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The Investigation of Securities Cost Using Methods of Spectral Analysis

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ABSTRACT

In this article short-term interest rates are studied by methods of spectral analysis, regular and singular perturbation theories which are described by Vasicek model with multidimensional stochastic volatility that has l-fast variables, r-slowly variables, $l \geq 1, r \geq 1, l \in N, r \in N$. The approximate value of securities and their rate of return are calculated. Applying the theory of Sturm-Liouville, Fredholm's alternative and analysis of singular and regular perturbations at different time scales have enabled us to obtain explicit formulas for the approximate value of securities and their yield on the basis of the development of their eigen functions and eigenvalues of self-adjoint operators using boundary value problems for singular and regular perturbations. The theorem of closeness estimates for bond prices approximation is proved.

JEL Classifications: G11, G13, G32.

Keywords: Vasicek model, spectral theory, singular perturbation theory, regular perturbation theory.

1. INTRODUCTION

Short-term interest rate dynamics models were considered in the paper by Vasicek (1977) for derivatives pricing. Significant contributions to the theory of rate of interest were made by Brennan et al. (1979), Hull et al. (1987), J. Cox et al. (1985), Ho et al. (1986), Merton (1973), namely: finding a credit spread of credit market instruments, calculating option prices for interest rates, determining the risk and derivatives' rate of return of the stock market financial instruments. The models developed by these scholars have their advantages and disadvantages, but each of them is used to increase the liquidity of financial markets. Applying more sophisticated models, despite their theoretical justification, leads to obtaining of complex multi-parameter functions of the yield curve, which results in significant errors in calculations.

Using spectral analysis, Linetsky (2004) applied a spectral theory of self-adjoint operators to different models, and to the Vasicek model, in particular. Lorig (2014) considered the short-term interest rates described by the Vasicek model with stochastic volatility, depending on two factors, one of which is fast, and the other is slowly variable. In our paper, the spectral theory and the theory of singular and regular perturbations are applied to self-adjoint operators in Hilbert spaces that describe processes with multidimensional stochastic volatility having l-fast variables, r-slowly variables, $l \geq 1, r \geq 1, l \in N, r \in N$. This theory is applied to the short-term interest rates described by the Vasicek model, in particular. The approximate value of securities and their yield are calculated. Applying the Sturm-Liouville theory, Fredholm alternatives, as well as analysis of singular and regular perturbations at different time scales, we obtained explicit formulas for convergence of bond prices and their yield. To obtain explicit formulas, we need to solve 2l Poisson equations.

The purpose of the article is to establish bond indicative prices and their profitability by methods of spectral theory and theory of perturbations.

2. RESULTS

Let (Q, F, P) be the probability space that supports a correlated Brownian motion $(W^x, W^{y_1}, \dots, W^{y_l}, W^{z_1}, \dots, W^{z_r})$ and an exponential random variable ε~Exp (1), which is independent of $(W^x, W^{y_1}, \dots, W^{y_l}, W^{z_1}, \dots, W^{z_r})$. We will assume that the economy with (l+r+1) factors is described by the homogeneous time and continuous Markov process $\chi = (X, Y_1, \dots, Y_l, Z_1, \dots, Z_r)$, which is defined in some state space $E = I \times R^l \times R^r$, where $(Y_1, \dots, Y_l) \in R^l$, $(Z_1, \dots, Z_r) \in R^r$ I is the interval at R with points $e_1$and e_2, such that $-\infty < e_1 < e_2 < \infty$. We assume that χ has the beginning at E and instantly disappears once X goes beyond I. In particular, the dynamics of χ with physical measure $\mathbb{P}$ is as follows:

$$\chi_t = \begin{cases} (X_t, Y_{1t}, \dots, Y_{lt}, Z_{1t}, \dots, Z_{rt})), \tau_I > t, \\ \Delta, \ \tau_I \leq t, \end{cases} \quad \tau_I = \inf(t > 0 \colon X_t \notin I),$$

where, $(X, Y_1, \dots, Y_l, Z_1, \dots, Z_r)$, are set

$$dX_t = v(X_t)dt + a(X_t)f(Y_{1t}, \dots, Y_{lt}, Z_{1t}, \dots, Z_{rt})dW_t^x,$$

$$dY_{jt} = \frac{1}{\epsilon_j}\alpha_j(Y_{jt})dt + \frac{1}{\sqrt{\epsilon_j}}\beta_j(Y_{jt})dW_t^{y_j}, j = \overline{1, l}.$$

$$dZ_{it} = \delta_i c_i(Z_{it})dt + \sqrt{\delta_i} g_i(Z_{it})dW_t^{z_i}, i = \overline{1, r}.$$

$$d(W^x, W^{y_j})_t = \rho_{xy_j}dt, j = \overline{1, l}.$$

$$d(W^x, W^{z_i})_t = \rho_{xz_i}dt, i = \overline{1, r}.$$

$$d(W^{y_j}, W^{z_i})_t = \rho_{y_j z_i}dt, j = \overline{1, l}, i = \overline{1, r}.$$

$$d(W^{y_j}, W^{y_r})_t = \rho_{y_j y_s} dt, j = \overline{1,l}, s = \overline{1,l}.$$

$$d(W^{z_i}, W^{z_k})_t = \rho_{z_i z_k} dt, i = \overline{1,n}, k = \overline{1,r}.$$

$$(X_0, \mathrm{Y}_{10}, \dots, \mathrm{Y}_{l0}, \mathrm{Z}_{10}, \dots, \mathrm{Z}_{r0}) = (x, y_{10}, \dots y_{l0}, z_{10}, \dots, z_{r0}) \in E.$$

where, $\rho_{y_j y_s} = 0, j \neq r, \rho_{z_i z_k} = 0, i \neq k$ $\rho_{xy_j}, \rho_{xz_i}, \rho_{y_j z_i}$, meet the conditions $\left|\rho_{xy_j}\right|, \left|\rho_{xz_i}\right|, \left|\rho_{y_j z_i}\right| \leq 1$, and correlation matrices of the form:

$$\begin{pmatrix} 1 & \rho_{xy_j} & \rho_{xz_i} \\ \rho_{y_j x} & 1 & \rho_{y_j z_i} \\ \rho_{z_i x} & \rho_{z_i y_j} & 1 \end{pmatrix}$$

semipositively defined, that is $1 + 2\rho_{xy_j}\rho_{xz_i}\rho_{y_j z_i} - {\rho_{xy_j}}^2 - {\rho_{xz_i}}^2 - {\rho_{y_j z_i}}^2 \geq 0, j = \overline{1,l}, i = \overline{1,r}.$

Process X may represent many economic phenomena and processes.

For example, the reserve size, the index price and reliable short-term interest rates, etc. Even more broadly, X is an external factor that characterizes the value of any of the above-mentioned processes. Physical measure $\mathbb{P}$ of process X is understood as the process X, which has an instant drift $v(X_t)$ and stochastic volatility $a(X_t) f(\mathrm{Y}_{1t}, \dots, \mathrm{Y}_{lt}, \mathrm{Z}_{1t}, \dots, \mathrm{Z}_{nt}) > 0$, which contains both components: local $a(X_t)$ and nonlocal $f(\mathrm{Y}_{1t}, \dots, \mathrm{Y}_{lt}, \mathrm{Z}_{1t}, \dots, \mathrm{Z}_{nt})$. Note that infinitesimal generators (infinitized) for Y_j and Z_i have the form $\forall\ i, j$

$$\mathfrak{L}_{Y_j}^{\epsilon_j} = \frac{1}{\epsilon_j}\left(\frac{1}{2}{\beta_j}^2(y_j)\partial^2_{y_j y_j} + \alpha_j(y_j)\partial_{y_j}\right), \mathfrak{L}_{Z_i}^{\delta_i} = \delta_i\left(\frac{1}{2}{g_i}^2(z_i)\partial^2_{z_i z_i} + c_i(z_i)\partial_{z_i}\right),$$

are characterized by the measures $\frac{1}{\epsilon_j}$ and δ_i, respectively. Thus, $\mathrm{Y}_1, \dots, \mathrm{Y}_l$ and $\mathrm{Z}_1, \dots, \mathrm{Z}_n$ have an internal time scale $\epsilon_j > 0$ and $\frac{1}{\delta_i} > 0$. We consider $\epsilon_j << 1$ and $\delta_i << 1$, so that the internal time scale Y_j is small, and the internal time scale Z_i is large. Consequently, Y_j , $j = \overline{1,l}$, are fast variables, and Z_i , $i = \overline{1,n}$ are slowly variables. Note that $\mathfrak{L}_{Y_j}^{\epsilon_j}$ and $\mathfrak{L}_{Z_i}^{\delta_i}$ have the form

$$L = \tfrac{1}{2}a^2(x)\partial^2_{xx} + b(x)\partial_x - k(x), x \in (e_1, e_2), \text{з}\ k(x) = 0$$

for all $x \in I$, are always self-adjoint in the Hilbert space $H = L^2(I, m)$, where $I \in R$ is the interval with the points e_1 and e_2 and m is the diffusion density rate. $\mathrm{Dom}(\mathfrak{L})\{f \in L^2(I, \mathfrak{m}): f, \partial_x f \in AC_{\mathrm{loc}}(I), \mathfrak{L} f L^2(I, \mathfrak{m}), BCs\ on\ e_1\ and\ e_2\}$ where $AC_{\mathrm{loc}}(I)$ is the space of functions which are absolutely continuous on each compact subinterval I Linetsky (2007). The boundary conditions for e_1 and e_2 are applied on the output, input, and regular bounds.

We will evaluate the derivatives with payoff at time $t > 0$, which may depend on the trajectory X. In particular, we will consider the forms of payoff:

$$\mathrm{Payoff} = H(X_t)\mathbb{I}_{(\tau > t)}$$

where, τ is a random moment of time during which there is a failure to make a payment of premium. Since we are interested in the derivatives estimation, we must determine the dynamics $(X, Y_1, \ldots, Y_l, Z_1, \ldots, Z_r)$, under the evaluation of the degree of neutral risk, which we denote as $\widetilde{\mathbb{P}}$. We have the following dynamics:

$$\begin{gathered}
dX_t = \left(b(X_t) - a(X_t) f(Y_{1t}, \ldots, Y_{lt}, Z_{1t}, \ldots, Z_{rt}) \Omega(Y_{1t}, \ldots, Y_{lt}, Z_{1t}, \ldots, Z_{rt})\right) dt \\
+ a(X_t) f(Y_{1t}, \ldots, Y_{lt}, Z_{1t}, \ldots, Z_{rt}) d\widetilde{W}_t^x, \\
dY_{jt} = \left(\frac{1}{\epsilon_j} \alpha_j(Y_{jt}) - \frac{1}{\sqrt{\epsilon_j}} \beta_j(Y_{jt}) \Lambda_j(Y_{1t}, \ldots, Y_{lt}, Z_{1t}, \ldots, Z_{rt})\right) dt + \frac{1}{\sqrt{\epsilon_j}} \beta_j(Y_{jt}) d\widetilde{W}_t^{y_j}, \\
dZ_{it} = \left(\delta_i c_i(Z_{it}) - \sqrt{\delta_i} g_i(Z_{it}) \Gamma_i(Y_{1t}, \ldots, Y_{lt}, Z_{1t}, \ldots, Z_{rt})\right) dt + \sqrt{\delta_i} g_i(Z_{it}) d\widetilde{W}_t^{z_i}, \\
d\langle \widetilde{W}^x, \widetilde{W}^{y_j} \rangle_t = \rho_{x y_j} dt, j = \overline{1, l}. \\
d\langle \widetilde{W}^x, \widetilde{W}^{z_i} \rangle_t = \rho_{x z_i} dt, i = \overline{1, r}. \\
d\langle \widetilde{W}^{y_j}, \widetilde{W}^{z_i} \rangle_t = \rho_{y_j z_i} dt, j = \overline{1, l}, i = \overline{1, r}. \\
d\langle \widetilde{W}^{y_j}, \widetilde{W}^{y_s} \rangle_t = \rho_{y_j y_s} dt, j = \overline{1, l}, s = \overline{1, l}. \\
d\langle \widetilde{W}^{z_i}, \widetilde{W}^{z_k} \rangle_t = \rho_{z_i z_k} dt, i = \overline{1, n}, k = \overline{1, n}. \\
(X_0, Y_{10}, \ldots, Y_{l0}, Z_{10}, \ldots, Z_{r0}) = (x, y_{10}, \ldots y_{l0}, z_{10}, \ldots, z_{r0}) \in E,
\end{gathered} \tag{1}$$

where

$$\begin{gathered}
d\widetilde{W}_t^x := dW_t^x + \left(\frac{v(X_t) - b(X_t)}{a(X_t) f(Y_{1t}, \ldots, Y_{lt}, Z_{1t}, \ldots, Z_{nt})} + \Omega(Y_{1t}, \ldots, Y_{lt}, Z_{1t}, \ldots, Z_{rt})\right) dt, \\
d\widetilde{W}_t^{y_j} := dW_t^{y_j} + \Lambda_j(Y_{1t}, \ldots, Y_{lt}, Z_{1t}, \ldots, Z_{rt}) dt, \\
d\widetilde{W}_t^{z_i} := dW_t^{z_i} + \Gamma_i(Y_{1t}, \ldots, Y_{lt}, Z_{1t}, \ldots, Z_{rt}) dt.
\end{gathered}$$

where $\rho_{y_j y_s} = 0, j \neq s, \rho_{z_i z_k} = 0, i \neq k$.

We establish such conditions so that the system (1) has the only strong solution.

Random time τ is the time of the derivative asset. In our case, default can occur in one of two ways:

when X fall outside the interval I,

at random time τ_h, which is managed by the risk level $h(X_t) \geq 0$.

This can be expressed as follows:

$$\begin{gathered}
\tau = \tau_I \wedge \tau_h, \\
\tau_I = inf\{t \geq 0 : X_t \notin I\}, \\
\tau_h = inf\left\{t \geq 0 : \int_0^t h(X_s) ds \geq \varepsilon(X, Y_1, \ldots, Y_l, Z_1, \ldots, Z_n)\right\}, \\
\varepsilon \sim Exp(1) \perp\!\!\!\perp.
\end{gathered}$$

Note that the random variable ε is independent of $(X, Y_1, \ldots, Y_l, Z_1, \ldots, Z_n)$.

To track τ_h, we use the process indicator: $D_t = \mathbb{I}_{\{t \geq \tau_h\}}$, where $\mathbb{D} = \{\mathfrak{D}_t, t \geq 0\}$, is a filter generated by D and $\mathbb{F} = \{\mathfrak{F}_t, t \geq 0\}$ is filter's generator $(W^x, W^{y_1}, \dots W^{y_l}, W^{z_1}, \dots, W^{z_n})$. We use the filtering $\mathbb{G} = \{\mathfrak{G}_t, t \geq 0\}$, where $\mathfrak{G}_t = \mathfrak{F}_t \vee \mathfrak{D}_t$. Note that $(X, \mathrm{Y}_1, \dots, \mathrm{Y}_l, \mathrm{Z}_1, \dots, \mathrm{Z}_n)$ are applied to $\mathbb{G}$ and τ is a stopping time $\left(\{\{\tau \leq t\}\} \in \mathfrak{G}_t \text{ for all } t \geq 0\right)$.

We will evaluate the derivative asset of some payoff (payment) using the neutral pricing risk and Markovian chain X, the price $u^{\overline{\epsilon},\overline{\delta'}}(t, x, y_1, \dots y_l, z_1, \dots, z_r)$ of some derivative assets at the initial moment of time has the form:

$$u^{\overline{\epsilon},\overline{\delta'}}(t, x, y_1, \dots y_l, z_1, \dots, z_r) = \widetilde{\mathbb{E}}_{x, y_1, \dots y_l, z_1, \dots, z_r}\left[exp\left(-\int_0^t r(X_s)ds\right) H(X_t \mathbb{I}_{\{t > \tau\}})\right],$$

Where, $\overline{\epsilon} = (\epsilon_1, \dots, \epsilon_l), \overline{\delta'} = (\delta_1, \dots, \delta_r)$, and $(x, y_1, \dots y_l, z_1, \dots, z_r) \in E$ is a starting point of the process $(X, \mathrm{Y}_1, \dots, \mathrm{Y}_l, \mathrm{Z}_1, \dots, \mathrm{Z}_r)$. By Feynmann-Kac formulas, we can show that $u^{\overline{\epsilon},\overline{\delta'}}(t, x, y_1, \dots y_l, z_1, \dots, z_r)$ satisfies the following Cauchy problem Linetsky (2007):

$$\left(-\partial_t + \mathfrak{L}^{\overline{\epsilon},\overline{\delta'}}\right) u^{\overline{\epsilon},\overline{\delta'}} = 0, (y_1, \dots y_l, z_1, \dots, z_r) \in E, t \in \mathbb{R}^+ \quad (2)$$

$$u^{\overline{\epsilon},\overline{\delta'}}(0, x, y_1, \dots y_l, z_1, \dots, z_r) = H(x) \quad (3)$$

where the operator $\mathfrak{L}^{\overline{\epsilon},\overline{\delta'}}$ has the form:

$$\mathfrak{L}^{\overline{\epsilon},\overline{\delta'}} = \sum_{j=1}^{l} \frac{1}{\epsilon_j} \mathfrak{L}_{0j} + \sum_{j=1}^{l} \frac{1}{\sqrt{\epsilon_j}} \mathfrak{L}_{1j} + \mathfrak{L}_{2j} + \sum_{i,j} \sqrt{\frac{\delta_i}{\epsilon_j}} \mathfrak{M}_{3ij} + \sum_i \sqrt{\delta_i} \mathfrak{M}_{1i} + \sum_i \delta_i \mathfrak{M}_{2i},$$

$$\mathfrak{L}_{0j} = \frac{1}{2} \beta_j^2(y_j) \partial_{y_j y_j}^2 + \alpha_j(y_j) \partial_{y_j}, j = \overline{1, l}.$$

$$\mathfrak{L}_{1j} = \beta_j(y_j)(\rho_{xy_j} a(x) f(y_1, \dots y_l, z_1, \dots, z_r) \partial_x - \Lambda_j(y_1, \dots y_l, z_1, \dots, z_r)) \partial_{y_j},$$

$$\mathfrak{L}_{2j} = \frac{1}{2} a^2(x) f^2(y_1, \dots y_l, z_1, \dots, z_r) \partial_{xx}^2 + (b(x) - a(x) \Omega(y_1, \dots y_l, z_1, \dots, z_r) f(y_1, \dots y_l, z_1, \dots, z_r)) \partial_x - k(x),$$

$$\mathfrak{M}_{3ij} = \rho_{xz_i} \beta_j(y_j) g_i(z_i) \partial_{y_j z_i}^2,$$

$$\mathfrak{M}_{1i} = g_i(z_i) \left(\rho_{xz_i} a(x) f(y_1, \dots y_l, z_1, \dots, z_r) \partial_x - \Gamma_i(y_1, \dots y_l, z_1, \dots, z_r)\right) \partial_{z_i},$$

$$\mathfrak{M}_{2i} = \frac{1}{2} g_i^2(z_i) \partial_{z_i z_i}^2 + c_i(z_i) \partial_{z_i}, k(x) = r(x) + h(x), \mathfrak{L}_{0j} = \mathfrak{L}_{Y_j}^1.$$

We assume that the diffusion with the infinitesimal generator $\mathfrak{L}_{Y_j}^1$ has an invariant distribution Π with density $\pi_j(y_j)$.

$$\pi_j(y_j) = \frac{2}{\beta_j^2(y_j)} \exp\left\{\int_{y_{j0}}^{y_j} \frac{2\alpha_j(\theta)}{\beta_j^2(\theta)} d\theta\right\}, \forall j = \overline{1, l}.$$

Besides the initial condition (3), the function $u^{\bar{\epsilon},\overline{\delta'}}(t,x,y_1,\ldots y_l,z_1,\ldots,z_r)$ must meet boundary conditions at the points of e_1 and e_2 of the interval I. The boundary conditions at points e_1 and e_2 belong to the domain $\mathfrak{L}^{\bar{\epsilon},\overline{\delta'}}$ and will depend on the nature of process X on the points of I and are classified as natural, output, input or regular Borodin et al. (2002). The Cauchy problem (2)-(3) for $(f,\alpha_1,\ldots\alpha_l,\beta_1,\ldots,\beta_r,\Lambda_1,\ldots\Lambda_l,c_1,\ldots,c_r,g_1,\ldots g_r,\Gamma_1,\ldots,\Gamma_r)$ has no analytical solution. However, for fixed $\overline{\delta'}$, the conditions containing $\bar{\epsilon}$ and are arbitrarily dviated in the $\bar{\epsilon}$-axis, which causes singular perturbations. For a fixed ϵ_j condition containing δ_i are small for some small $\overline{\delta'}$-axis, which causes regular perturbations. Thus, the $\bar{\epsilon}$-axis and $\overline{\delta'}$-axis yields the combined singular-regular perturbation of $\mathcal{O}(1)$ of the operator $\mathfrak{L}_2$.To find the asymptotic solution of the Cauchy problem (2) ¬ (3), we develop $u^{\bar{\epsilon},\overline{\delta'}}$ in orders $\sqrt{\epsilon_j}$ and $\sqrt{\delta_i}$ Burtnyak et al. (2014):

$$u^{\bar{\epsilon},\overline{\delta'}} = \sum_{i_1\geq 0}\ldots\sum_{i_l\geq 0}\sum_{j_1\geq 0}\ldots\sum_{j_r\geq 0}\sqrt{\epsilon_1}^{j_1}\ldots\sqrt{\epsilon_l}^{j_l}\sqrt{\delta_1}^{i_1}\ldots\sqrt{\delta_r}^{i_r}u_{j_1,\ldots,j_n,i_1,\ldots i_l}$$

where,

$$\sum_{i_1\geq 0}\ldots\sum_{i_l\geq 0}\sum_{j_1\geq 0}\ldots\sum_{j_r\geq 0}\sqrt{\epsilon_1}^{j_1}\ldots\sqrt{\epsilon_l}^{j_l}\sqrt{\delta_1}^{i_1}\ldots\sqrt{\delta_r}^{i_r}u_{j_1,\ldots,j_r,i_1,\ldots i_l}$$

$$= \lim\sum_{i_1\geq 0}^{m_1}\ldots\sum_{i_l\geq 0}^{m_l\geq 0}\sum_{j_1\geq 0}^{m_{l+1}\geq 0}\ldots\sum_{j_n\geq 0}^{m_{l+r}}\sqrt{\epsilon_1}^{j_1}\ldots\sqrt{\epsilon_l}^{j_l}\sqrt{\delta_1}^{i_1}\ldots\sqrt{\delta_r}^{i_r}u_{j_1,\ldots,j_r,i_1,\ldots i_l},\ m_1\to\infty,\ \ldots,m_{l+r}\to\infty.$$

The approximate price is calculated

$$u^{\bar{\epsilon},\overline{\delta'}} \approx u_{\bar{0},\overline{0'}} + \sum_{j=1}^{l}\sqrt{\epsilon_j}u_{\bar{1}_j,\overline{0'}} + \sum_{i=1}^{r}\sqrt{\delta_i}u_{\bar{0},\overline{1'_i}}.$$

The choice of development in half-integer orders ϵ_j and δ_i are natural for $\mathfrak{L}^{\bar{\epsilon},\overline{\delta'}}$.

By conducting an analysis of singular perturbations at the corresponding levels, we obtain that $u_{\bar{0},\overline{0'}}, u_{\bar{1}_j,\overline{0'}}, u_{\bar{0},\overline{1'_i}}$ do not depend on $y_1,\ldots y_l$. The basic findings of the asymptotic analysis are given using the following formulas

$$\mathcal{O}(1)\colon \sum_{j=1}\mathfrak{L}_{0j}\,u_{\bar{2}_j,\overline{0'}} + (-\partial_t + \langle\mathfrak{L}_2\rangle)u_{\bar{0},\overline{0'}} = 0, u_{\bar{0},\overline{0'}}(0,x,z_1,\ldots,z_r) = H(x), \tag{4}$$

$$\mathcal{O}(\sqrt{\epsilon_j})\colon \mathfrak{L}_{0j}u_{\bar{3}_j,\overline{0'}} + \mathfrak{L}_{1j}u_{\bar{2}_j,\overline{0'}} + (-\partial_t + \langle\mathfrak{L}_2\rangle)u_{\bar{1}_j,\overline{0'}} + \sum_{k\neq j}\mathfrak{L}_{1k}u_{\bar{1}_{kj},\overline{0'}} + \sum_{i\neq j}\mathfrak{L}_{1i}$$

$$= \mathcal{A}_j u_{\bar{0},\overline{0'}},\ u_{\bar{1}_j,\overline{0'}}(0,x,z_1,\ldots,z_n) = 0, \tag{5}$$

$$\bar{1}_{kj} = \left(\underbrace{0,\ldots 1}_{k}\ \underbrace{0,1,0,\ldots.0}_{j}\right)$$

According to the analysis of regular perturbations we have

$$\mathcal{O}\left(\sqrt{\delta_i}\right): (-\partial_t + \langle\mathfrak{L}_2\rangle)u_{\overline{0},\overline{1'_i}} = \mathcal{B}_i\partial_{z_i}u_{\overline{0},\overline{0'}}, u_{\overline{0},\overline{1'_i}}(0, x, z_1, \dots, z_r) = 0, i = \overline{1, r}. \quad (6)$$

Operators $\langle\mathfrak{L}_2\rangle$, $\mathcal{A}_j$, $\mathcal{B}_i$ and ∂_{z_i} are defined by the formulas

$$\langle\mathfrak{L}_2\rangle = \frac{1}{2}\bar{\sigma}^2 a^2(x)\partial_{xx}^2 + \left(b(x) - \overline{f\Omega}a(x)\right)\partial_x - k(x), x \in (e_1, e_2),$$

$$\mathcal{A}_j = -\nu_{3j}a(x)\partial_x a^2(x)\partial_{xx}^2 - \nu_{2j}a^2(x)\partial_{xx}^2 - \mathcal{U}_{2j}a(x)\partial_x a(x)\partial_x - \mathcal{U}_{1j}a(x)\partial_x, ,$$

$\mathcal{B}_i = -\nu_{1i}a(x)\partial_x - \nu_{0i}$ and $\partial_{z_i} = \partial_{z_i}\bar{\sigma}\partial_{\bar{\sigma}} + \overline{f\Omega}'\partial_{\overline{f\Omega}}, \nu_{1i} := g_i\rho_{xz_i}\langle f\rangle, \nu_0 = g_i\langle\Gamma_i\rangle,$

$\forall\, i = \overline{1, n}$ and norm function

$$\langle\mathcal{X}\rangle_j := \int \mathcal{X}(y_1, \dots y_l)\pi_j(y_j)dy_j, \forall j = \overline{1, l},$$

$$\langle\mathcal{X}\rangle_{1,2} = \int_{R^2}\mathcal{X}(y_1, \dots y_l)\pi_1(y_1)\pi_2(y_2)dy_1dy_2, \dots \langle\mathcal{X}\rangle_{l-1,l}$$

$$= \int_{R^l}\mathcal{X}(y_1, \dots y_l)\pi_1(y_1)\dots\pi_l(y_l)dy_1\dots dy_l, \langle\mathcal{X}\rangle_{l-1,l} = \langle\mathcal{X}\rangle, \langle f\Omega\rangle := \overline{f\Omega}, \langle f^2\rangle = \bar{\sigma}^2.$$

We find solutions to the equations (4)-(6) on the basis of eigenfunctions, eigenvalues of the operator $\langle\mathfrak{L}_2\rangle$, each of which meets a corresponding Poisson equation

$$\mathfrak{L}_{01}\varphi_1 = f^2 - \langle f^2\rangle_1, \mathfrak{L}_{02}\varphi_2 = \langle f^2\rangle_1 - \langle f^2\rangle_{1,2}, \dots, \mathfrak{L}_{0l}\varphi_l = \langle f^2\rangle_{l-2,l-1} - \langle f^2\rangle_{l-1,l}.$$

$$\mathfrak{L}_{01}\eta_1 = f\Omega - \langle f\Omega\rangle_1, \dots, \mathfrak{L}_{0j}\eta_j = \langle f\Omega\rangle_{j-2,j-1} - \langle f\Omega\rangle_{j-1,j}, \dots, \mathfrak{L}_{0l}\eta_l = \langle f\Omega\rangle_{l-2,l-1} - \langle f\Omega\rangle_{l-1,l}$$

Theorem 1: Assume that we can solve the following equation to find an eigenvalue:

$$-\langle\mathfrak{L}_2\rangle\psi_n = \lambda_n\psi_n, \psi_n \in dom(\langle\mathfrak{L}_2\rangle), \quad (7)$$

and also that $H \in \mathcal{H}$. Then the solution $u_{\overline{0},\overline{0'}}$ has the form:

$$u_{\overline{0},\overline{0'}} = \sum_{n=1}^{\infty} c_n\psi_n T_n, c_n = (\psi_n, H), T_n = e^{-t\lambda_n}.$$

Theorem 2: Let c_n, ψ_n, T_n be described using Theorem 1. We define

$$\mathcal{A}_{jk,n} := \left(\psi_k, \mathcal{A}_j\psi_n\right), U_{k,n} := \frac{T_k - T_n}{\lambda_k - \lambda_n}.$$

Then the solution $u_{\overline{1}_j,\overline{0'}}$ of equation (5) has the form:

$$u_{\overline{1}_j,\overline{0'}} = \sum_n\sum_{k\neq n} c_n A_{jk,n}\psi_k U_{k,n} - \sum_n c_n\mathcal{A}_{jn,n}\psi_n t T_n$$

Note that $u_{\overline{1}_j,\overline{0'}}$ is linear in the parameter group $(\vartheta_{3j}, \vartheta_{2j}, \mathcal{U}_{2j}, \mathcal{U}_{1j})$.

Theorem 3: Let c_n, ψ_n and T_n be defined with Theorem 1, and $U_{k,n}$ with Theorem 2, we have

$$\widetilde{\mathcal{B}}_{ik,n} := \left(\psi_k, \mathcal{B}_i\partial_{Z_i}\psi_n\right), \mathcal{B}_{ik,n} := (\psi_k, \mathcal{B}_i\psi_n), V_{ik,n} := \frac{T_k - T_n}{(\lambda_k - \lambda_n)^2} + \frac{tT_n}{\lambda_k - \lambda_n}.$$

Then the solution $u_{\overline{0,1'_i}}$ has the form:

$$u_{\overline{0,1'_i}} = \sum_n \sum_{k \neq n} c_n \widetilde{\mathcal{B}}_{ik,n} \psi_k U_{ik,n} - \sum_n c_n \widetilde{\mathcal{B}}_{in,n} \psi_n t T_n +$$
$$+ \sum_n \sum_{k \neq n} (\partial_{Z_i} c_n) \mathcal{B}_{ik,n} \psi_k U_{ik,n} - \sum_n (\partial_{Z_i} c_n) \mathcal{B}_{in,n} \psi_n t T_n$$
$$+ \sum_n \sum_{k \neq n} c_n \mathcal{B}_{ik,n} \psi_k (\partial_{Z_i} \lambda_n) V_{ik,n} - \sum_n c_n \mathcal{B}_{in,n} \psi_n (\partial_{Z_i} \lambda_n) \frac{1}{2} t^2 T_n.$$

We draw attention to the fact that $u_{\overline{0,1'_i}}$ is linear in $(\nu_{1i}\bar{\sigma}', \nu_{1i}\overline{f\Omega'}, \nu_{0i}\bar{\sigma}', \nu_{0i}\overline{f\Omega'})$.

Having obtained the approximate solution $u^{\bar{\epsilon},\overline{\delta'}} \approx u_{\overline{0,0'}} + \sum_{j=1}^{l} \sqrt{\epsilon_j} u_{\overline{1_j,0'}} + \sum_{i=1}^{n} \sqrt{\delta_i} u_{\overline{0,1'_i}}$ for the derivative asset pricing.

For a more exact result we assume that the Payoff function $H(x)$ and its derivative are smooth and limited functions. Thus, we limit our derivative analysis to a smooth and limited payoff; in this case, the closeness estimates is based on the following theorem:

Theorem 4: For the fixed $(t, x, y_1, \dots, y_l, z_1, \dots, z_r)$ there exists an invariable C such that for any $\epsilon_j \le 1$, $\delta_i \le 1$ we have:

$$\left| u^{\bar{\epsilon},\overline{\delta'}} - \left(u_{\overline{0,0'}} + \sum_{j=1}^{l} \sqrt{\epsilon_j} u_{\overline{1_j,0'}} + \sum_{i=1}^{n} \sqrt{\delta_i} u_{\overline{0,1'_i}} \right) \right| \le C\left(\sum_{j=1}^{l} \epsilon_j + \sum_{i=1}^{r} \delta_i\right)$$

Theorem 4 gives us information on how the approximate price behaves when $\epsilon_j \to 0$ and $\delta_i \to 0$

Let X be short interest rates. One of the most widely known models of short interest rates is the Vasicek model, in which X is modeled as the Ornshtein-Ulenbeek process with multidimensional stochastic volatility. $\mathbb{P}$ dynamics of X are given, in particular

$$dX_t = (\kappa(\theta - X_t) - f(\mathrm{Y}_1, \dots, \mathrm{Y}_l, \mathrm{Z}_1, \dots, \mathrm{Z}_r)\Omega(\mathrm{Y}_1, \dots, \mathrm{Y}_l, \mathrm{Z}_1, \dots, \mathrm{Z}_r))dt + f(\mathrm{Y}_1, \dots, \mathrm{Y}_l, \mathrm{Z}_1, \dots, \mathrm{Z}_r)$$
$$+ d\widetilde{W}_t^x, r(X_t) = X_t, h(X_t) = 0,$$

where, $\mathrm{Y}_1, \dots, \mathrm{Y}_l$, and $\mathrm{Z}_1, \dots, \mathrm{Z}_r$ are fast and slowly variable volatility factors as described. We calculate the approximate price for a zero coupon bond.

We write the operator $\langle \mathfrak{L}_2 \rangle$ and the density associated with it at a rate $m(x)$

$$\langle \mathfrak{L}_2 \rangle = \frac{1}{2}\bar{\sigma}^2 \partial_{xx}^2 + \kappa(\bar{\theta} - x)\partial_x - x, (8)$$

$$m(x) = \frac{2}{\bar{\sigma}^2} exp\left(\frac{-k}{\bar{\sigma}^2}(\bar{\theta} - x)^2\right), \bar{\theta} = \theta - \frac{1}{\kappa}\overline{f\Omega}.$$

$$m(x) = \frac{2}{\bar{\sigma}^2} exp\left(\frac{-k}{\bar{\sigma}^2}(\bar{\theta} - x)^2\right), \bar{\theta} = \theta - \frac{1}{\kappa}\overline{f\Omega}.$$

To find a bond price with a payoff $H(X_t) = \mathbb{I}_{\{\tau > t\}} = 1$, we need to solve the equation (7) to find the eigenvalues for the segment $I = (-\infty, \infty)$ with $\langle \mathfrak{L}_2 \rangle$ in compliance with (8). Since both points ∞ and ∞ are natural limits, then the solution has the form Gorovy et al. (2004).

$$\psi_n = \mathcal{N}_n \exp\left(-A\xi - \frac{1}{2}A^2\right) H_n(\xi + A),$$

$$\mathcal{N}_n = \left(\sqrt{\frac{\kappa}{\pi}} \frac{\bar{\sigma}}{2^{n+1} n!}\right)^{1/2}$$

$$A = \frac{\bar{\sigma}}{\kappa^{3/2}}, \xi = \frac{\sqrt{\kappa}}{\bar{\sigma}}(x - \bar{\theta}),$$

$$\lambda_n = \lambda_n = \bar{\theta} - \frac{\bar{\sigma}^2}{2\kappa^2} + \kappa n, n = 0,1,2, \dots .$$

Here, H_n are Hermite polynomials. We will write the expressions for the operators $\mathcal{A}_j$ and $\mathcal{B}_j$:

$$\mathcal{A}_j = -\vartheta_{j3}\partial^3_{xxx} - \left(\vartheta_{j2} + \mathfrak{U}_{j2}\right)\partial^2_{xx} - \mathfrak{U}_{j1}\partial_x, \mathcal{B}_j = \vartheta_{j1}\partial_x - \vartheta_{j0}.$$

Operators $\mathcal{A}_{jk,n}, \mathcal{B}_{jk,n}$, and $\widetilde{\mathcal{B}}_{jk,n}$ are written on the basis of recurrence relations:

$$\partial_x H_n = 2nH_{n-1}, 2xH_n = H_{n+1} + \partial_x H_n, \mathcal{A}_{jk,n} = -\vartheta j_3$$

$$\left\{\sum_{m=0}^{3\wedge n} \binom{3}{m} \left(\frac{-1}{\kappa}\right)^{3-m} \left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)^m \frac{n!\,\mathcal{N}_n}{(n-m)!\,\mathcal{N}_{n-m}} \delta_{k,n-m}\right\}$$

$$-(\vartheta_2 + \mathfrak{U}_{j2})$$

$$\left\{\sum_{m=0}^{3\wedge n} \binom{2}{m} \left(\frac{-1}{\kappa}\right)^{2-m} \left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)^m \frac{n!\,\mathcal{N}_n}{(n-m)!\,\mathcal{N}_{n-m}} \delta_{k,n-m}\right\}$$

$$-\mathfrak{U}_{j1}\left\{\left(\frac{-1}{\kappa}\right)\delta_{k,n} + \left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1}\right\},$$

$$\mathcal{B}_{jk,n} = -\vartheta_1\left\{\left(\frac{-1}{\kappa}\right)\delta_{k,n} + \left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1}\right\} - \vartheta_{j0}\delta_{k,n},$$

$$\widetilde{\mathcal{B}}_{jk,n} = -\vartheta_{j1}\bar{\sigma}'^{\{[(\frac{-1}{\kappa})(\frac{1}{2\bar{\sigma}} - \frac{\bar{\sigma}}{\kappa^3} - \frac{n}{\bar{\sigma}})]\delta_{k,n}} + \left[\left(\frac{-1}{\kappa}\right)\left(\frac{4}{\kappa^{\frac{3}{2}}}\right) + \left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)\left(\frac{1}{2\bar{\sigma}} - \frac{\bar{\sigma}}{\kappa^3} - \frac{n}{\bar{\sigma}}\right)\right]\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1}$$

$$+\left[\left(\frac{-1}{\kappa}\right)\left(\frac{-2}{\bar{\sigma}}\right) + \left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)\left(\frac{4}{\kappa^{\frac{3}{2}}}\right)\right]\frac{n!\,\mathcal{N}_n}{(n-2)!\,\mathcal{N}_{n-2}}\delta_{k,n-2}$$

$$+\left[\left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)\left(\frac{-2}{\bar{\sigma}}\right)\right]\frac{n!\,\mathcal{N}_n}{(n-3)!\,\mathcal{N}_{n-3}}\delta_{k,n-3}\Bigg\}$$

$$-\vartheta_{i0}\bar{\sigma}'\left\{\left(\frac{1}{2\bar{\sigma}}-\frac{\bar{\sigma}}{\kappa^3}-\frac{n}{\bar{\sigma}}\right)\delta_{k,n}+\left(\frac{4}{\kappa^{\frac{3}{2}}}\right)\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1}\right.$$

$$\left.+\left(\frac{-2}{\bar{\sigma}}\right)\frac{n!\,\mathcal{N}_n}{(n-2)!\,\mathcal{N}_{n-2}}\delta_{k,n-2}\right\}$$

$$-\vartheta_{i1}\overline{f\Omega}'\left\{\left(\frac{1}{\kappa^3}\right)\delta_{k,n}+\left(\frac{-4}{\bar{\sigma}\kappa^{\frac{3}{2}}}\right)\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1}+\left(\frac{4}{\bar{\sigma}^2}\right)\frac{n!\,\mathcal{N}_n}{(n-2)!\,\mathcal{N}_{n-2}}\delta_{k,n-2}\right\}$$

$$-\vartheta_{i0}\overline{f\Omega}'\left\{\left(\frac{-1}{\kappa^2}\right)\delta_{k,n}+\left(\frac{2}{\bar{\sigma}\sqrt{\kappa}}\right)\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1}\right\}.$$

Calculation of c_n can be found in Gorovy and Linetsky (2004) [12]

$$c_n=(\psi_n,1)=\frac{2}{\bar{\sigma}}\sqrt{\frac{\pi}{\kappa}}\mathcal{N}_nA^ne^{-A^2/4}.$$

The approximate price of a bond can now be calculated applying the theorems 1-3.

For zero-coupon bonds, the bond curve is often considered, and not the bond price itself. The yield $R^{\bar{\epsilon},\overline{\delta'}}$ for a zero-coupon bond, on which one dollar is paid at time t is defined by the relation:

$$u^{\bar{\epsilon},\overline{\delta'}}=\exp(-R^{\bar{\epsilon},\overline{\delta'}}t)\cdot$$

We obtain the approximation for a zero-coupon bond, developing in series as bond prices $u^{\bar{\epsilon},\overline{\delta'}}$, and the yield $R^{\bar{\epsilon},\overline{\delta'}}$ in orders $\sqrt{\epsilon_j}$ and $\sqrt{\delta_i}$:

$$u_{\bar{0},\overline{0'}}+\sum_{j=1}^{l}\sqrt{\epsilon_j}u_{\overline{1_j},\overline{0'}}+\sum_{i=1}^{r}\sqrt{\delta_i}u_{\bar{0},\overline{1'_i}}+\cdots=e^{-\left(R_{\bar{0},\overline{0'}}+\sum_{j=1}^{l}\sqrt{\epsilon_j}R_{\overline{1_j},\overline{0'}}+\sum_{i=1}^{n}\sqrt{\delta_i}R_{\bar{0},\overline{1'_i}}\right)t}$$

$$=e^{-R_{\bar{0},\overline{0'}}t}+\sum_{j=1}^{l}\sqrt{\epsilon_j}R_{\overline{1_j},\overline{0'}}e^{-R_{\bar{0},\overline{0'}}t}+\sum_{i=1}^{r}\sqrt{\delta_i}R_{\bar{0},\overline{1'_i}}e^{-R_{\bar{0},\overline{0'}}t}+\cdots.$$

Grouping in orders $\sqrt{\epsilon_j}$ and $\sqrt{\delta_i}$ we obtain:

$$R^{\bar{\epsilon},\overline{\delta'}}\approx R_{\bar{0},\overline{0'}}+\sum_{j=1}^{l}\sqrt{\epsilon_j}R_{\overline{1_j},\overline{0'}}+\sum_{i=1}^{r}\sqrt{\delta_i}R_{\bar{0},\overline{1'_i}},$$

$$R_{\bar{0},\overline{0'}}=-\tfrac{1}{t}\ln(u_{\bar{0},\overline{0'}})\,,\;R_{\overline{1_j},\overline{0'}}=\frac{-u_{\overline{1_j},\overline{0'}}}{tu_{\bar{0},\overline{0'}}},R_{\bar{0},\overline{1'_i}}=\frac{-u_{\bar{0},\overline{1'_i}}}{tu_{\bar{0},\overline{0'}}},$$

Note that figures are constructed component-wise on each corresponding time scale, in much the same way as components in Lorig (2014) and Burtnyak et al. (2014).

3. CONCLUSIONS

The spectral theory and the theory of singular and regular perturbations are applied to investigate the short-term interest rates described by the Vasicek model. The approximate price of bonds and their yield are calculated. Applying the Sturm-Liouville theory, Fredholm's alternatives, as well as analysing singular and regular perturbations in different time scales, we obtained explicit formulas for convergence of bond prices and yields. To obtain explicit formulas, we need to solve 2l Poisson equations. The main advantage of our pricing methodology is that by combining methods from spectral theory, regular perturbation theory, and the theory of singular perturbations, we reduce everything to the solution of the equations to find their eigenfunctions and eigenvalues.

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