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WEIGHTED COMPOSITION OPERATORS WITH CLOSED RANGES

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Abstract: In the present paper we study weighted composition operators which have closed ranges. We also study compact weighted composition operators acting on a space of operators.

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INTRODUCTION AND PRELIMINARIES

Let X and Y be two non-empty sets and let $V(X)$ and $V(Y)$ be topological vector spaces of complex-valued functions defined on X and V respectively. If $T : Y \to X$ is a mapping such that $f \circ T \in V(Y)$ whenever $f \in V(X)$, then we can define a composition transformation $C_T : V(X) \to V(X)$ By

$$C_T F = f \circ T \text{ for every } f \in V(X).$$

A continuous linear operator C_T is called a composition operator induced by T. If $U : X \to C$ is a function such that $U.f \in V(X)$ whenever $f \in V(X)$, then a multiplication operator $M_U : V(X) \to V(X)$ is a continuous linear transformation defined by

$$M_U f = U.f \text{ for every } f \in V(X).$$

The class of composition operators and the class of multiplication operators under suitable situation breed another class of operators called weighted composition operators. Thus a weighted composition operator is an operator of the type $M_U C_T$ or $C_T M_U$ and we usually denote it by $M_{u,T}$..

So far as we know composition operator made its first appearance in 1871 in a paper of Schroeder [8] in which it was asked to find a function f and a number such that $(f \circ T)(z) = \alpha f(z)$ for every given T and for every z in an appropriate domain. In 1884, Koenighs [2] solved this problem. Afterwards these operators found their application in Littlewood's [5] subordination theory. But a systematic study of composition operators is started in 1969 with the paper of Nordgren [6] followed by Schwartz [9], Ridge [7]. For more details about composition operators and weighted composition operators we refer to ([1], [10], [3], [4]) and references therein.

Let H be a Hilbert space. By $B_L(H)$ we denote the Banach space of all bounded linear operators from H into itself under the norm defined as

$$\|A\| = \sup_{\substack{x \neq 0 \\ x \in H}} \frac{\|Ax\|}{\|x\|}.$$

The symbol $B_b(H)$ denotes the Banach space of all bounded functions defined on H into itself under the norm

$$\|f\| = \sup\{\|f(x)\| : x \in H\}.$$

We point out that no non-zero element of $B_b(H)$ is a member of $B_L(H)$. For, if $A \in B_L(H)$ is such that $A(x) \neq 0$ for some $x \in H$, then $A(nx) = nA(x) \to \infty$ as $n \to \infty$.. Thus A is not bounded as a function. Hence $B_L(H) \cap B_b(H) = \{0\}$. If $A \in B_L(H)$, then the range of A is defined to be the set $\{Af : f \in H\}$ and it is denoted by ran A. The kernel of A is defined to be the set $\{f \in H : Af = 0\}$ and it is denoted by ker A. By BL_b *(H;H)* we denote the set of all bounded linear operators from $B_L(H)$ into $B_b(H)$.

A continuous linear operator $A : H \to H$ is called a compact operator, if $A(B_1)$ is precompact i.e. $A(B_1)$ is a compact subset of H, where $B_1 = \{x \in H : \|x\| \leq 1\}$ is the closed unit ball of H.

CHARACTERIZATION OF WEIGHTED COMPOSITION OPERATORS

In this section we characterize weighted composition operators with closed ranges.

Theorem 2.1. If *abs*(c_0(*ran u:T*)) has an interior point as a subset of span(*ran u:T*), then $M_{u,T}$ has closed range.

Proof. Suppose that zero is the interior point of $\bar{abs}(c_0(ran \;\; u.T))$. If the sequence $\{M_{u,T}A_n\}$ is convergent, then $\{A_n\}$ is uniformly convergent on ranu:T and consequently uniformly Cauchy. That is, for any $\epsilon > 0$ there is some positive integer n0 such that for any $x \in ran \;\; u.T$,

$$\|(A_m - A_n)(x)\| < \epsilon \tag{2.1}$$

for any positive integers m and n larger than n0. By the triangular inequality, the same inequality is valid if in equation (2.1) x is replaced by any absolutely convex combination of points in *ran u:T*. It follows that the sequence $\{A_n\}$ is uniformly Cauchy on $\bar{abs}(c_0(ranu.T))$. Let $r > 0$ be such that for any $x \in span(ranu.T)$ with $\|x\| < r,$, we have $x \in \bar{abs}(c_0(ran \;\; u.T))$. Then equation (2.1) holds for any such x and consequently the sequence $\{A_n|span(ran \;\; (u.T))\}$ regarded as a sequence of continuous operators between *span (ran u.T)* and H is a norm Cauchy sequence and hence tends to A_0. Set $Ax = A_0x$ for $x \in span(ran \;\; u.T)$ and $Ax = 0$ for $x \in (span(ran \;\; u.T))^{\perp}$. We obtain $A \in B_L(H)$ such that $A_n|span(ran \;\; u.T) \to A|span(ran \;\; u.T)$ in the norm topology. We shall have that the sequence $(u.A_n \circ T)(x)$ tends to $(u.A \circ T)(x)$ for any $x \in H$ and so the limit in $B_b(H)$ of $M_{u,T}A_n$ is $M_{u,T}A$. Hence $M_{u,T}$ has closed range. Since in a convex set which has interior point, any internal point of any absolutely convex set, including $\bar{abs}(c_0(ranu.T))$. This completes the proof.

Example 2.1. Let $H = l^2$, the Hilbert space of square summable sequences of complex numbers. Let $u : l^2 \to l^2$ be defined by

$$u(x) = \begin{cases} 0, & \text{if } \|x\| > 1 \\ 1, & \text{elsewhere} \end{cases}.$$

Let $T : H \to H$ be defined as $T(x) = x$, for every $x \in H$. Then ran $u.T = B_1$ and so $\bar{span}(B_1) = H$. Clearly 0 is an interior point of $\bar{abs}(c_0(ran\ u.T))$. Hence M_u, T has closed range.

In next example, we show that if H is an innite dimensional separable Hilbert space, then weighted composition operators having non-closed ranges exist.

Example 2.2. Let H^2 be the classical Hardy space, that is, the space of all functions analytic on the open unit disc having square summable Taylor coecients and H^∞, the algebra of all bounded analytic functions on the unit disc. For any $\phi \in H^\infty$, T_ϕ is the analytic Toeplitz operator induced by ϕ, that is, the operator $T_\phi f = \phi f$ for every $f \in H^2$.

We recall that if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two series, their product $\sum_{n=0}^{\infty}\sum_{n=0}^{\infty} b_n$ is the series having summands $c_n = a_0 b_n + a_1 b_{n-1} + ... + a_n b_0$. Choose $\phi \in H^2$, $\phi(z) \sum_{n=0}^{\infty} \lambda_n z^n$ such that $\sum_{n=0}^{\infty} \lambda_n^2$ fails having square summable summands.

Using the logarithm test, after straight forward calculations one can show that $\phi(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}$, where $\alpha \in (1/2, 3/4)$ is such a choice.

Set $A_n = T_{\phi_n}$, where $\{\phi_n\}$ is a sequence in H^∞ which tends to $\phi \in H^2$ For any $T : H^2 \to H^2$ such that *ran* $u.T = \{1, z, z^2, ..., z^n, ...\}$.

It is obvious that

$$\sup_{f \in H^2} \|(M_{u,T}A_n - M_{u,T}A_m)(f)\|_2 = \sup_k \|(\phi_m - \phi_n)z^k\|_2 = \|(\phi_m - \phi_n)\|_2.$$

Since $\{\phi_n\}$ tends to, ϕ, $\{M_{u,T}A_n\}$ is uniformly Cauchy, hence there is $f \in B_b(H^2)$, which is uniform limit of the sequence $\{M_{u,T}A_n\}$.

If we suppose $f = M_{u,T}A$ for some $A \in B_L(H)$, we get $(A \circ u.T)(h) = \lim_{n\to\infty} \phi_n(u.T)h$ for any $h \in H^2$. Since $\|(\phi_n - \phi)z^k\|_2 \to 0$ as $n \to$ it follows that $Az^k = \phi z^k$ for any positive integer k.

Denote g by A_ϕ. Obviously g is in H^2.Therefore,

$$g = \sum_{k=0}^{\infty} \langle \phi, z^k \rangle z^k = \sum_{k=0}^{\infty} \langle 0, z^k \rangle (z^k \phi).$$

If we calculate the Fourier coecients $\langle g, z^k \rangle$ we obtain exactly the summands of $\sum_n \lambda_n^2$ which are not square summable and hence g is not in H^2, which is absurd. Consequently f is not in the range of $M_{u,T}$ and so $M_{u,T}$ has not closed range.

Theorem 2.2. Let $M_{u,T} \in B_{L_b}(H, H)$. If span(*ran u.T*) is nite dimensional, then M_u,T has a closed range.

Proof. Let $\{x_1, x_2, ..., x_n\}$ be a basis for *span* (*ran u.T*). If for a sequence $\{A_n\}$ in $B_L(H)$, $\{M_{u,T}A_n\}$ is convergent, then $u.A_n \circ T$ is uniformly convergent on H. Let Y_i denote the limit of $A_n x_i$ for $_i = 1, 2, ..., m$. We get $Ax_i = Y_i$, $i = 1, 2, ..., m$. Extend A to *span* (*ran u.T*) by linearity. we set $Ax = 0$ for any $x \in [span(ranu.T)]^{\perp}$.. Then $A \in B_L(H)$ and

$$\lim_{n\to\infty} (u.A_n \circ T)(x) = \lim_{n\to\infty} u(x)A_n(T(x)) = \lim_{n\to\infty} u(x)A(T(x)).$$

Hence the limit of $M_{u,T}A_n$ is $M_{u,T}A$ and thus the range of $M_{u,T}$ is closed.

COMPACT WEIGHTED COMPOSITION OPERATORS

In this section we compact weighted composition operators acting on a space of operators. It is proved that if H is innite dimensional, then the only compact weighted composition operator is the zero operator.

Theorem 3.1. Let H be an innite dimensional Hilbert space. Let $M_{u,T} : B_L(H) \to B_b(H)$ be a non-zero weighted composition operator. Then $M_{u,T}$ is never compact.

Proof. Since Mu;T is a non-zero weighted composition operator, there exists $x_0 \in H$ such that $u(x_0) \neq 0$ and $T(x_0) \neq 0$. Since $T(x_0) \neq 0$, by Hahn Banach theorem there exists a bounded linear functional f on H such that $f(T(x_0)) = \|T(x_0)\|$ and $\|f\| = 1$.

Let $B_1 = \{x \in H : \|x\| \leq 1\}$. Then B_1 is not compact in H. Let $\{x_n\}$ be a sequence in B_1 such that no subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent. That is, there exists $\epsilon > 0$ such that $\|x_{n_k} - x_{n_j}\| \geq \epsilon$,, for every k, j. For each n_k, $A_{n_k} : H \to H$ by $A_{n_k}(x) = f(x)x_{n_k}$.

Then $\{A_{n_k}\}$ is a sequence of bounded linear operators in $B_L(H)$. Now

$$\begin{aligned}
\|M_{u,T}A_{n_k} - M_{u,T}A_{n_j}\| &= \|u.A_{n_k} \circ T - u.A_{n_j} \circ T\| \\
&= \sup\{\|u(x)A_{n_k}(T(x)) - u(x)A_{n_j}(T(x))\| : x \in H\} \\
&\geq \|u(x_0)A_{n_k}(T(x_0)) - u(x_0)A_{n_j}(T(x_0))\| \\
&= \|u(x_0)(A_{n_k}(T(x_0)) - A_{n_j}(T(x_0))\| \\
&= |u(x_0)|\|A_{n_k}(T(x_0)) - A_{n_j}(T(x_0))\| \\
&= |u(x_0)|\|f(T(x_0))x_{n_k} - f(T(x_0))x_{n_j}\| \\
&= |u(x_0)||f(T(x_0))|\|x_{n_k} - x_{n_j}\| \\
&= |u(x_0)|\|T(x_0)\|\|x_{n_k} - x_{n_j}\| \\
&\geq \epsilon|u(x_0)|\|T(x_0)\|.
\end{aligned}$$

Thus $\|M_{u,T}A_{n_k} - M_{u,T}A_{n_j}\| \geq \epsilon|u(x_0)|\|T(x_0)\|$. Therefore, the sequence $\{M_{u,T}A_{n_k}\}$ has no convergent subsequence. Hence M_u,T is not compact.

Theorem 3.2. Let $M_{u,T} : B_L(H, C) \to B_b(H, C)$ be a continuous linear transformation. Then M_u,T is compact if and only if $(u.T)(H)$ is totally bounded.

Proof. Suppose that M_u,T is a compact operator. Let B_1 be the closed unit ball of $B_L(H)$*.

Then $M_{u,T}B_1$ is a relatively compact subset of $B_b(H)^*$. Hence for given $\epsilon > 0$ there exists a nite partition $\{E_i\}$ of $B_b(H)*$ such that $s_i \in E_i$

$$\sup_{s\in E_i} \|M_{u,T}A(s_i) - M_{u,T}A(s)\| \leq \epsilon, \tag{3.1}$$

for each i = 1, 2, ..., n.

If A is an isometry, then equation (3.1) implies that

$$\|u(s_i)T(s_i) - u(s)T(s)\| < \epsilon, \text{for } \ i = 1, 2, ..., n \ \text{ and for all } \ s \in E_i.$$

Therefore, $u.T(E_i) \subset s_\epsilon(u.T(s_i))$, for each i = 1, 2, ..., n, where $s_\epsilon(T(s_i))$ is a sphere of radius ϵ with centre at $(u.T)(s_i)$ and so

$$\cup_{i=1}^n (u.T)(E_i) \subset \cup_{i=1}^n s_\epsilon((u.T)(s_i)).$$

That is,

$$(u.T)\Big(\cup_{i=1}^n (E_i)\Big) \subset \cup_{i=1}^n s_\epsilon(T(s_i))$$

or $(u.T)(H) \subset \cup_{i=1}^n s_\epsilon(T(s_i))$, which shows that $(u.T)(H)$ is totally bounded.

Conversely, suppose that $(u.T)$ (H) is a totally bounded subset of H. Let E be a bounded subset of $B_L(H)^*$. Then $\|A\| \leq k$ for every $A \in E$ and for some $k > 0$. We prove that $M_{u,T}(E)$ is relatively compact. Let $\epsilon > 0$ be given. Then by hypothesis there is a nite set $\{s_1, s_2, ..., s_n\}$ in H such that

$$(u.T)(H) \subset \cup_{i=1}^n s_{\frac{\epsilon}{2k}}(T(s_i)).$$

Let $E_i = (u.T)s_{\frac{\epsilon}{2k}}(T(s_i))$. Consider

$$\begin{aligned} \|(M_{u,T}A)s_i - (M_{u,T})(s)\| &= \|u(s_i)A(T(s_i)) - u(s)A(T(s))\| \\ &\leq \|A\|\|(u.T)(s_i) - u(s)T(s)\| \\ &\leq k.\frac{\epsilon}{2k}, \end{aligned}$$

for each $s \in E_i$. Hence

$$\sup_{s\in E_i} \|(M_{u,T}A)(s_i) - (M_{u,T}A)(s)\| \leq \frac{\epsilon}{2}, \ \text{ for each } \ i = 1, 2, ..., n.$$

Clearly, $H = \cup_{i=1}^n E_i$. This shows that $M_{u,T}$ is relatively compact.

Example 3.3 Let $u : \ell^2 \to \mathbb{C}$ and $T : \ell^2 \to \ell^2$ be defined by

$$T(x) = \begin{cases} x, & \text{if } x \in \{e_n : n \in N\} \\ 0, & \text{if } x \notin \{e_n : n \in N\} \end{cases}$$

and $u(x) = 1$; for all $x \in \ell^2$. Then $\{(u.T)(x) : x \in \ell^2\} = \{e_n : n \in N\}$ which is not a totally bounded subset of ℓ^2.

Hence $M_{u,T}$ is not a compact operator.

Example 3.4. Let H be a Hilbert space and e_1 be a basis vector of H. Define

$T : H \to H$ by $T(x) = e_1$ for all $x \in H$ and $u : H \to \mathbb{C}$ by

$$u(x) = \begin{cases} \|x\|, & \text{if } x \in B_1(H) \\ 0, & \text{if } x \notin B_1(H) \end{cases}$$

Then $(u.T)(H) = \{\alpha e_1 : \alpha \in D\}$, where $\underline{D}$ is the closed unit disc in $\mathbb{C}$. Now $(u.T)(H)$ is a totally bounded subset. Hence $M_{u,T}$ is a compact operator.

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