(1) = 3I_{0,0}\{N_1'(1)s_1''(1)+N_2'(1)s_2''(1)+[\frac{\lambda_h+\lambda_l}{(1-p)\overline{V}(\lambda_h+\lambda_l)}]D'(1)s_3''(1)\}$$
$$+I_{0,0}\{N_1(1)s_1'''(1)+N_2(1)s_2'''(1)+[\frac{\lambda_h+\lambda_l}{(1-p)\overline{V}(\lambda_h+\lambda_l)}]D(1)s_3'''(1)\}$$
$$N_1'(1) = -(\lambda_h+\lambda_l)\{\{\sigma_1'(1)+(\lambda_l q+\alpha)[\frac{\sigma_1'(1)E(V)}{(1-p)\overline{V}(\lambda_h+\lambda_l)}]\}\{(\lambda_h+\lambda_l q+\alpha)$$
$$-\lambda_l q[1-\overline{B}_2(\lambda_h+\lambda_l q+\alpha)]\}+(\lambda_l q+\alpha)\{(\lambda_h+\lambda_l q+\alpha)-2\lambda_l q$$
$$\times[1-\overline{B}_2(\lambda_h+\lambda_l q+\alpha)]+\psi_1'(1)[1+\lambda_l q\overline{B}_2'(\lambda_h+\lambda_l q+\alpha)]\}\}$$
$$+\lambda_l\{\{\psi_1'(1)+(\lambda_h+\lambda_l q+\alpha)E(X)\}\{\alpha+\lambda_l q\overline{B}_2(\lambda_l q+\alpha)\}$$
$$+(\lambda_h+\lambda_l q+\alpha)\{\sigma_1'(1)\overline{B}_2(\lambda_l q+\alpha)+\lambda_l q\sigma_1'(1)\overline{B}_2'(\lambda_l q+\alpha)$$
$$-\alpha[1-\overline{B}_2(\lambda_l q+\alpha)]\sigma_2'(1)[E(W_2)+E(R)]\}\}$$

$$N_2'(1) = \{(\lambda_l q+\alpha)\psi_1'(1)+(\lambda_h+\lambda_l q+\alpha)\sigma_1'(1)\}\{\lambda_l\{1-\lambda_h[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\}$$
$$-(\lambda_h+\lambda_l)\{\overline{M}(\lambda_h+\lambda_l)+\lambda_l[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\}\}+(\lambda_h+\lambda_l q+\alpha)(\lambda_l q+\alpha)$$
$$\times\{\lambda_l E(X)\{1-\lambda_h[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\}-\lambda_h\lambda_l E(X_1)E(X)[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]$$
$$-(\lambda_h+\lambda_l)[\frac{\sigma_2'(1)E(V)}{(1-p)\overline{V}(\lambda_h+2)}]\{\overline{M}(\lambda_h+\lambda_l)+\lambda_l[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\}$$
$$-(\lambda_h+\lambda_l)\lambda_l E(X)[\frac{1-\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}]\}$$

$$D'(1) = (\lambda_l q + \alpha)\{(\lambda_h + \lambda_l q + \alpha) - 2\lambda_l q[1 - \overline{B}_2(\lambda_h + \lambda_l q + \alpha)] + \psi_1'(1)[1 + \lambda_l q \times \overline{B}_2'(\lambda_h + \lambda_l q + \alpha)]\}\{1 - \lambda_h[\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\} + (\lambda_l q + \alpha)\{(\lambda_h + \lambda_l q + \alpha) - \lambda_l q[1 - \overline{B}_2(\lambda_h + \lambda_l q + \alpha)]\}\{-\lambda_h E(X_1)E(X)[\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\} + \sigma_1'(1)\{1 - \lambda_h[\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\}\{-\lambda_h E(X_1)E(X)\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\} - \psi_1'(1)\{\overline{M}(\lambda_h + \lambda_l) + \lambda_l[\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\}\{\alpha + \lambda_l q \overline{B}_2(\lambda_l q + \alpha)\} - (\lambda_h + \lambda_l q + \alpha)\{\alpha + \lambda_l q \overline{B}_2(\lambda_l q + \alpha)\}\{\lambda_l E(X)[\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\} - (\lambda_h + \lambda_l q + \alpha)\{\overline{M}(\lambda_h + \lambda_l) + \lambda_l[\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\}\{\sigma_1'(1)\overline{B}_2(\lambda_l q + \alpha) + \lambda_l q \sigma_1'(1) \times \overline{B}_2'(\lambda_l q + \alpha) - \alpha[1 - \overline{B}_2(\lambda_l q + \alpha)]\sigma_2'(1)[E(W_2) + E(R)]\}$$

$$s_1''(1) = (\lambda_l q + \alpha)^2[\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\{-2\lambda_h E(X_1)E(X) + 2\underline{\phi}_2'(1)\}\{\underline{\phi}_2'(1)[1 - \overline{B}_1(\alpha)] \times (1 + \alpha[E(W_1) + E(R)])\}$$

$$s_1'''(1) = [\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\{6(\lambda_l q + \alpha)(\underline{\phi}_1'(1) + \sigma_1'(1))\{\underline{\phi}_2'(1) - \lambda_h E(X_1)E(X)\} \times \{\underline{\phi}_2'(1)[1 - \overline{B}_1(\alpha)](1 + \alpha[E(W_1) + E(R)])\} + 3(\lambda_l q + \alpha)^2\{\underline{\phi}_2''(1) - \lambda_h E(X_1^2)E(X) - \lambda_h E(X_1)E(X^2)\}\{\underline{\phi}_2'(1)[1 - \overline{B}_1(\alpha)](1 + \alpha[E(W_1) + E(R)])\} + 3(\lambda_l q + \alpha)^2 \times \{\underline{\phi}_2'(1) - \lambda_h E(X_1)E(X)\}\{\underline{\phi}_2''(1)[1 - \overline{B}_1(\alpha)](1 + \alpha[E(W_1) + E(R)]) - 2(\underline{\phi}_2'(1))^2\overline{B}_1'(\alpha)(1 + \alpha[E(W_1) + E(R)]) - \alpha(\underline{\phi}_2'(1))^2[1 - \overline{B}_1(\alpha)] \times (E(W_1^2) + E(R^2) + 2E(W_1)E(R))\}\}$$

$$s_3''(1) = 2(\lambda_l q + \alpha)^2\phi_2'(1)\sigma_2'(1)[1 - \overline{B}_1(\alpha)](1 + \alpha[E(W_1) + E(R)])E(V)$$

$$s_3'''(1) = 6(\lambda_l q + \alpha)(\underline{\phi}_1'(1) + \sigma_1^1)\underline{\phi}_2'(1)\sigma_2'(1)[1 - \overline{B}_1(\alpha)](1 + \alpha[E(W_1) + E(R)])E(V) + 3(\lambda_l q + \alpha)^2\underline{\phi}_2'(1)[1 - \overline{B}_1(\alpha)](1 + \alpha[E(W_1) + E(R)])\{\sigma_2''(1)E(V) + (\sigma_2'(1))^2E(V^2)\} + 3(\lambda_l q + \alpha)^2\sigma_2'(1)E(V)\{\underline{\phi}_2''(1)[1 - \overline{B}_1(\alpha)](1 + \alpha[E(W_1) + E(R)]) - 2(\underline{\phi}_2'(1))^2\overline{B}_1'(\alpha) \times (1 + \alpha[E(W_1) + E(R)]) - \alpha(\underline{\phi}_2'(1))^2[1 - \overline{B}_1(\alpha)](E(W_1^2) + E(R^2) + 2E(W_1)E(R))\}$$

$$s_2''(1) = 2\{\underline{\phi}_2'(1)[1 - \overline{B}_1(\alpha)](1 + \alpha[E(W_1) + E(R)])\}\{\{\lambda_l q(\lambda_l q + \alpha)(\underline{\phi}_1'(1) - \sigma_1'(1)) \times \overline{B}_2'(\lambda_l q + \alpha) - \alpha(\underline{\phi}_1'(1) - \sigma_1'(1))[1 - \overline{B}_2(\lambda_l q + \alpha)] - \alpha(\lambda_l q + \alpha)(\underline{\phi}_2'(1) - \sigma_2'(1)) \times [1 - \overline{B}_2(\lambda_l q + \alpha)](E(W_2) + E(R))\} + \{(\lambda_l q + \alpha)\underline{\phi}_2'(1)[1 - \overline{B}_2(\lambda_l q + \alpha)] \times (1 + \alpha[E(W_2) + E(R)])\}\}$$

$$\begin{aligned} s_2^{'''}(1) = {} & 3\{\underline{\phi}_2^{''}(1)[1-\overline{B}_1(\alpha)](1+\alpha[E(W_1)+E(R)]) - 2(\underline{\phi}_2^{'}(1))^2\overline{B}_1^{'}(\alpha)(1+\alpha[E(W_1) \\ & +E(R)]) - \alpha(\underline{\phi}_2^{'}(1))^2[1-\overline{B}_1(\alpha)](E(W_1^2)+E(R^2)+2E(W_1)E(R))\} \\ & \times\{\{\lambda_l q(\lambda_l q+\alpha)(\underline{\phi}_1^{'}(1)-\sigma_1^{'}(1))\overline{B}_2^{'}(\lambda_l q+\alpha) - \alpha(\underline{\phi}_1^{'}(1)-\sigma_1^{'}(1))[1-\overline{B}_2(\lambda_l q+\alpha)] \\ & -\alpha(\lambda_l q+\alpha)(\underline{\phi}_2^{'}(1)-\sigma_2^{'}(1))[1-\overline{B}_2(\lambda_l q+\alpha)](E(W_2)+E(R))\} + \{(\lambda_l q+\alpha)\underline{\phi}_2^{'}(1) \\ & \times[1-\overline{B}_2(\lambda_l q+\alpha)](1+\alpha[E(W_2)+E(R)])\}\} + 3\{\underline{\phi}_2^{'}(1)[1-\overline{B}_1(\alpha)] \\ & \times(1+\alpha[E(W_1)+E(R)])\}\{\{2(\lambda_l q+\alpha)((\underline{\phi}_1^{'}(1))^2-(\sigma_1^{'}(1))^2)\overline{B}_2^{'}(\lambda_l q+\alpha) \\ & +\lambda_l q(\lambda_l q+\alpha)(\underline{\phi}_1^{''}(1)-\sigma_1^{''}(1))\overline{B}_2^{'}(\lambda_l q+\alpha) + \lambda_l q(\lambda_l q+\alpha)((\underline{\phi}_1^{'}(1))^2-(\sigma_1^{'}(1))^2) \\ & \times\overline{B}_2^{''}(\lambda_l q+\alpha) - \alpha(\underline{\phi}_1^{''}(1)-\sigma_1^{''}(1))[1-\overline{B}_2(\lambda_l q+\alpha)] - 2\alpha(\sigma_1^{'}(1)\underline{\phi}_2^{'}(1) \\ & -\underline{\phi}_1^{'}(1)\sigma_2^{'}(1))[1-\overline{B}_2(\lambda_l q+\alpha)](E(W_2)+E(R)) + 2\alpha(\lambda_l q+\alpha)(\underline{\phi}_1^{'}(1)\underline{\phi}_2^{'}(1) \\ & -\sigma_1^{'}(1)\sigma_2^{'}(1))\overline{B}_2^{'}(\lambda_l q+\alpha)(E(W_2)+E(R)) - \alpha(\lambda_l q+\alpha)(\underline{\phi}_2^{''}(1)-\sigma_2^{''}(1)) \\ & \times[1-\overline{B}_2(\lambda_l q+\alpha)](E(W_2)+E(R)) + \alpha(\lambda_l q+\alpha)((\underline{\phi}_2^{'}(1))^2-(\sigma_2^{'}(1))^2) \\ & \times[1-\overline{B}_2(\lambda_l q+\alpha)](E(W_2^2)+E(R^2)+2E(W_2)E(R))\} + \{2\sigma_1^{'}(1)\underline{\phi}_2^{'}(1) \\ & \times[1-\overline{B}_2(\lambda_l q+\alpha)](1+\alpha[E(W_2)+E(R)]) - 2(\lambda_l q+\alpha)\underline{\phi}_1^{'}(1)\underline{\phi}_2^{'}(1)\overline{B}_2^{'}(\lambda_l q+\alpha) \\ & \times(1+\alpha[E(W_2)+E(R)]) + (\lambda_l q+\alpha)\underline{\phi}_2^{''}(1)[1-\overline{B}_2(\lambda_l q+\alpha)] \\ & \times(1+\alpha[E(W_2)+E(R)]) - \alpha(\lambda_l q+\alpha)[1-\overline{B}_2(\lambda_l q+\alpha)](\underline{\phi}_2^{'}(1))^2 \\ & \times(E(W_2^2)+E(R^2)+2E(W_2)E(R))\}\}. \end{aligned}$$

5.2 The Expected Waiting Time

Expected waiting time of high priority customers is

$$W_{q_1} = \frac{L_{q_1}}{\lambda_h} \tag{52}$$

Expected waiting time of low priority customers is

$$W_{q_2} = \frac{L_{q_2}}{\lambda_l} \tag{53}$$

6. STOCHASTIC DECOMPOSITION

A study of stochastic decomposition for retrial model is done in Yang et al. [19] and Yang and Templeton [18]. The existence of the stochastic decomposition property of our model can be considering

$$\Phi(z) = \Psi(z)\Pi(z)$$

where Φ (z) is the PGF of the queue size distribution of an $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ queue with priority services, customers abandonment, working breakdown, repair and randomized vacation policy, which can be obtained for equation (46) by putting $\overline{M}(\lambda_h + \lambda_l) = 1$. Thus we have

$$\Phi(z) = I_{0,0}\frac{n(z_h, z_l)}{d(z_h, z_l)}$$

where,

$$\begin{aligned} n(z_h, z_l) = {} & \{(\lambda_h + \lambda_l)\sigma_1(z_l)\{\frac{[\overline{V}(\sigma_2(z_l)) - 1]}{(1-p)\overline{V}(\lambda_h + \lambda_l)} - 1\}\{z_l\psi_1(z) - z_l^2\lambda_l q[1 - \overline{B}_2(\psi_1(z))]\} \\ & + \lambda_l C(z_l)\psi_1(z)\{\sigma_1(z_l)\overline{B}_2(\sigma_1(z_l)) + \alpha[1 - \overline{B}_2(\sigma_1(z_l))]\overline{W}_2(\sigma_2(z_l))\overline{R}(\sigma_2(z_l))\}\} \\ & \times S_1(z_h, z_l) + \{\psi_1(z)\sigma_1(z_l)\{\lambda_l C(z_l) + (\lambda_h + \lambda_l)\{\frac{[\overline{V}(\sigma_2(z_l)) - 1]}{(1-p)\overline{V}(\lambda_h + \lambda_l)} - 1\}\}\}S_2(z_h, z_l) \\ & + [\frac{\lambda_h + \lambda_l}{(1-p)\overline{V}(\lambda_h + \lambda_l)}]\{\sigma_1(z_l)\{z_l\psi_1(z) - z_l^2\lambda_l q[1 - \overline{B}_2(\psi_1(z))]\} \\ & \psi_1(z)\{\sigma_1(z_l)\overline{B}_2(\sigma_1(z_l)) + \alpha[1 - \overline{B}_2(\sigma_1(z_l))]\overline{W}_2(\sigma_2(z_l))\overline{R}(\sigma_2(z_l))\}\}S_3(z_h, z_l), \\ d(z_h, z_l) = {} & \phi_1(z_h, z_l)\phi_2(z_h, z_l)\sigma_1(z_l)\{\phi(z_h, z_l)[z_h - \overline{B}_1(\phi(z_h, z_l))] - \alpha[1 - \overline{B}_1(\phi(z_h, z_l))] \\ & \times \overline{W}_1(\phi_2(z_h, z_l))\overline{R}(\phi_2(z_h, z_l))\}\{\sigma_1(z_l)\{z_l\psi_1(z) - z_l^2\lambda_l q[1 - \overline{B}_2(\psi_1(z))]\} \\ & - \psi_1(z)\{\sigma_1(z_l)\overline{B}_2(\sigma_1(z_l)) + \alpha[1 - \overline{B}_2(\sigma_1(z_l))]\overline{W}_2(\sigma_2(z_l))\overline{R}(\sigma_2(z_l))\}\}. \end{aligned}$$

$\Pi(z)$ is the PGF of the conditional distribution of the number of customers in the orbit given that the system is idle. That is,

$$\Pi(z) = \frac{I_{0,0} + I_0(z_l)}{I_{0,0} + I_0(1)},$$

which is equal to

$$\Pi(z) = \frac{\left\{\begin{aligned} & D(1)\{\sigma_1(z_l)\{z_l\psi_1(z) - z_l^2\lambda_l q[1 - \overline{B}_2(\psi_1(z))]\}\{\overline{M}(\lambda_h + \lambda_l) \\ & -\{\frac{\lambda_h C(g[z_h])}{\lambda_h + \lambda_l} - \frac{\overline{V}(\sigma_2(z_l)) - 1}{(1-p)\overline{V}(\lambda_h + \lambda_l)}\}[1 - \overline{M}(\lambda_h + \lambda_l)]\}\} \end{aligned}\right\}}{\left\{\begin{aligned} & D(z_l)(\lambda_l q + \alpha)\{(\lambda_h + \lambda_l q + \alpha) + \lambda_l q[1 - \overline{B}_2(\lambda_h + \lambda_l q + \alpha)]\} \\ & \times \{\overline{M}(\lambda_h + \lambda_l) - \lambda_h[\frac{1 - \overline{M}(\lambda_h + \lambda_l)}{\lambda_h + \lambda_l}]\}. \end{aligned}\right\}}$$

7. PARTICULAR CASES

Case 1: If there is no high priority queue, no retrial, no breakdown, no abandoned customers, low priority customers arrive in single. i.e. $\lambda_h = 0$, $C(z_l) = z$, $\bar{M}(\lambda_h + \lambda_l) = 1$, $\alpha = 0$, $q = 0$. Then, our model reduced to M/G/1 queue with general randomized vacation policy.

$$P^{(2)}(z) = \frac{I_{0,0}[1 - \overline{B}_2(\lambda_l(1-z))]\{\overline{V}(\lambda_l(1-z)) - (1-z)(1-p)\overline{V}(\lambda_l) - 1\}}{(1-z)(1-p)\overline{V}(\lambda_l)\{z - \overline{B}_2(\lambda_l(1-z))\}},$$

$$V(z) = \frac{I_{0,0}[1 - \overline{V}(\lambda_l(1-z))]}{(1-p)(1-z)\overline{V}(\lambda_l)}.$$

The above result similar to the result of Chen et al. [3] without second optional service.

Case 2: If there is no high priority queue, no breakdown, no vacation, low priority customers arrive in single. i.e. $\lambda_h = 0$, $C(z_l) = z$, $p = 0$, $\bar{V}(\lambda_h + \lambda_l) = 1$, $\alpha = 0$. Then, our model reduced to M/G/1 queue with abandoned customers.

$$I_0(z) = \frac{I_{0,0} z(1-z)[1-\bar{B}_2(\lambda_h(1-z)+\lambda_l qz)][1-\bar{M}(\lambda_l)]}{D(z)},$$

$$P^{(2)}(z) = \frac{I_{0,0}\bar{M}(\lambda_l)(1-z)[1-\bar{B}_2(\lambda_l(1-z)+\lambda_l qz)]}{D(z)},$$

where,

$$D(z) = \bar{B}_2(\lambda_l(1-z)+\lambda_1 qz)\{(1-z+qz)(z+(1-z)\bar{M}(\lambda_1)) - z^2 q\} - z(1-z).$$

The above result coincides with the result of Krishna Kumar et al.[12] without second optional service.

8. NUMERICAL RESULTS

We present a set of numerical examples which were computed using MATLAB software. The system's parameter and distributions are as follows:

1. The high and low priority arrival rate $\lambda_h = 1$ and $\lambda_l = 3.2$.
2. The normal service time is exponentially distributed with parameter $\mu_1 = \mu_2 = 8$.
3. The working breakdown service time, repair time and vacation time follows exponential and Erlang-2 distribution with parameters $\omega_1 = \omega_2 = 5$, $\eta=4$ and $\gamma = 0.6$ respectively.
4. $p = 0.5$ and $q = 0.9$.
5. The retrial time for different distributions are as follows:

(a) Exponential distribution with parameter $\beta = 23$.

(b) Erlang-2 distribution with parameter $\beta = 23$.

(c) Hyper-exponential distribution with parameters $c = 0.5$, $\beta_1 = 5$ and $\beta_2 = 15$.

(d) Phase-type distribution with PH(τ, T), where

$$\tau = [1/3 \quad 2/3] T = \begin{bmatrix} -2 & 0.5 \\ 0.3 & -4.8 \end{bmatrix} t_0 = \begin{bmatrix} 1.5 \\ 4.5 \end{bmatrix}$$

Table 2, shows that when the breakdown rate (α) increases, then the expected queue length of high and low priority increases with working breakdown service, repair, vacation are exponential distribution and various retrial distributions. The expected queue length of high priority and low priority increases for increasing breakdown (α) with working breakdown service, repair, vacation are Erlang-2 distribution and various retrial distributions as shown in table 3.

In Figs. 1-2 two dimensional graphs are illustrated. Figure 1, presents the expected queue length of low priority (L_{q_2}) increases for increasing breakdown (α) with working breakdown service, repair, vacation are exponential distribution and various retrial distributions. The expected queue length of low priority (L_{q_2}) increases for increasing breakdown (α) with working breakdown service, repair, vacation are Erlang-2 distribution and various retrial distributions as shown in Fig. 2.

In Figs. 3-4 three dimensional graphs are illustrated. In Fig. 3, the surface displays an upward trend for expected queue length of high priority (L_{q_1}) against increasing high priority arrival rate (λ_h) and breakdown rate (α). The surface displays a downward trend for idle against increasing high priority arrival rate (λ_h) and breakdown rate (α) as shown in Fig. 4.

Table 2: Impact of α on Mean Values of Queues length

	Exponential distribution							
α	Expected queue length of high priority				Expected queue length of low priority			
	EXP	**ERL**	**HEY**	**PH(2)**	**EXP**	**ERL**	**HEY**	**PH(2)**
0.1	11.2809	11.2802	11.2572	11.2230	28.2430	28.4562	35.1601	44.8123
0.2	16.3003	16.2989	16.2523	16.1826	29.0597	29.2760	36.0778	45.8733
0.3	22.0229	22.0206	21.9486	21.8408	29.8691	30.0884	36.9862	46.9222
0.4	28.4699	28.4668	28.3675	28.2189	30.6697	30.8920	37.8834	47.9569
0.5	35.6619	35.6579	35.5296	35.3373	31.4597	31.6849	38.7674	48.9749

Table 3: Impact of α on mean values of queues length

	Erlang-2 distribution							
α	Expected queue length of high priority				Expected queue length of low priority			
	EXP	**ERL**	**HEY**	**PH(2)**	**EXP**	**ERL**	**HEY**	**PH(2)**
0.1	11.5903	11.5897	11.5721	11.5458	26.6654	26.8778	33.5546	43.1663
0.2	16.9289	16.9278	16.8923	16.8392	27.4930	27.7084	34.4828	44.2372
0.3	22.9790	22.9773	22.9225	22.8405	28.3139	28.5323	35.4022	45.2965
0.4	29.7600	29.7577	29.6822	29.5693	29.1263	29.3477	36.3109	46.3421
0.5	37.2907	37.2877	37.1903	37.0443	29.9287	30.1529	37.2068	47.3715

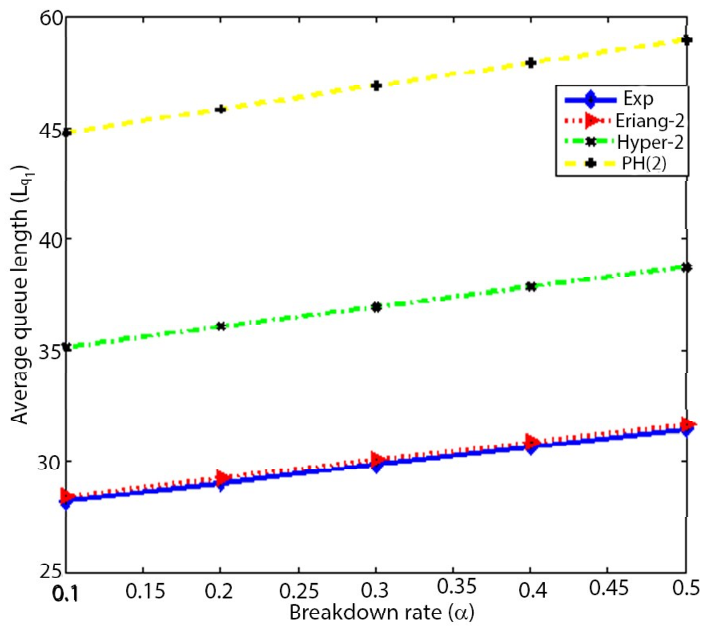

Figure 1: L_{q2} Versus α for Various Retrial Distributions

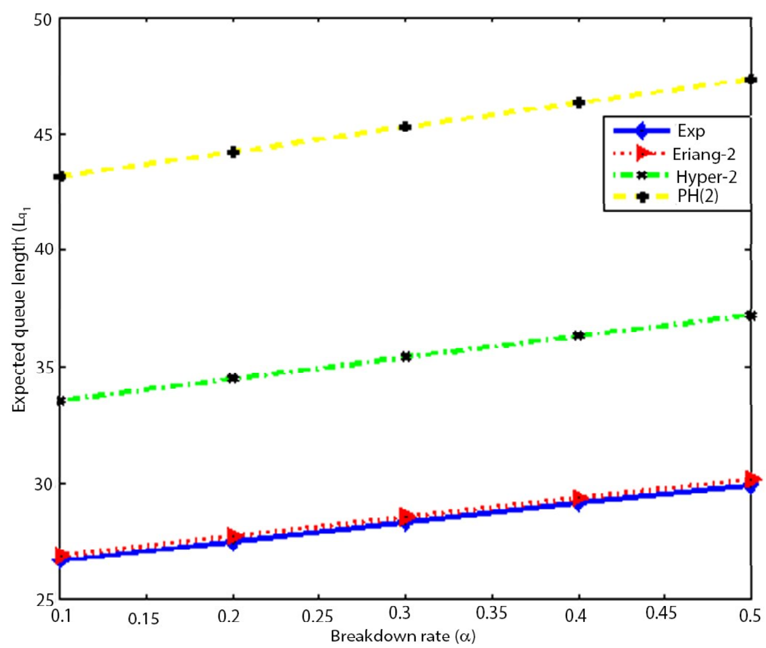

Figure 2: L_{q2} Versus α for Various Retrial Distributions

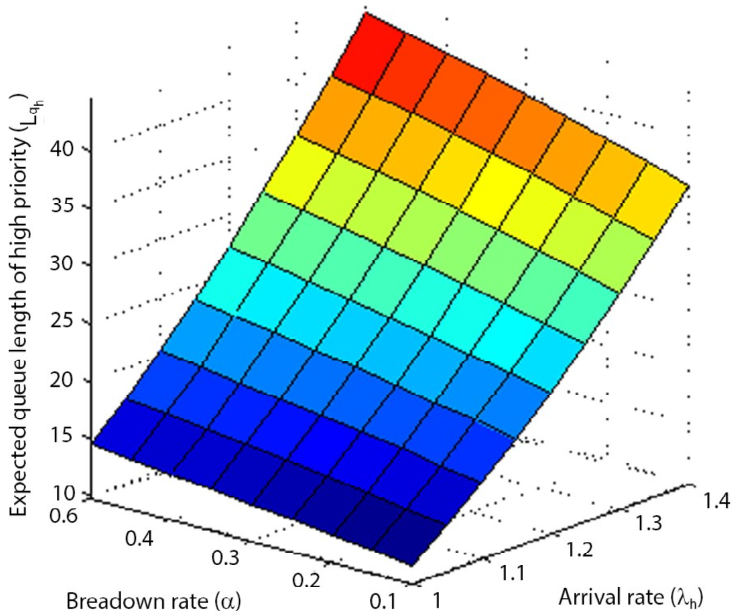

Figure 3: L_{q1} Versus α and λ_h

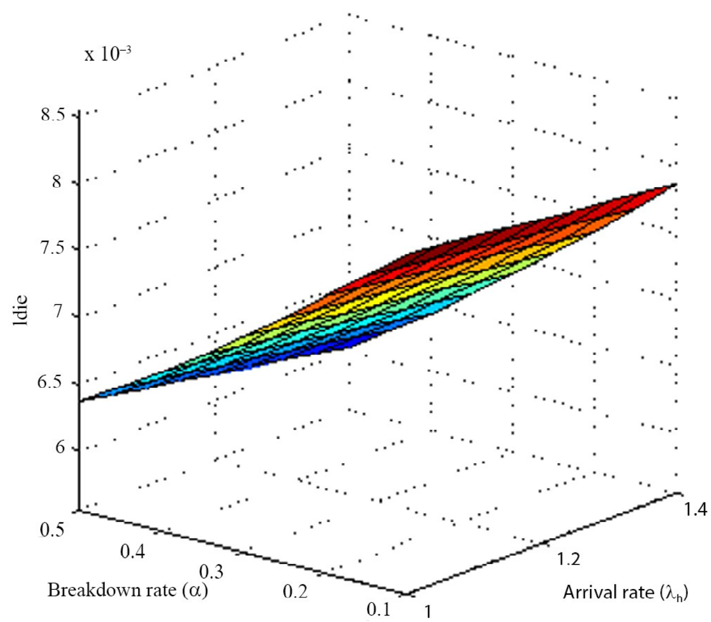

Figure 4: Idle versus α and λ_h.

9. CONCLUSIONS

In this paper, a single server batch arrival nonpreemptive priority based retrial queueing system with abandoned customers, working breakdown and repair under general randomized vacation policy is analyzed. The probability generating function of the queue size distribution at an arbitrary time is obtained and some performance measures are calculated. Finally, we present some numerical examples to study the effect of various parameters. For future research, the discretionary priority based on service consider the similar model.

REFERENCES

[1] Artalejo, J.R., Dudin, A.N., and Klimenok, V.I. (2001). Stationary analysis of a retrial queue with preemptive repeated attempts, Operations Research Letters. 28(4), 173-180.

[2] Bong Dae Choi and Dong Bi Zhu. (1998). The M_1,M_2/G/1/K retrial queueing systems with priority, J. Korean Math. Soc., 35(3) 691-712.

[3] Chen Y., Lin XW., Wei CM., Fan Z. (2018). An M/G/1 queue with second optional service and general randomized vacation policy. In: Cao BY. (eds) Fuzzy Information and Engineering and Decision. IWDS 2016. Advances in Intelligent Systems and Computing, vol 646. Springer, Cham.

[4] Dong-Yuh Yang & Yi-Hsuan Chen.(2018). Computation and optimization of a working breakdown queue with second optional service, Journal of Industrial and Production Engineering, 35(3), 181-188.

[5] Gautam Choudhury and Jau-Chuan Ke. (2012). A batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair,Applied Mathematical Modelling, 36(1), 255-269.

[6] Gomez-Corral. A. (1999). Stochastic analysis of a single server retrial queue with general retrial times, Naval Research Logistics, 46, 561-581.

[7] Jau-Chuan Ke, Kai-Bin Huang and Wen Lea Pearn. (2010). The randomized vacation policy for a batch arrival queue, Applied Mathematical Modelling, 34, 1524-1538.

[8] Jau-Chuan Ke,Fu-Min Chang. (2009). Modified vacation policy for M/G/1 retrial queue with balking and feedback, Computers & Industrial Engineering, 57(1), 433-443.

[9] Kalidass. K and Kasturi. R.(2012). A queue with working breakdowns, Computers and Industrial Engineering, 63(4), 779-783.

[10] Kapyrim, V.A. (1977). A study of stationary distributions of a queueing system with recurring demands, Cybernatics, 13, 548-590.

[11] Krishna Kumar B. and Arivudainambi D. (2002) The M/G/1 retrial queue with Bernoulli schedules and general retrial times, Computers & Mathematics with Applications, 43(1-2), 15-30.

[12] Krishna Kumar B., Vijayakumar A. and Arivudainambi. D. (2002). An M/G/1 Retrial Queueing System with Two-Phase Service and Preemptive Resume, Annals of Operations Research, 113(1-4), 61-79.

[13] Krishnamoorthy A., Joshua V.C., Mathew A.P. (2017) A Retrial Queueing System with Abandonment and Search for Priority Customers. In: Vishnevskiy V., Samouylov K., Kozyrev D. (eds) Distributed Computer and Communication Networks. DCCN 2017. Communications in Computer and Information Science, vol 700. Springer, Cham.

[14] Madhu Jain, Amita Bhagat and Chandra Shekhar.(2015). Double orbit finite retrial queues with priority customers and service interruptions, Applied Mathematics and Computation, 253,324-344.

[15] Shan Gao, Bin Liu and Jinting Wang. (2018). On a retrial queue with abandoned customers and multi-optional vacations, International Journal of Computer Mathematics: Computer Systems Theory, 3(3), 177-195.

[16] Shan Gao and Yunfei Yao. (2014). An M^X/G/1 queue with randomized working vacations and at most J vacations, International Journal of Computer Mathematics, 91(3), 368-383.

[17] Wang, J. (2008). On the single server retrial queue with priority subscribers and server breakdowns, Journal of Systems Science and Complexity. 21(2), 304-315.

[18] Yang. T and Templeton.J.G.C.(1987). A survey on retrial queue, Queueing Systems. Vol 2, 201-233.

[19] Yang. T, Posner.M.J.M, Templeton.J.G.C and Li.H. (1994). An approximation method for the M/G/1 retrial queue with general retrial times, Eur.J.Oper.Res., Vol 76, 552-562

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