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AUTO AND EXTERNALLY INDUCED REGIME SWITCHING DIFFUSIONS.

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Abstract. In the current literature we can find mainly two approaches to the SDE regime switching modeling. The traditional one, the *externally induced regime switching diffusions* is described by the switching being derived from a separate continuous time Markov process, with a finite, or denumerable, state space – indexing the regimes – the random times of the regime switches being exactly the jump times of the finite valued Markov process. There is a first alternative approach in which the regime switching occurs whenever the trajectory *enters* in some prescribed region on the state space; the regions we consider will be mainly open intervals defined by unknown thresholds for the trajectories; thresholds that, in principle, should also be estimated. In this approach the partitioning of the the state space is already defined in the drift and volatility of the SDE. In a second alternative approach the switching occurs in a random way but at some random times defined when the trajectories hit some prescribed thresholds, that again, must be estimated. We may designate these two alternative approaches as *auto-induced regime switching diffusions* as there is no external noise source to force the switching occurrence. We prove a generalization of an existence result of the existence of auto-induced regime switching SDE solutions for irregular coefficients and a result that encompasses some of the cases of both externally and auto-induced regime switching SDE solutions.

The first author dedicates this work to his teacher Eric Langlart to whom he owes a first opportunity to answer a non trivial problem in Probability theory and a consequent lifelong passion.

1. Introduction

It is a natural assumption, in modeling, that phenomena that are shown to present oscillations – around some observable stable equilibrium – must be subjected to different types of influences that match each other in such a way as to originate the equilibrium. For unidimensional variables, for instance, the equilibria we mention can take several forms, to wit: the observed values oscillate around a

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certain unknown value; the observed values are contained in some band of values in the state space.

If the model we consider is given by a stochastic differential equation (SDE) of Ito's type a natural way to model different influences at work at a given observed phenomena is to consider that there is some regime switching.

The following is a first fundamental idea: a regime switching process may be defined by an increasing sequence of distinct random times at which the regime switches occur. A natural way to imagine a regime switch – in a process solution of an SDE – is to consider that, for a given trajectory of the process, at a certain date – a random time – the drift and the volatility of the SDE change; the changes can be either structural – the drift and volatility changing their functional form – or merely parametric in nature, that is, the functional form remains but some parameters changes as the regime switches.

A second fundamental idea is that the regimes may correspond to the partitioning in several distinct regions of the state space of the SDE solution process. For instance, when the process oscillates around some value there are essentially two regions: the points above the value and the points below the value; when the process is confined to some band of values in the state space there are essentially three regions: above the upper level of the band, below the lower level of the band and in between the upper and the lower levels of the band. Let us stress that due to the behavior – on a level set – of a process with continuous trajectories – given as a solution of some SDE – the points at the boundary of the regions mentioned may require a special treatment. We will detail this question further on as there are several ways of addressing this question.

Following these two fundamental ideas, in the current literature we can find mainly two approaches to the SDE regime switching modeling. The traditional one is roughly described by the idea that the switching is derived from a separate continuous time Markov process, with a finite, or denumerable, state space – indexing the regimes – the random times of the regime switches being exactly the jump times of the finite valued Markov process. We will designate this case by *externally induced regime switching diffusions.* There is a first alternative approach in which the regime switching occurs whenever the trajectory *enters* in some prescribed region on the state space; the regions we consider will be mainly open intervals defined by unknown thresholds for the trajectories; thresholds that, in principle, should also be estimated. In this approach the partitioning of the state space is already defined in the drift and volatility of the SDE. In a second alternative approach the switching occurs in a random way but at some random times defined when the trajectories hit some prescribed thresholds, that again, must be estimated. We may designate these two alternative approaches as *auto-induced regime switching diffusions* as there is no external noise source to force the switching occurrence. The main subject of this paper is a synthesis of the the first approach and the second alternative approach. Let us briefly describe the plan.

- In Section 2 we summarize the main features – and some of the most important bibliographic references – of the *externally induced regime switching diffusions* alternative approach of regime switching diffusions.
- In Section 3 we describe the *auto-induced regime switching diffusions* highlighting the main published results and references as well as the evolution of the subject. We prove a generalization of an existence result of the

existence of auto-induced regime switching SDE solutions for irregular coefficients and a result that encompasses some of the cases of both externally and auto-induced regime switching SDE solutions.
- In Section 4 we present an analysis of some of the published articles that either refer to our work or explicitly use the results in our publications.

2. Externally induced RSD

Let us briefly expose some fundamental concepts related to this approach. The literature on the subject of *externally induced regime switching diffusions* is vast and we may have a comprehensive view of the subject in the reference work [YZ10] where there are 190 bibliographic items referred to. The contents of the book is perfectly described by the following words in the preface, and we quote: *This book encompasses the study of hybrid switching diffusion processes and their applications. The word hybrid signifies the coexistence of continuous dynamics and discrete events, which is one of the distinct features of the processes under consideration. Much of the book is concerned with the interactions of the continuous dynamics and the discrete events.*

As far as we know, [MB85], [Mar91], [GAM93] and [GAM97], the first published references to this approach, are related to control problems. In the reference text [Mar90, p. 141] the author briefly states *...the interpretation of this diffusion with markovian switching coefficients is a collection of piecewise defined Ito stochastic differential equations* which is a broad sense definition of regime switching diffusions. A technical and most important aspect of the subject is the stability. The work [KZY07] is an authoritative reference.

2.1. Externally induced RSD – a recent existence result. The more recent works [Zha20] and [XYZ19] address the existence and uniqueness of strong solutions and also the strong Feller property of solutions of externally induced regime-switching processes whose coefficients may be singular, for instance, non Lipschitz. In order to gave a flavor of the hypothesis and results of this approach we may briefly describe the first theorem in the first work and comment on it. By definition, an externally induced RSD is, according to [Zha20], a two component stochastic process $(X_t, \Lambda_t)_{t\geq 0}$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that,

$$dX_t = b(X_t, \Lambda_t, t)dt + \sigma(X_t, \Lambda_t, t)dB_t \tag{2.1}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion with respect to some right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$, Λ_t is a continuous time Markov chain taking integer values – indexing the regimes – independent of $(B_t)_{t\geq 0}$, and so, verifying:

$$\mathbf{P}\left[\Lambda_{t+\Delta t} = j \mid \Lambda_t = i \ , X_s, \Lambda_s \ , s \leq t\right] = \begin{cases} q_{ij}(X_t)\Delta t + o(\Delta t) & \text{if } i \neq j \\ 1 + q_{ii}(X_t)\Delta t + o(\Delta t) & \text{if } i = j \end{cases} \tag{2.2}$$

with, b, σ and q_{ij} measurable functions and the Q-matrix $[q_{ij}]$ is *conservative*, that is:

$$i \neq j \ , q_{ij}(x) \geq 0 \text{ and } \sum_{j\neq i} q_{ij}(x) = -q_{ii}(x) \ .$$

An example of a quite general existence result in this context is given by Theorem 2.1. Essentially, this theorem says that if there are unique strong solutions in a – possibly random – time interval for each of the regimes then there exist a strong solution obtained by *glueing* together solutions of the SDEs – corresponding to different regimes – obtained between the consecutive jump times of the continuous time Markov chain $(\Lambda_t)_{t\geq 0}$. We observe, for further reference, that this idea is close to the main idea in our work [EM14].

Theorem 2.1. *Suppose that for each integer $i \geq 1$ and for each $t_0 \geq 0$ and initial value x_0, there is an unique local strong solution for the SDE*

$$X_t^i = x_0 + \int_0^t b(X_s^i, i, t_0 + s)ds + \int_0^t \sigma(X_s^i, i, t_0 + s)dB_s \tag{2.3}$$

defined for all $t \in [0, \tau[$ for a stopping time $\tau > 0$. Then, if for each integer i, the function $\sum_{j\neq i} q_{ij}(x)$ is locally bounded, there exist for each initial state k integer and initial value x_0 a process $(X_t)_{t\in[0,\tau[}$ that coincides – between two consecutive jump times of Λ_t – almost surely, with a solution of some of the SDE given by formula (2.3).

The proof of this theorem, for a multidimensional state space and for an infinite valued continuous time Markov chain, relies essentially, as in [EM14], in using the jump – stopping – times of the Markov chain to define solutions of SDE, starting anew at each stopping time and then glueing these solutions as a well defined process.

We stress that there are two independent noise sources in a model built upon a process like $(X_t, \Lambda_t)_{t\geq 0}$; the one associated with the process $(B_t)_{t\geq 0}$ and the one associated with the process $(\Lambda_t)_{t\geq 0}$. As a consequence it not possible, in general, to partition the state space of $(X_t)_{t\geq 0}$ – the SDE part of the two component process – in disjoint regions with each region corresponding to a given regime. This is a major difference with the first alternative formulation of auto-induced RSD we will present in Section 3, a difference that may matter for the applications of regime switching diffusions to stock prices modeling.

3. Auto-induced RSD

The approach to which we contributed the most to the subject of switching diffusions is described now. For us, the main motivation for considering auto-induced RSD – instead of externally induced RSD – is related to the *efficient market* hypothesis stating that *asset prices reflect all available information of the market.* In fact, assuming this hypothesis and in the case we may observe a regime switch in the stock price there should be no external shock causing this regime switch. As so, the uncertainty in the regime switch should just result from the implicit noise source in the stochastic market model. There is another possible restrictive natural assumption which is to consider that whenever the trajectories of the process occupy some prescribed region in the state space of the process, these trajectories may be modeled by a correspondent uniquely defined regime; this leads to partition the state space in intervals and to the definition the regions for the different regimes as the topological interiors of these intervals. As previously stated when the partition has two regions the model may be parametrized by, essentially, one threshold and whenever the partition has three regions, we may consider, essentially, two threshold parameters.

The analysis of these kind of models was started, as far as we know, in the PhD thesis [Mot07] of Professor Pedro P. Mota developed around a problem posed by his thesis co-director Professor Mathieu Kessler from the University of Cartagena in Spain. The almost immediate next step was to try an application of the auto-induced RSD to the stock market, first in a condensed form for the geometric Ornstein-Uhlenbeck functional form model [Mot13] and then with full development for the geometric Brownian model [ME14] which, after completion, was published only in 2014 after several years development. In the mean time, in the publication [NLPC09] – done in collaboration with a MSc student Nelson Rianço – we considered a first approach to a model with a coupled system of SDE – one for prices and another for liquidities – in the stock market, thus having a two dimensional state space and with a large set of varied possible combinations of regimes. The general existence result for auto-induced RSD – published in [EM14] – was obtained as an answer to one of the referee's questions to one of the first versions of [ME14]; in this work, having in mind an application to stock market modeling, we also showed that if the SDE models – for each one of the regimes – are arbitrage free and complete then the RSD model would also be arbitrage free and complete. Finally, in [ME16] a thorough statistical study was performed investigating the usual geometric Brownian motion (gBm) against the auto-induced RSD having as regimes gBm processes with different parameters; the general conclusion being that under the usual statistical criteria there is a clear advantage in using the gBm auto-induced RSD.

3.1. Auto-induced RSD – an existence result. In this section we formulate an existence result which is a generalization of the result in [EM14] which was proved by resorting to classical existence and unicity of strong solutions for SDE as in [LS01, p. 134] or [Øksendal03, p. 68]. This new formulation will be in terms of natural increasing sequences of stopping times that can be seen as associated with the regime switches.

For completeness we briefly expose some necessary essential results for existence and uniqueness of SDE, following the masterful expositions of [LS01, pp. 132] and [Kal02, pp. 412–426].

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ a right-continuous complete filtration on it. Let $\mathcal{B}([0,T])$ be the Borel σ-algebra on $[0,T])$.

Definition 3.1 (Progressively measurable processes)**.** A process $(X_t)_{t\geq 0}$ is progressively measurable – with respect to $\mathbf{F}$ – if for every $T>0$ the process $(X_t)_{t\leq T}$ is such that for every $t \in [0,T]$ we have that $X_t : \Omega \times [0,T] \mapsto \mathbf{R}$ is measurable with respect to $\mathcal{F}_t \times \mathcal{B}([0,t])$.

We observe that as sections of functions defined on a product measure space are measurable, a progressive measurable process is adapted to the filtration $\mathbf{F}$. Now, consider a SDE given by:

$$\begin{cases} dX_t = \mu(t, X_t, \theta)dt + \sigma(t, X_t, \theta)dB_t \ , \quad t \in [0,T] \ , \\ X_0 = Z \ , \end{cases} \tag{3.1}$$

with $(B_t)_{t\in[0,T]}$ a Brownian motion with respect to some filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$ such that, almost surely, for all $t>0$

$$\int_0^t |\mu(u, X_u, \theta)| + |\sigma(u, X_u, \theta)|^2 \, du < +\infty \ , \tag{3.2}$$

in order for the Lebesgue and Ito's integrals in Formula (3.1) to exist, and such that the integrands in Formula (3.1) are progressively measurable.

Definition 3.2 (Strong solutions)**.** Let us suppose to be given the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, the filtration $\mathbf{F}$, $(B_t)_{t\in[0,T]}$ a Brownian motion with respect $\mathbf{F}$ and Z an $\mathcal{F}_0$ measurable random variable. A **strong solution** – for a SDE such the one in Formulas (3.1) – specified by its coefficients μ and σ, is a $\mathbf{F}$-adapted stochastic process satisfying Formula (3.3), which is equivalent to Formula (3.1),

$$X_t = Z + \int_0^t \mu(u, X_u, \theta)du + \int_0^t \sigma(u, X_u, \theta)dB_u \ , \ \ t \in [0, T] \ , \tag{3.3}$$

in particular, such that $X_0 = Z$.

In case of strong solutions the probability space, the filtration and the Brownian motion are prescribed. If $\mathbf{F}$ is the natural filtration associated with the Brownian motion then, as the solution is $\mathbf{F}$-adapted, we have that:

$$\sigma\left(X_s : s \leq t\right) \subseteq \sigma\left(B_s : s \leq t\right) =: \mathcal{F}_t \ , \ t \geq 0 \ .$$

We may say that there is *strong existence* – for a SDE specified by its coefficients μ and σ – if given some random variable Z then for every prescribed probability space, filtration and Brownian motion there exists a strong solution according to Definition 3.2.

Definition 3.3 (Weak solutions)**.** Let us suppose to be given a probability law $\mathcal{L}$ and (progressive) functions μ and σ such that, for every $x \in \mathbf{R}$, Formula (3.2) is verified. A process $(X_t)_{t\geq 0}$ is a **weak solution**, for a SDE specified by its coefficients μ and σ, if there exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ a filtration $\mathbf{F}$ a $(B_t)_{t\in[0,T]}$ a Brownian motion with respect $\mathbf{F}$ such that $(X_t)_{t\geq 0}$ is $\mathbf{F}$ adapted, condition given by Formula (3.2) is verified, equation given by Formula (3.1) – or equivalently by Formula (3.3)– is verified and the probability law of X_0 is $\mathcal{L}$.

In the case of weak solutions, if $\mathbf{F}$ is the natural filtration associated with $(X_t)_{t\geq 0}$ then as $(B_t)_{t\in[0,T]}$ a Brownian motion with respect $\mathbf{F}$ we have that:

$$\sigma\left(B_s : s \leq t\right) \subseteq \sigma\left(X_s : s \leq t\right) =: \mathcal{F}_t \ , \ t \geq 0 \ .$$

For some prescribed law $\mathcal{L}$, we may say that there is *weak existence* – for a SDE specified by its coefficients μ and σ – if there exists a weak solution according to Definition 3.3.

There are also associated notions of uniqueness that are essential for the developments ahead.

Definition 3.4 (Uniqueness in Law)**.** There is *uniqueness in law* if the processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ being *weak solutions* of an SDE – specified by its coefficients μ and σ – for a given initial law $\mathcal{L}$ we have that both processes have the same distribution (that is have the same finite dimension probability laws).

Definition 3.5 (Pathwise uniqueness)**.** Given an initial law $\mathcal{L}$, we have *pathwise uniqueness* for solutions of an SDE – specified by its coefficients μ and σ – if given two processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ defined on a common filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$ being solutions of the SDE in formula (3.1) with a common Brownian process such that $X_0 = Y_0$ almost surely with law $\mathcal{L}$, we have that the processes are **indistinguishable** that is: $\mathbf{P}\left[X_t = Y_t : \forall t \geq 0\right] = 1$.

Example 3.6 (Tanaka equation)**.** (see [KS91, pp. 301–302].) The equation given by:

$$dX_t = \text{ sign } (X_t) dB_t \ , \ X_0 \equiv 0 \ ,$$

has no strong solution. In fact the volatility is not Lipschitz and so the Ito's existence and unicity theorem does not apply. Nevertheless, it can be shown that it admits a weak solution, uniqueness in law holds, but pathwise uniqueness cannot hold for this equation.

It is well known that if the processes are modifications of one another – that is if for every $t \geq 0$ we have $\mathbf{P}[X_t = Y_t] = 1$ – and if the trajectories are right-continuous, then the processes are indistinguishable (see [KS91, p. 2]).

It is also well known that pathwise uniqueness implies uniqueness in law (Proposition 1 in [YW71, p. 158] or Proposition 3.20 in [KS91, p. 309]). Furthermore, there is a remarkable result connecting strong existence and pathwise uniqueness due to Yamada and Watanabe [YW71] that may also be read with a thorough treatment in [Kal02, p. 424].

Theorem 3.7 (Strong solutions and pathwise uniqueness)**.** *Let an SDE – specified by its coefficients μ and σ – have **weak** solutions and **pathwise uniqueness** for a given initial law $\mathcal{L}$. Then **strong existence** and **uniqueness in law** hold for such solutions.*

We now follow the indication given in [Pig19] pointing to results presented in [LG84] in which there is an important result on the existence and uniqueness of strong solutions of SDE with possibly non regular coefficients; regular, in this context means at least Lipschitz with sub-linear growth as in the classical existence and unicity of Ito. Theorem 3.7 is essential to establish Theorem 3.8. For that, as in [LG83], we consider the following two hypothesis.

A There exists $\rho : [0, +\infty[\mapsto [0, +\infty[$, **increasing and bounded**, possibly dependent of each $\theta \in \Theta$, such that:

$$\lim_{\epsilon \to 0} \int_\epsilon \frac{du}{\rho(u)} = +\infty \ ,$$

and

$$\forall t, x, y, \theta \quad (\sigma(t, x, \theta) - \sigma(t, y, \theta))^2 \leq \rho(|x - y|) \ .$$

B There exists a **increasing and bounded** function f, possibly dependent of each $\theta \in \Theta$, such that:

$$\forall t, x, y, \theta \quad (\sigma(t, x, \theta) - \sigma(t, y, \theta))^2 \leq |f(x) - f(y)|$$

and

$$\exists \epsilon > 0 \ \ \forall t, x, y, \theta \ \ \sigma(t, x, \theta) > \epsilon \ .$$

The next result, Theorem 3.8, is a powerful tool for dealing with SDE with irregular coefficients; for this purpose there are alternatives. A very thorough presentation but with almost no proofs – by S. V. Anulova and A. Yu. Veretnnikov – of the main results on solutions of SDE with irregular coefficients is in collective book [PS98, pp- 39–75].

Theorem 3.8 (Strong solutions for irregular drifts and volatilities)**.** *Consider a parameter set $\mathbf{\Theta} = \{\theta_1, \theta_2, \ldots \theta_m\}$with m regime parameters. Let $\mu(t, x, \theta)$ and $\sigma(t, x, \theta)$ be two real valued **measurable** functions defined on $\mathbf{R} \times \mathbf{\Theta}$ such that, either, for every $\theta \in \mathbf{\Theta}$:*

(1) $\sigma(t,x,\theta)$ *verifies hypothesis* $\boldsymbol{A}$ *and* $\mu(t,x,\theta)$ *is Lipschitz.*
(2) $\sigma(t,x,\theta)$ *verifies* $\boldsymbol{A}$ *and there exists* $\epsilon > 0$ *such that* $|\sigma(t,x,\theta)| > \epsilon$
(3) $\sigma(t,x,\theta)$ *verifies* $\boldsymbol{B}$.

Then for any random variable Z, $\theta \in \boldsymbol{\Theta}$ *and* $t_0 \in [0,T]$ *the following stochastic differential equation*

$$\begin{cases} dX_t = \mu(t, X_t, \theta)dt + \sigma(t, X_t, \theta)dB_t \ , & t \in [t_0, T] \ , \\ X_{t_0} = Z \ , \end{cases} \tag{3.4}$$

has an unique strong solution on any filtered probability $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$ *space, with* $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$, *carrying a Brownian motion.*

The proof, in [LG84, pp. 55–56] (see also [LG83]) – following [YW71], [Nak72] and [OS75] – is firstly based on a clever use of local times to prove pathwise uniqueness results, implying weak existence results and then by using the Theorem 3.7 – first published in [YW71] – ensuring that there exists an unique – in law – strong solution with a given random variable as initial value.

Let us observe that, according to [LP19b], this result covers the case – of the partition of the state space in two regimes – where we have, for $\gamma_+ > 0$ and $\gamma_- > 0$:

$$\mu(t,x,\beta) = \beta_+ 1\!\!1_{\{x\geq 0\}} + \beta_- 1\!\!1_{\{x<0\}} \text{ and } \sigma(t,x,\gamma) = \gamma_+ 1\!\!1_{\{x\geq 0\}} + \gamma_- 1\!\!1_{\{x<0\}} \ ,$$

and so, also, the case in which the state space of the diffusion is partitioned in two intervals, each one corresponding to some regime. In fact, supposing that $\gamma_+ > \gamma_- > 0$, for instance, and as, by a quick examination, we have always,

$$(\sigma(t,x,\gamma) - \sigma(t,y,\gamma))^2 \leq |\gamma_+ - \gamma_-|^2 \leq \gamma_+^2 + \gamma_-^2 \ ,$$

by considering, the increasing bounded function:

$$f(z) := \left(\gamma_+^2 + \gamma_-^2\right) 1\!\!1_{\{z\geq 0\}} + \left(\gamma_-^2 - \gamma_+^2\right) 1\!\!1_{\{z<0\}} \ ,$$

and observing that:

$$|f(x) - f(y)| = \begin{cases} \left|\gamma_+^2 + \gamma_-^2 - \left(\gamma_-^2 - \gamma_+^2\right)\right| = 2\gamma_+^2 & x \geq 0 \ , y < 0 \\ \left|\gamma_-^2 - \gamma_+^2 - \left(\gamma_+^2 + \gamma_-^2\right)\right| = 2\gamma_+^2 & x < 0 \ , y \geq 0 \\ 0 & x \geq 0 \ , y \geq 0 \text{ or } x < 0 \ , y < 0 \ , \end{cases}$$

we have that, whenever $x \geq 0$, $y \geq 0$ or $x < 0$, $y < 0$,

$$(\sigma(t,x,\gamma) - \sigma(t,y,\gamma))^2 = 0 = |f(x) - f(y)| \ ,$$

and whenever, $x \geq 0$, $y < 0$ or $x < 0$, $y \geq 0$,

$$(\sigma(t,x,\gamma) - \sigma(t,y,\gamma))^2 \leq \gamma_+^2 + \gamma_-^2 \leq 2\gamma_+^2 = |f(x) - f(y)| \ ,$$

and, as $\sigma(t,x,\gamma) \geq \min(\gamma_+, \gamma_-) = \gamma_- > 0$, we have that the third condition of Theorem 3.8 is verified.

Remark 3.9 (Partition of the state space in three regimes). We suppose that, together with a similar expression for the drift, we have:

$$\sigma(t,x,\gamma) = \gamma_- 1\!\!1_{\{x\leq 0\}} + \gamma_0 1\!\!1_{\{0<x<1\}} + \gamma_+ 1\!\!1_{\{x\geq 1\}} \ ,$$

that, for instance, $\gamma_- < \gamma_+ < \gamma_0$ and we define the increasing bounded function f by:

$$f(z) := \left(-\gamma_-^2 - \gamma_+^2 - \gamma_0^2\right) 1\!\!1_{\{z\leq 0\}} + \left(-\gamma_+^2 + 2\gamma_0^2\right) 1\!\!1_{\{0<z<1\}} + 3\gamma_0^2 1\!\!1_{\{z\geq 1\}}.$$

By observing that,

$$|f(x)-f(y)| =$$

$$= \begin{cases} \left|-\gamma_-^2-\gamma_+^2-\gamma_0^2-\left(-\gamma_+^2+2\gamma_0^2\right)\right| = \gamma_-^2+3\gamma_0^2 & x\leq 0\ , 0<y<1 \\ \left|-\gamma_-^2-\gamma_+^2-\gamma_0^2-3\gamma_0^2\right| = \gamma_-^2+\gamma_+^2+4\gamma_0^2 & x\leq 0\ , y\geq 1 \\ \left|-\gamma_+^2+2\gamma_0^2-\left(-\gamma_-^2-\gamma_+^2-\gamma_0^2\right)\right| = 3\gamma_0^2+\gamma_-^2 & 0<x<1\ , y\leq 0 \\ \left|-\gamma_+^2+2\gamma_0^2-3\gamma_0^2\right| = \gamma_0^2+\gamma_+^2 & 0<x<1\ , y\geq 1 \\ \left|3\gamma_0^2-\left(-\gamma_-^2-\gamma_+^2-\gamma_0^2\right)\right| = 4\gamma_0^2+\gamma_+^2+\gamma_-^2 & x\geq 1\ , y<0 \\ \left|3\gamma_0^2-\left(-\gamma_+^2+2\gamma_0^2\right)\right| = \gamma_0^2+\gamma_+^2 & x\geq 1\ , 0<y<1 \\ 0 & \text{all other cases}\ , \end{cases}$$

and that, by a quick inspection,

$$\begin{aligned}(\sigma(t,x,\gamma)-\sigma(t,y,\gamma))^2 &\leq \max\left(|\gamma_- - \gamma_0|^2\,;|\gamma_- - \gamma_+|^2\,,|\gamma_0-\gamma_+|^2\right) \leq \\ &\leq \max\left(\gamma_-^2+\gamma_0^2;\gamma_-^2+\gamma_+^2;\gamma_0^2+\gamma_+^2\right) \leq \\ &\leq \gamma_0^2+\gamma_+^2\ ,\end{aligned}$$

as we have that $(\sigma(t,x,\gamma)-\sigma(t,y,\gamma))^2 = 0$, in the cases where $|f(x)-f(y)| = 0$, we then have that, for all possible combinations of cases,

$$(\sigma(t,x,\gamma)-\sigma(t,y,\gamma))^2 \leq |f(x)-f(y)|\ .$$

Finally, as $\sigma(t,x,\gamma) \geq \min(\gamma_0,\gamma_+,\gamma_-) = \gamma_- > 0$, we have, again, that the third condition of Theorem 3.8 is also verified and so we may consider solved the existence of regime switching SDE solutions that correspond to the partition of the state space in three different regions, having in each region a different regime.

In order to describe an alternative model for a two regime SDE – corresponding to the partition of the state space – we now address the problem of building some process by glueing the solutions of two different types of SDE – corresponding to two alternating regimes – in time intervals of random length defined by an increasing sequence of stopping times. We observe that the subject of building a process from a collection of random variables indexed by a family of stopping times has roots in the early eighties of the twentieth century (see for instance [DL81] and [DL82]).

For our purpose we formulate a third hypothesis.

C Let $0 \equiv \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ be an increasing sequence of $\mathbf{F}$-stopping times, denoted by $\mathcal{T}$, such that: we have, almost surely, $\lim_{n\to+\infty}\tau_n = +\infty$ and, for any $T \in \mathbf{R}_+$ and almost all $\omega \in \Omega$:

$$\#\left\{k \geq 1 : \tau_k(\omega) \leq T\right\} < +\infty\ . \tag{3.5}$$

Now for any $T \in \mathbf{R}_+$ define $n(T) : \Omega \mapsto \mathbf{N}$ such that, almost surely $\tau_{n(T)(\omega)}(\omega) < T$ and for all $k \geq n(T)(\omega)+1$ we have that $\tau_k(\omega) \geq T$.

Theorem 3.10 (Existence of RS SDE). *Let $(\Omega,\mathcal{F},\mathbf{P},\mathbf{F})$ be a filtered complete probability space, with $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$, carrying a Brownian motion $(B_t)_{t\geq 0}$. Consider a parameter set $\mathbf{\Theta} = \{\theta_1,\theta_2\}$ with two regime parameters, some fixed $T > 0$ and $\mu(t,x,\theta)$ and $\sigma(t,x,\theta)$ two real valued measurable functions defined on $\mathbf{R}\times\mathbf{\Theta}$ satisfying at least one of the conditions of Theorem 3.8. Define, for $n \geq 0$,*

$$\widehat{n} := \frac{1-(-1)^n}{2}+1\ .$$

Then, for any increasing sequence of $\mathbf{F}$ *stopping times* $(\tau_n)_{n\geq 0}$ *satisfying hypothesis* $\boldsymbol{C}$*, and taking alternatively the two values of* $\boldsymbol{\Theta}$*, and for any* $T>0$*, the stochastic differential equation defined with* $X_0^{\theta}=x_0$ *for* $t\in[0,T]$*, by*

$$X_t^{\theta} = x_0 + \int_0^t \mu\left(u, X_u^{\theta}, \theta\right)(u)du + \int_0^t \sigma\left(t, X_u^{\theta}, \theta\right) dB_u \ , \tag{3.6}$$

has an unique – in law – strong solution which is a regime switching process $(X_t^{\theta})_{t\in[0,T]}$ *given by:*

$$X_t^{\theta} = \sum_{n=0}^{n(T)-1} X_{n,t}^{\theta_{\widehat{n}}} \mathbb{I}_{[\tau_n,\tau_{n+1}[}(t) + \left[X_{n,t}^{\theta_{\widehat{n}}} \mathbb{I}_{[\tau_n,T[}(t)\right]_{n=n(T)} \ , \tag{3.7}$$

with the excursion process $(X_{n,t}^{\theta_{\widehat{n}}})_{t\in[\tau_n,\tau_{n+1}[}$ *given by the unique – in law – strong solution of the stochastic differential equation in Formula* (3.8)*, that exists by Theorem 3.8,*

$$\begin{cases} dX_{n,t}^{\theta_{\widehat{n}}} = \mu(t, X_{n,t}^{\theta_{\widehat{n}}}, \theta_{\widehat{n}})dt + \sigma(t, X_{n,t}^{\theta_{\widehat{n}}}, \theta_{\widehat{n}})dB_t \ , \quad \tau_n \leq t < \tau_{n+1} \ , \\ X_{n,0}^{\theta_{\widehat{n}}} = X_{n,\tau_n}^{\theta_{\widehat{n-1}}} \ \ \text{and} \ X_{n,\tau_n}^{\theta_{\widehat{n-1}}} \ |_{n=0} = x_0 \ , \end{cases} \tag{3.8}$$

SDE that may be interpreted, for instance, by Formula 3.9.

Proof. Let the initial condition $x_0\in\mathbf{R}$ be such that the process $(X_{0,t}^{\theta_{\widehat{0}}})_{t\in[\tau_0,T]}$ is the strong solution of the SDE in Formula (3.4) given by:

$$\begin{cases} dX_t = \mu(t, X_t, \theta_{\mathbf{1}})dt + \sigma(t, X_t, \theta_{\mathbf{1}})dB_t \ , \quad t\in[0,T] \ , \\ X_0 = x_0 \ , \end{cases}$$

thus considering that – as $(X_{0,t}^{\theta_{\widehat{0}}})_{t\in[\tau_0,T]} \equiv (X_{0,t}^{\theta_1})_{t\in[0,T]}$ – we start in the first regime. Supposing that, almost surely, $\tau_1\leq T$, solve the SDE:

$$\begin{cases} dX_t = \mu(t+\tau_1, X_t, \theta_{\mathbf{2}})dt + \sigma(t+\tau_1, X_t, \theta_{\mathbf{2}})dB_t \ , \quad t\in[0,T] \ , \\ X_0 = X_{\tau_1}^{\theta_1} \ , \end{cases}$$

to obtain the process $(X_{1,t}^{\theta_{\widehat{1}}})_{t\in[\tau_1,T]} \equiv (X_{1,t}^{\theta_2})_{t\in[\tau_1,T]}$, a process in the second regime. Supposing, again that, almost surely, $\tau_2\leq T$ solve the SDE:

$$\begin{cases} dX_t = \mu(t+\tau_2, X_t, \theta_{\mathbf{1}})dt + \sigma(t+\tau_2, X_t, \theta_{\mathbf{1}})dB_t \ , \quad t\in[0,T] \ , \\ X_0 = X_{\tau_2}^{\theta_2} \ , \end{cases}$$

to obtain the process $(X_{2,t}^{\theta_{\widehat{2}}})_{t\in[\tau_2,T]} \equiv (X_{2,t}^{\theta_1})_{t\in[\tau_2,T]}$, a process, again, in the first regime. By induction, as long as, almost surely, $\tau_n\leq T$ we have that there exists a process $(X_{n,t}^{\theta_{\widehat{n}}})_{t\in[\tau_n,T]}$ which is a solution of:

$$\begin{cases} dX_t = \mu(t+\tau_n, X_t, \theta_{\widehat{n}})dt + \sigma(t+\tau_n, X_t, \theta_{\widehat{n}})dB_t \ , \quad t\in[0,T] \ , \\ X_0 = X_{\tau_n}^{\theta_{\widehat{n-1}}} \ . \end{cases} \tag{3.9}$$

We may now *glue* together the processes $(X_{n,t}^{\theta_{\widehat{n}}})_{t\in[\tau_n,\tau_{n+1}[}$, for $n\geq 0$ to define the regime switching process $(X_t^{\theta})_{t\in[0,T]}$ by:

$$X_t^{\theta} = \sum_{n=0}^{+\infty} X_{n,t}^{\theta_{\widehat{n}}} \mathbb{I}_{[\tau_n,\tau_{n+1}[}(t) := \sum_{n=0}^{n(T)-1} X_{n,t}^{\theta_{\widehat{n}}} \mathbb{I}_{[\tau_n,\tau_{n+1}[}(t) + \left[X_{n,t}^{\theta_{\widehat{n}}} \mathbb{I}_{[\tau_n,T[}(t)\right]_{n=n(T)} ,$$

that can interpreted to satisfy the SDE 3.10 given, for $t \in [0,T]$, by:

$$
\begin{aligned}
X_t^{\theta} = x_0 + \int_0^t \left(\sum_{n=0}^{+\infty} \mu\left(t, X_u^{\theta_{\widehat{n}}}, \theta_{\widehat{n}}\right) \mathbb{1}_{[\tau_n,\tau_{n+1}[}(u) \right) du \\
+ \int_0^t \left(\sum_{n=0}^{+\infty} \sigma\left(t, X_u^{\theta_{\widehat{n}}}, \theta_{\widehat{n}}\right) \mathbb{1}_{[\tau_n,\tau_{n+1}[}(u) \right) dB_u \ ,
\end{aligned}
\tag{3.10}
$$

which, in turn, may be taken as the full interpretation of formula 3.6. □

We now present some comments on this result.

- Theorem 2.1 in [ME14, p. 763] is a particular case of Theorem 3.10 in the case where the coefficients are Lipschitz and have sub-linear growth ensuring, by the classical Ito's theorem, existence and unicity of strong solutions. In this case the solution has also the strong Markov property (see Theorem 5.4 in [Kle99, p. 127]); extensions of this result for the case of weak solutions can be stated, by means of the solution of a martingale problem in the sense of Stroock-Varadhan (see [Kle99, p. 130] and for the case where the – multidimensional – volatility is continuous and the drift is measurable, Theorem 10.4 in [Pin95, p. 34]).
- For stock price modeling Theorem 3.10 may be applied to a partition of the state space in three disjoint intervals with two very close thresholds – closer than the least significative difference between two stock price values – in such a way that we never observe price values between the two thresholds. Then to each of the remaining intervals corresponds a distinct parameter value in $\mathbf{\Theta} = \{\theta_1, \theta_2\}$. There is a most simple transition rule for the values taken by the successive terms of the sequence of stopping times, namely:

  $$\theta_1 \to \theta_2 \to \theta_1 \ ,$$

  that allows an immediate application of Theorem 3.10 considering the strictly increasing sequence of stopping times defined by the crossing of the lower threshold – when the trajectory starts in the upper region – or the crossing of the upper threshold when the trajectory comes from the lower region. In [ME14, p. 763], for the diffusion case, we also show that the increasing sequence of stopping times is a finite sequence in every bounded interval and so, by construction, it satisfies hypothesis **C** as in page 9.
- A simple generalization of Theorem 3.10 may also be applied to a parameter set with a finite number of elements with a similar proof as long as we prescribe a transition rule, for the stopping times – over the parameter set. For instance for $\mathbf{\Theta} = \{\theta_1, \theta_2, \theta_3\}$ we could have the following transition rules:

  $$\theta_1 \to \theta_2 \to \theta_3 \to \theta_1 \ .$$

  corresponding in an obvious way to a stochastic transition matrix,

  $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

  or, the following transition rule,

  $$\theta_1 \to \theta_3 \to \theta_2 \to \theta_1$$

with correspondent stochastic transition matrix,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

If this is the case, it does not seem possible to partition the phase space into three regions, each region corresponding to a different parameter value; in fact, when the trajectory is getting out of the central region there will be two possibilities: the upper region and the lower region and so there is no transition rule of the type just described encompassing this kind of evolution.

- In order to be able to *glue* the excursions in between stopping times we need the existence of strong solutions. There are several results on the existence of strong solutions under hypothesis different from the Lipschitz continuity and sub-linear growth of the coefficients, for instance the beautiful result in [Sko65, p. 129]; we opted for Theorem 3.8 as it allows us to consider under a general framework, our result in [ME14], the results of Lejay and Pigato – [Pig19], [LP19a] and [LP19b] – and also as we will see next, a result on externally induced RSD thus providing an unified approach to regime switching diffusions.
- With full rigor, Theorem 3.8 should be formulated up to T, a stochastic *explosion time* of the solution. When considering one dimensional processes of diffusion type, that is with the SDE being associated with functions $\mu(x, \theta)$ and $\sigma(x, \theta)$ – that is, not depending on time – the condition $\mu(x, \theta)$ bounded prevents the explosion of any solution of the SDE (see, for instance [KS91, p. 341] for a very thorough discussion of the subject) and we have (see, for instance [IW89, p. 173]) with the notations of Theorem 3.8,

$$T = \lim_{n \to +\infty} \inf\{t \geq 0 : X_t \geq n\} .$$

This is the situation most currently found in applications.

3.2. A companion result on (generally) induced RS SDE. Let us state in broad terms our aim. We want to formulate an existence theorem for SDE with possible irregular coefficients and with a determined random variable as an initial condition; the goal is for this theorem to cover both the externally induced and the auto induced cases, if not for all at least in some interesting instances. For that purpose we will collect some fundamental results. We start with weak existence result obtained via the solution of a martingale problem in the sense of Stroock-Varadhan (see Theorem 6..16 in [SV79, p. 143]) in the formulation of Kallenberg (see [Kal02, p. 418]). For completeness we state an important definition.

Definition 3.11 ((Local) Martingale problem)**.** Consider the semi-elliptic differential operator defined for $\phi \in \mathcal{C}_K^\infty$ – the space of indefinitely differentiable functions with compact support – by:

$$\mathcal{A}(\phi) = \sum_{i=1}^{d} \mu_i \frac{\partial}{\partial x_i} \phi + \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \phi ,$$

with the coefficients μ_i and a_{ij} being locally bounded measurable functions on $\mathbf{R}^d$. A process $(X_t)_{t \geq 0}$ with law given by the probability measure $\mathbf{Q}_{\mathcal{L}}$ over the canonical space of continuous functions $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$ solves the (local) martingale problem for $\mathcal{A}$ with initial law $\mathcal{L}$ – or starting at X_0 a random variable with law $\mathcal{L}$ – if the process defined by:

$$M_t = \phi(X_t) - \phi(X_0) - \int_0^t \mathcal{A}(\phi(X_s))ds \ ,$$

is a (local) martingale, for every $\phi \in \mathcal{C}_K^\infty$. If the coefficients are bounded we have a *bona fides* martingale problem.

The following result shows the important connection between the existence of a week solution for some SDE and the existence of a solution for a (local) martingale problem with an operator naturally associated to the coefficients of the SDE.

Theorem 3.12 (Stroock-Varadhan)**.** *Consider the canonical space of continuous functions $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$ some Wiener process on it and its correspondent natural filtration. Let μ, σ be progressively measurable functions (in the sense of Definition 3.1). Let $\mathbf{Q}$ be some probability measure over $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$. Then the SDE with drift μ and volatility σ has a weak solution with distribution $\mathbf{Q}$ if and only if $\mathbf{Q}$ solves the (local) martingale problem for $\sigma \cdot \sigma^t$ and μ.*

With extra regularity on the coefficients of the SDE we have an weak existence result (see [Kal02, p. 419]).

Theorem 3.13 (Skorokhod)**.** *Within the same context of Theorem 3.12, suppose additionally that for any fixed $t \geq 0$ the functions $\mu(t, \cdot)$ and $\sigma(t, \cdot)$ are continuous in $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$. Then for any initial distribution $\mathcal{L}$ the martingale problem for μ and σ has a solution $\mathbf{Q}_\mathcal{L}$.*

The following result of Stroock-Varadhan show how to extend a solution for the martingale problem with a real initial value to a solution with any initial distribution.

Theorem 3.14 (Stroock-Varadhan)**.** *Within the same context of Theorem 3.12, suppose that the (local) martingale problem for μ and σ has an unique solution $\mathbf{Q}_x$ for any initial law $\mathcal{L} \equiv \delta_x$. Then for any initial law $\mathcal{L}$ the (local) martingale problem for μ and σ has an unique solution $\mathbf{Q}_\mathcal{L}$.*

The Markov property is one of the main properties of a stochastic process $(X_t)_{t\geq 0}$. Essentially, as it is wonderfully written in [Øksendal03, p. 115], *...the future behaviour of the process given what has happened up to time t is the same as the behaviour obtained when starting the process at X_t*. In [Kha12, p. 60–63] we find the following rigorous and yet very simple formulation of the concept of Markov process.

Definition 3.15 (Markov property)**.** A process $(X_t)_{t\geq 0}$ with state space in $\mathbf{R}^d$ is a **Markov process** if for all $B \in \mathcal{B}(\mathbf{R}^d)$, the Borel σ-algebra and $0 \leq s < t$, we have:

$$\mathbf{P}\left[X_t \in B \mid \mathcal{F}_s^X\right] = \mathbf{P}\left[X_t \in B \mid X_s\right] \ ,$$

with $\mathcal{F}_s^X := \sigma\left(X_u : u \leq s\right)$ the natural σ-algebra of the process $(X_t)_{t\geq 0}$.

The strong Markov property is the extension of the Markov property for random times (see for a reasonable simple formulation [Øksendal03, p. 117]).

Definition 3.16 (Strong Markov property)**.** A process $(X_t)_{t\geq 0}$ with state space in $\mathbf{R}^d$ is a **strong Markov process** if for all bounded Borel measurable functions f on $\mathbf{R}^d$, for all $B \in \mathcal{B}(\mathbf{R}^d)$, the Borel σ-algebra, and for any stopping time τ with respect to $\mathcal{F}_s^X := \sigma(X_u : u \leq s)$ – the natural σ-algebra of the process $(X_t)_{t\geq 0}$ – and $0 \leq s$, we have:

$$\mathbf{P}\left[f(X_\tau + s) \in B \mid \mathcal{F}_\tau\right] = \mathbf{P}\left[f(X_s) \in B \mid X_\tau\right] ,$$

with $\mathcal{F}_\tau := \{B \in \mathcal{F}_\infty^X : B \cap \{\tau \leq t\} \in \mathcal{F}_t^X\}$ and $\mathcal{F}_\infty^X := \bigvee_{s\geq 0} \mathcal{F}_s^X$ the smaller σ-algebra containing all the $\mathcal{F}_s^X$.

Finally the following result (see[Kal02, p. 421] and [SV79, p. 146]) ensures the strong Markov property for the solution process distribution.

Theorem 3.17 (Stroock-Varadhan)**.** *Within the same context of Theorem 3.12, suppose that the (local) martingale problem for measurable functions μ and σ has an unique solution $\mathbf{Q}_x$ for any initial law $\mathcal{L} \equiv \delta_x$. Then the family $\mathbf{Q}_x$ satisfies the strong Markov property.*

We may now state the fundamental result which was initially formulated for homogeneous SDE but is also valid in the non homogeneous case (see [RW00, p. 266]). In [PS98, p. 40] (and in [KS91, p. 291] with essentially the classical proof of Yamada-Watanabe result) there is a formulation of Theorem 1 in [YW71, p. 164] for non homogeneous drift and volatilities that – for dimension one – suits better our purposes of having processes with continuous trajectories.

Theorem 3.18 (Yamada-Watanabe 1971)**.** *Suppose that μ and σ are progressively measurable, bounded and satisfy the following hypothesis.*

D *There exists $\rho_1 : [0, +\infty[\mapsto [0, +\infty[$, an* ***increasing****, continuous function possibly dependent of each $\theta \in \Theta$, such that $\rho_1(0) = 0$ such that:*

$$\lim_{\epsilon\to 0} \int_\epsilon \frac{du}{\rho_1(u)} = +\infty ,$$

and

$$\forall t, x, y, \theta \quad (\sigma(t, x, \theta) - \sigma(t, y, \theta))^2 \leq \rho_1(|x - y|) .$$

E *There exists $\rho_2 : [0, +\infty[\mapsto [0, +\infty[$, an* ***increasing****, concave function possibly dependent of each $\theta \in \Theta$, such that $\rho_2(0) = 0$ such that:*

$$\lim_{\epsilon\to 0} \int_\epsilon \frac{du}{\rho_2(u)} = +\infty ,$$

and

$$\forall t, x, y, \theta \quad (\mu(t, x, \theta) - \mu(t, y, \theta)) \leq \rho_2(|x - y|) .$$

Then for any random variable Z, and $\theta \in \boldsymbol{\Theta}$ and $t_0 \in [0, T]$ the following stochastic differential equation

$$\begin{cases} dX_t = \mu(t, X_t, \theta)dt + \sigma(t, X_t, \theta)dB_t , & t \in [t_0, T] , \\ X_{t_0} = Z , \end{cases} \tag{3.11}$$

pathwise uniqueness holds. Moreover, by the continuity of both μ and σ applying Skorokhod's theorem 3.13 we have that there is a weak solution. Finally, by Theorem 3.7 there is strong existence and uniqueness in law holds.

We observe next that boundedness of the coefficients imply the continuity of trajectories.

Corollary 3.19. *Let $(X_t)_{t\geq 0}$ be a strong solution of the SDE* (3.11) *given by Theorem 3.18. Then $(X_t)_{t\geq 0}$ has continuous trajectories.*

Proof. We apply Kolmogorov criteria with K_μ and K_σ bounds of μ and σ.

$$\begin{aligned}
&\mathbf{E}\left[|X_t - X_s|^4\right] = \mathbf{E}\left[\left|\int_s^t \mu(u,X_u,\theta)du + \int_s^t \sigma(u,X_u,\theta)dB_u\right|^4\right] \leq \\
&\leq 8\left(\mathbf{E}\left[\left(\int_s^t |\mu(u,X_u,\theta)|\,du\right)^4\right] + \mathbf{E}\left[\left(\int_s^t |\sigma(u,X_u,\theta)|\,dB_u\right)^4\right]\right) \leq \\
&\leq 8\left(K_\mu^4\,|t-s|^4 + K_\sigma^4\mathbf{E}\left[(B_t - B_s)^4\right]\right) \leq \\
&\leq 8\max\left(K_\mu^4, K_\sigma^4\right)\left(|t-s|^4 + d(d+2)\,|t-s|^2\right) .
\end{aligned}$$

for the Brownian motion in dimension d, as $(a+b)^4 \leq 2^2(a^2+b^2)^2 \leq 8(a^4+b^4)$, by Jensen's inequality and by the properties of the Brownian process. □

In order to cover the externally induced RSD, we first detail the structure of a continuous time process that induces the regime switching. For that we will look for general processes that can be written as jump processes driven by a strictly increasing sequence of non accumulating stopping times. Consider a sequence of random variables $(Z_n)_{n\geq 0}$ taking values in a finite parameter space $\mathbf{\Theta} = \{\theta_1, \theta_2, \ldots, \theta_m\}$ and an increasing sequence of $\mathbf{F}$-stopping times $0 \equiv \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ verifying hypothesis $\mathbf{C}$ as in page 9. Let us define the continuous time process $(Y_t)_{t\geq 0}$ by:

$$Y_t = \sum_{n=0}^{+\infty} Z_n \mathbb{1}_{[\tau_n, \tau_{n+1}[}(t) \ . \tag{3.12}$$

It is well known – see [Res92, pp. 367–379] and [RSST99, pp. 317–320] – that if $(Z_n)_{n\geq 0}$ is a Markov chain and the time intervals $(\tau_{n+1} - \tau_n)_{n\geq 0}$ are exponentially distributed then $(Y_t)_{t\geq 0}$ can be taken to be a continuous time homogeneous Markov chain. In case if $(Z_n)_{n\geq 0}$ is a Markov chain and the time intervals $(\tau_{n+1} - \tau_n)_{n\geq 0}$ have a distribution that can depend on the present state as well as on the one visited next then $(Y_t)_{t\geq 0}$ can be taken to be a *semi-Markov* process (see [Ios80, pp. 261, 262] and [IT73, pp. 295–299], for brief references. In the case of semi-Markov processes a nice result of Ronald Pyke (see [Pyk61, p. 1236] guarantees that when the state space if finite the process is *regular* implying that almost all paths of such a semi-Markov process are step-functions over $[0, +\infty[$.

In another more general case (see [Ios80, pp. 262–266] and [IT73, pp. 195–244]), adequate hypothesis on the distribution of the stopping times and on the sequence $(Z_n)_{n\geq 0}$ implies that $(Y_t)_{t\geq 0}$ will be a non homogeneous Markov chain process in continuous time, whose trajectories are step functions.

Theorem 3.20 (On the existence of (generally) induced RS SDE)**.** *Consider a finite set $\mathbf{\Theta} = \{\theta_1, \theta_2, \ldots, \theta_m\}$ as the parameters of the different regimes. Let the functions $\mu(t,x,\theta)$ and $\sigma(t,x,\theta)$ satisfy the hypothesis of theorem 3.18. Let $(Y_t)_{t\geq 0}$ be an $\mathbf{F}$ adapted continuous time process defined in Formula* (3.12) *and consider that $\mathcal{T}$ the increasing sequence of $\mathbf{F}$-stopping times $0 \equiv \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ verifies hypothesis **C**. There exists a process $(X_t)_{t\geq 0}$ which is an unique in law **regime switching** process associated with the parameter space $\mathbf{\Theta}$ and the Markov process $(Y_t)_{t\geq 0}$ with continuous trajectories.*

Proof. Now, consider some random variable X_0 measurable with respect to $\mathcal{F}_0$ and consider the SDE given by:

$$\begin{cases} dX^1_t = \mu(t, X^1_t, Y_t)dt + \sigma(t, X^1_t, Y_t)dB_t \ , \quad t \in [0, \tau_1] \ , \\ X^1_{t_0} = X_0 \ , \end{cases} \tag{3.13}$$

where the functions $\mu(t, x, \theta)$ and $\sigma(t, x, \theta)$ satisfy the hypothesis of theorem 3.18. We observe that as in each trajectory of $(Y_t)_{t\geq 0}$ takes one only value $\theta \in \mathbf{\Theta}$ in $[0, \tau_1[$ the strong solution $(X^1_t)_{t\in[0,\tau_1[}$ exists and is unique in law. Now, consider again the SDE

$$\begin{cases} d\widetilde{X}^2_t = \mu(t + \tau_1, \widetilde{X}^2_t, Y_{t+\tau_1})dt + \sigma(t, \widetilde{X}^1_t, Y_t)dB_t \ , \quad t \in [0, \tau_2 - \tau_1] \ , \\ \widetilde{X}^2_0 = X^1_{\tau 1} \ . \end{cases} \tag{3.14}$$

Again, as before, the strong solution and unique in law solution $(\widetilde{X}^2_t)_{t\in[0,\tau_2-\tau_1]}$ exists and we may define $(X^2_t)_{t\in[\tau_1,\tau_2[} \equiv (\widetilde{X}^2_{t-\tau_1})_{t\in[\tau_1,\tau_2]}$. By induction, we may so have a sequence $((X^n_t)_{t\in[\tau_{n-1},\tau_n[})_{n\geq 1}$ of unique in law processes and so – using the hypothesis of the sequence of stopping times stated in Formula (3.5), in page 9 – we may define the process, for almost all $\omega \in \Omega$:

$$X_t = \sum_{n=1}^{+\infty} X^n_t \mathbb{1}_{[\tau_{n-1},\tau_n[}(t) \ ,$$

which is the unique in law process obtained by glueing together excursions corresponding to regimes prescribed by the initially given sequence of stopping times and the continuous time Markov chain. As a consequence of Corollary 3.19 we have the continuity of the glued trajectories. □

Remark 3.21 (The strong Markov property of the solutions). As a consequence of the strong Markov property given by Theorem 3.17 and then by Theorem 3.14, we have that, given the sequence of stopping times $\mathcal{T}$, the excursions given by $((X^n_t)_{t\in[\tau_{n-1},\tau_n[})_{n\geq 1}$ have independent laws and so are independent processes.

Remark 3.22. The idealized behavior of a typical trajectory of the regime switching process obtained in Theorem 3.20 is as expected; the trajectory is governed by a specific regime, until a change of regime occurs at a certain random time then, again, during an interval of time of random duration the regime stays the same until the next change dictated by the following random time in the sequence. Let us stress that this behavior does not correspond to the partition of the state space of the process in a finite union of intervals. This is due to the fact that the level sets of the Brownian process are perfect sets, that is, closed sets in which all points are accumulation points of the set (see [Str94, pp. 247–249] and [RY99, p. 109]). Moreover, the level sets have null Lebesgue measure, have the cardinality of the continuum and the complement is a countable union of disjoint intervals. The same applies to any continuous local martingale as such process is, in law, a Brownian process with the time changed by a continuous increasing process (see [BS15, p. 155]).

3.3. Auto-induced RSD: a two threshold example. A natural set of open questions is to identify the auto-induced RSD threshold models that can be brought under some of the hypothesis of Theorem 3.20 concerning the structure of process

$(Y_t)_{t\geq 0}$. We know that if the Markov process is to be homogeneous then the duration times in each state must be exponentially distributed (see [IT73, p. 221]). In general it will not be the case, even for a simple diffusion, like the Brownian process itself. Nevertheless we know the distribution of hitting times of many interesting processes, for instance, for the geometric Brownian motion and that can be put to some use; we will present a more general result.

For illustration of our purpose we consider the case of a parameter space $\mathbf{\Theta} = \{\theta_1, \theta_2\}$ and some SDE solution, for $\theta \in \mathbf{\Theta}$ given by:

$$\begin{cases} dX_t^\theta = \mu(t, X_t^\theta, \theta)dt + \sigma(t, X_t^\theta, \theta)dB_t \ , \quad t \in [0, T] \ , \\ X_0^\theta = x_0 \ , \end{cases} \tag{3.15}$$

with the drift $\mu \in \mathbf{R}$ and the volatility $\sigma > 0$ and such that the existence Theorem 3.8 can be applied. We consider the case of two regimes corresponding to two thresholds $M_1 < M_2$ in the state space with, for instance, θ_1 being associated with the lower threshold M_1 and θ_2 being associated with the upper threshold M_2. If the process is in the regime corresponding to θ_1 then the regime switch occurs at the hitting time of M_2 and vice versa. For the initial condition x_0 at $t = 0$ we suppose that the process starts with the regime θ_1 if $|x_0 - M_1| \leq |x_0 - M_2|$ and it starts with the regime θ_2 if $|x_0 - M_1| > |x_0 - M_2|$.

We suppose, for instance, that $x_0 = M_1$ and so the first non trivial stopping time is $\tau_1 = \inf\{t > 0 : X_t^{\theta_1} = M_2\}$. Now, at the time τ_1 a regime witch occurs and we wave a second excursion, solution of:

$$\begin{cases} dX_t^{\theta_2} = \mu(t, X_t^{\theta_2}, \theta_2)dt + \sigma(t, X_t^{\theta_2}, \theta_2)dB_t \ , \quad t \in [\tau_1, T] \ , \\ X_{\tau_1}^{\theta_2} = X_{\tau_1}^{\theta_1} \ , \end{cases}$$

or in an equivalent way,

$$\begin{cases} dX_{t+\tau_1}^{\theta_2} = \mu(t + \tau_1, X_t^{\theta_2}, \theta_2)dt + \sigma(t + \tau_1, X_t^{\theta_2}, \theta_2)dB_{t+\tau_1} \ , \quad t \in [0, T] \ , \\ X_{\tau_1}^{\theta_2} = X_{\tau_1}^{\theta_1} \ , \end{cases}$$

with parameter θ_2 until this excursion hits M_1. The next random time for a regime switch to occur is $\tau_2 = \inf\{t > \tau_1 : X_t^{\theta_2} = M_1\}$.

Let us consider $\mathcal{T} = \{\tau_1, \dots, \tau_n, \dots\}$ the increasing sequence of hitting times that we can define by induction in this way. In this case of two thresholds and for a general SDE solution process such as the one given by Formula (3.15), in order to apply Theorem 3.20, we have to prove that the successive hitting times of the thresholds satisfy Hypothesis **C** given by Formula 3.1 (in page 9).

Remark 3.23. We observe that following the same reasoning exposed in [EM14, p. 763] we have that as the excursions process, given by Theorem 3.20, have continuous trajectories by Corollary 3.19, the hitting thresholds stopping times are isolated.

Theorem 3.24. *If σ is such that the Ito's stochastic integral part in the SDE given by Formula* (3.15) *is a martingale then $\mathcal{T}$ verifies Hypothesis **C**.*

Remark 3.25 (On deterministic times to hit a threshold). The proof is based on the following observation. Given a function $f : \mathbf{R} \mapsto \mathbf{R}$, such that we can apply the fundamental theorem of calculus, we have that:

$$t_2 - t_1 = \frac{1}{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f'(t)dt} \left(f(t_2) - f(t_1)\right)$$

and so the *time* $t_2 - t_1$ taken to go from $f(t_1)$ to $f(t_2)$ is proportional to $f(t_2) - f(t_1)$, the proportionality constant being the inverse of the average derivative of f between the *dates* t_1 and t_2.

Proof. We always have, with $t_1 < t_2$ and $X_{t_1} = M_1$ and $X_{t_2} = M_2$, for a general SDE:

$$M_2 - M_1 = \int_{t_1}^{t_2} \mu(s, X_s) ds + \int_{t_1}^{t_2} \sigma(s, X_s) dB_s$$

and so then, for instance, when the stochastic integral defines a martingale,

$$M_2 - M_1 = \int_{t_1}^{t_2} \mathbf{E}\left[\mu(s, X_s)\right] ds = (t_2 - t_1) \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mathbf{E}\left[\mu(s, X_s)\right] ds$$

and so,

$$t_2 - t_1 = \frac{M_2 - M_1}{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mathbf{E}\left[\mu(s, X_s)\right] ds}$$

and, furthermore if μ is continuous and bounded, say by K,

$$t_2 - t_1 = \left| \frac{M_2 - M_1}{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mathbf{E}\left[\mu(s, X_s)\right] ds} \right| \geq \frac{|M_2 - M_1|}{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mathbf{E}\left[|\mu(s, X_s)|\right] ds} \geq$$
$$\geq \frac{|M_2 - M_1|}{K} =: \Lambda \, .$$

Now, for random hitting times $\tau_1 < \tau_2$,

$$\begin{aligned}
&\mathbf{P}\left[\tau_2 - \tau_1 \geq \Lambda \middle| \tau_2 = t_2, \tau_1 = t_1\right] = \\
&= \mathbf{P}\left[\left| \frac{M_2 - M_1}{\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \mathbf{E}\left[\mu(s, X_s)\right] ds} \right| \geq \Lambda \middle| \tau_2 = t_2, \tau_1 = t_1\right] \geq \\
&= \mathbf{P}\left[\left| \frac{M_2 - M_1}{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mathbf{E}\left[\mu(s, X_s)\right] ds} \right| \geq \Lambda\right] \geq \\
&= \mathbf{P}\left[\frac{|M_2 - M_1|}{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mathbf{E}\left[|\mu(s, X_s)|\right] ds} \geq \Lambda\right] \geq \\
&\geq \mathbf{P}\left[\Lambda \geq \Lambda\right] = 1 \, .
\end{aligned}$$

and so, for random hitting times $\tau_1 < \tau_2$,

$$\mathbf{P}\left[\tau_2 - \tau_1 \geq \Lambda\right] = \mathbf{E}\left[\mathbf{P}\left[\tau_2 - \tau_1 \geq \Lambda \middle| \tau_2 = t_2, \tau_1 = t_1\right]\right] = 1 \, .$$

Now, for $N \geq 1$ we have that:

$$\{\tau_N \geq N\Lambda\} \supseteq \{\tau_N - \tau_{N-1} \geq \Lambda\} \cap \cdots \cap \{\tau_1 - \tau_0 \geq \Lambda\} \, ,$$

and so, as the excursions are independent given $\mathcal{T}$, we have that

$$\mathbf{P}\left[\tau_N \geq N\Lambda\right] \geq \mathbf{P}\left[\bigcap_{k=0}^{N-1} \{\tau_{k+1} - \tau_k \geq \Lambda\}\right] = \prod_{k=0}^{N-1} \mathbf{P}\left[\tau_{k+1} - \tau_k \geq \Lambda\right] = 1 \, .$$

We can then conclude, as the hitting times sequence is increasing that

$$\mathbf{P}\left[\sup_{N \geq 1} \tau_N = +\infty\right] = \lim_{N \to +\infty} \mathbf{P}\left[\tau_N \geq N\Lambda\right] = 1$$

and so we have

$$\mathbf{P}\left[\sup_{N\geq 1}\tau_N < +\infty\right] = 0$$

as claimed. □

4. Short analysis of related literature

We now proceed to a summary description of some publications that most relate to the main theme of this work.

- [LP17] considers an auto-induced regime switching geometric Brownian motion model, with two regimes – two distinct values for the drift and for the volatility – and one threshold dividing the state space of the diffusion in two regions; the region above and including the threshold value and the region strictly below the threshold. The existence theorem for this kind of diffusion may be deduced from the results in [LG84] – as pointed out in [Pig19] – and so, the author's claim to consider this model as an alternative to the model in [EM14] is vindicated; the estimation procedures they develop is based on occupation times estimators and local time estimators. The statistical analysis with this model uses the same set of 21 stock prices as in [EM14].
- [BW18] uses exactly the estimation procedure of [ME14] in an application of a regime switching diffusion model driven by an external Markov chain.
- [HS15] refers both to regime switching threshold models and to hybrid switching diffusion models in the context of atmospheric modeling establishing a connection between the two types of models.
- [Pig19] considers the same model for the volatility parameter as in [LP17] and refers to [ME14] for – and we quote – the *estimation of similar continuous-time models on empirical time series of prices.*
- [ES19] develops a *regime classification algorithm* aiming at determining the number of regimes in the time series. For that purpose it uses the Bayesian Information Criterion (BIC) in the sense of [ME16]. They consider the the switching among regimes *depends upon*the crossing of the thresholds as in [EM14], [ME14] and [BW18]. As in [ME14], in order to validate a given regime they consider that this regime must contain at least 15% of the total observations.
- [LP19a] performs a thorough comparison between the model in [ME14] and [EM14] and the model in [LP17] and [Pig19] by considering the same data and comparing the results of the parameters estimation, finding good agreement. In the section 4.1 of this work there is a rigorous description of the model in [ME14] and [EM14], they quote the estimation results in this model to finally compare the estimations results between the two approaches. This paper is the most adequate work to get a wide perspective between the similarities and the differences between the models. These two models are essentially different as it seems impossible to express the model of [LP19a] as a glueing procedure of processes between two successive terms of an increasing sequence of stopping times by reason of the level sets structure of continuous local martingales.
- [LP19b] considers the same model and in [LP17] and studies maximum likelihood estimators for the drift of the model complementing [LP18] where

the estimators for the volatility were studied. The reference made to [ME14] is, again, that the model considered in their work may be thought as an alternative to the model proposed in our work.

Conclusion

We showed that the subject of regime switching diffusions – both in the case of externally induced and the case of auto-induced in the case of two thresholds – may be unified if we consider that there is a sequence of stopping times that commands the regime switches. These stopping times may be either the jump times of a continuous time Markov chain or hitting threshold times for diffusions.

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