IJCTA, 9(9), 2016, pp. 4033-4040

Two dimensional polar coordinate system in airy stress functions

S. Senthil[1] and P. Sekar[2]

ABSTRACT

Satisfy the given equations, boundary conditions and biharmonic equation.In order to solve 2-dimensional airy stress function problems by using a polar coordinate reference frame, the equations of equilibrium, definition of Airy's stress function and one of the stress equations of compatibility must be recognized in terms of polar coordinates.

***Keywords*:** Form Cartesian Coordinates in to Polar Coordinates, Airy stress function for polar coordinates, Equation in polar coordinates, A stress field symmetric about an axis, A circular hole sheet under remote shear and Example

1. INTRODUCTION

The main purpose of this address is to bring to the attention of the workers in Airy stress function and related branches of applied mathematics a simple general method of solution of several important classes of 2-dimensional boundary value problem. A 2-dimensional polar coordinate study in rings and disks, curved bars of narrow rectangular cross section with a circular axis etc.Using the polar coordinates is advantageous to solve in airy stress function.

1.1. From Cartesian Coordinates in to Polar Coordinates

To transform equations from Cartesian to polar coordinates, first note the relations

$x = r\cos\theta,$ $\qquad y = r\sin\theta$

$r^2 = x^2 + y^2,$ $\qquad \theta = \tan{-1}\dfrac{y}{x}$

$r = \sqrt{x^2 + y^2},$ $\qquad \theta = \tan{-1}\dfrac{y}{x}$

$$\frac{\partial r}{\partial x} = \cos\theta \qquad \frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial r}{\partial y} = \sin\theta, \qquad \frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

1 Assistant professor, Department of Mathematics, Vel Tech Multi Tech Dr Rangarajan Dr sagunthala Engineering College, Avadi, Chennai, Tamil Nadu, India, *Email: senthil1986s@gmail.com*

2 Associate Professor, Department of Mathematics, C. Kandaswamy Naidu for Men College, Anna Nagar, Chennai, Tamil Nadu, India, *Email: cicesekar@yahoo.in*

$$\frac{\partial\varphi}{\partial x}=\frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial\varphi}{\partial\theta}\frac{\partial\theta}{\partial x}$$

$$\frac{\partial\varphi}{\partial x}=\cos\theta\frac{\partial\varphi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\varphi}{\partial\theta}$$

$$\frac{\partial\varphi}{\partial y}=\frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial\varphi}{\partial\theta}\frac{\partial\theta}{\partial y}$$

$$\frac{\partial\varphi}{\partial y}=\sin\theta\frac{\partial\varphi}{\partial r}+\frac{\cos\theta}{r}\frac{\partial\varphi}{\partial\theta}$$

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)=$$

$$\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial\varphi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

$$\frac{\partial^2\varphi}{\partial x^2}=(\cos\theta)^2\frac{\partial^2\varphi}{\partial r^2}+(\sin\theta)^2\left(\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right)+$$

$$\sin 2\theta\left(\frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}\right) \tag{1.1.1}$$

Similarly

$$\frac{\partial^2\varphi}{\partial y^2}=(\sin\theta)^2\frac{\partial^2\theta}{\partial r^2}+(\cos\theta)^2\left(\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right)-$$

$$\sin 2\theta\left(\frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}\right) \tag{1.1.2}$$

$$\frac{\partial^2\varphi}{\partial x\partial y}=-\sin\theta\ \cos\theta\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right)-$$

$$\cos 2\theta\left(\frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}\right) \tag{1.1.3}$$

From (1.1.1) and (1.1.2)

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \tag{A}$$

The Laplace equation is

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}=0 \tag{B}$$

1.2. Airy stress function for polar coordinates

The polar coordinate in plane elasticity problems such as the stresses in circular rings and disks, curved bars of narrow rectangular cross-section with a circular axis, etc. For a two-dimensional polar coordinate system, the solution to plane stress problems involves the determination of in plane stresses.The stress transformation from the Cartesian coordinate to the polar coordinate is

$$\sigma_{rr} = \sigma_x (\cos\theta)^2 + \sigma_y (\sin\theta)^2 + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_{\theta\theta} = \sigma_x (\sin\theta)^2 + \sigma_y (\cos\theta)^2 - 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{r\theta} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}\left((\cos\theta)^2 - (\sin\theta)^2\right) \quad (1.2.1)$$

1.3. Equation in polar coordinates

The airy stress function is a function of the polar coordinate $\varphi(r, \theta)$ the stress are expressed in term of the airy stress function

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r},\ \sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2},\ \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) \quad (1.3.1)$$

The bi harmonic equation is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right) = 0 \quad (C)$$

1.4. A stress field symmetric about an axis

Let the airy stress function be $\varphi(r)$. The bi harmonic equation is

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr}\right) = 0 \quad (1.4.1)$$

Each term in this equation has the same dimension in the independent variable r such an ODE is known as an equi-dimensional equation solution to an equi-dimensional equation is of the form

$$\varphi = r^m \quad (1.4.2)$$

Let equi-dimensional equation is

$$r^2 (r - 3)^2 = 0 \quad (1.4.3)$$

Into the bi harmonic equation, we obtain that the auxiliary equation is

$$m^2(m - 3)^2 = 0.$$
$$m^2 = 0 \text{ or } (m - 3)^2 = 0.$$

The fourth order algebraic equation has double root of 0 and a double root of 3.

The general solution is

$$\varphi(r) = A \log r + Br^3 \log r + C r^3 + D \tag{1.4.4}$$

Where A, B, C and D are constant of integration.

The components of the stress field are

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} = \frac{A}{r^2} + Br\left(1+3\log r\right)+3CR$$

$$\sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2} = -\frac{A}{r^2} + Br\left(5+2\log r\right)+6Cr$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) = 0 \tag{1.4.5}$$

The stress field is linear in A, B and C. The contribution due to A and C are familiar they are the same as the lame problem.

For example,

A hole of radius a in an infinite sheet subject to a remote biaxial stress S, the stress field in the sheet is

$$\sigma_{rr} = S\left[r - \left(\frac{a}{r}\right)^2\right], \quad \sigma_{\theta\theta} = S\left[2r + \left(\frac{a}{r}\right)^2\right] \tag{D}$$

The stress concentration factor of this hole is 2 we may compare this problem with that of a spherical cavity in an infinite elasticity solid under remote tension

$$\sigma_{rr} = S\left[r^2 - \left(\frac{a}{r}\right)^3\right], \quad \sigma_{\theta\theta} = S\left[2r^2 + \left(\frac{a}{r}\right)^3\right] \tag{E}$$

1.5. A circular hole in an sheet under remote shear

The sheet is a state of pure shear

$$\tau_{xy} = s,\ \sigma_{xx} = 0,\ \sigma_{yy} = 0. \tag{1.5.1}$$

The remote stresses in the polar coordinates are

$$\tau_{r\theta} = S\cos 2\theta,\ \sigma_{rr} = S\sin 2\theta,\ \sigma_{\theta\theta} = -S\sin 2\theta \tag{1.5.2}$$

We know that

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r},\ \sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2},\ \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

We guess that the stress function must be in the form

$$\varphi\left(r,\theta\right) = f\left(r\right)\sin 2\theta \tag{1.5.3}$$

The bi harmonic equation is

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{4f}{r^2}\right) = 0 \tag{F}$$

A solution to this equi-dimensional ODE takes the form

$$f(r) = r^m \tag{1.5.4}$$

Inserting this form into the ODE,

We obtain that

$$\left((m-2)^2 - 4\right)\left(\left(m^2 - 2\right)\right) = 0$$

$$\left((m-2)^2 - 4\right) = 0 \ or \ \left(m^2 - 2\right) = 0$$

The algebraic equation has four roots 2, –2, 0, –4.

The stress function is

$$\varphi(r,\theta) = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\sin 2\theta \tag{1.5.5}$$

The stress components inside the sheet are

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2} = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\sin 2\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) = \left(-2A - 6Br^2 + \frac{6C}{r^4} + \frac{2D}{r^2}\right)\cos 2\theta \tag{1.5.6}$$

Where A, B, C and D are constant.

To find the constant A, B, C, D.We invoke the boundary conditions:

1. Remote from the hole namely, $r \to \infty,\ \sigma_{rr} = S\sin 2\theta,\ \tau_{r\theta} = S\cos 2\theta,$ giving $A = \frac{-S}{2},\ B = 0.$

2. On the surface of the hole, namely, $r = a,\ \sigma_{rr} = 0,\ \tau_{r\theta} = 0,$ giving $D = \frac{-Sa^4}{2}.$

The stress field inside the sheet is

$$\sigma_{rr} = S\left(1 + 3\left[\frac{a}{r}\right]^4 - 4\left[\frac{a}{r}\right]^2\right)\sin 2\theta$$

$$\sigma_{\theta\theta} = -s\left(1 + 3\left[\frac{a}{r}\right]^4\right)\sin 2\theta$$

$$\tau_{r\theta} = s\left(1 - 3\left[\frac{a}{r}\right]^4 + 2\left[\frac{a}{r}\right]^2\right)\cos 2\theta \tag{1.5.7}$$

Example 2.1.1 A thin plate is subjected to uniform tensile stress σ_0 at its ends; Find the field of stress existing within the plate.

Solution:

The origin of coordinate axes at the centre of the plate

The state of stress in the plate is

$$\sigma_x = \sigma_0,\ \sigma_y = \tau_{xy} = 0$$

The stress function is $\varphi = \sigma_0 \frac{y^2}{2}$, satisfies the biharmonic equation.

The stress function φ may be transformed by substituting $y = r \sin \theta$.

$$\varphi = \frac{1}{2}\sigma_0 r^2 (\sin\theta)^2$$

$$\varphi = \frac{1}{4}\sigma_0 r^2 (1-\cos 2\theta)$$

The stresses in the plate

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} = \frac{1}{2}\sigma_0 (1+\cos 2\theta)$$

$$\sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2} = \frac{1}{2}\sigma_0 (1-\cos 2\theta)$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) = -\frac{1}{2}\sigma_0 \sin 2\theta$$

Example 2.1.2 A thin plate containing a small circular hole of radius α is subjected to simple tension. Find the field of stress and compare with those of a plate containing a small circular hole.

Solution: The boundary conditions appropriate to the circumference of the hole are

$$\sigma_r = \tau_{r\theta} = 0,\ r = a \tag{2.1.1}$$

For large distance away from the origin we set σ_r, σ_θ and $\tau_{r\theta}$ equal to the values found for a solid plate for $r = \infty$.

$$\sigma_r = \frac{1}{2}\sigma_0 (1+\cos 2\theta)$$

$$\sigma_\theta = \frac{1}{2}\sigma_0 (1-\cos 2\theta)$$

$$\tau_{r\theta} = -\frac{1}{2}\sigma_0 \sin 2\theta \tag{2.1.2}$$

We assume a stress function

$$\varphi = f_1(r) + f_2(r)\cos 2\theta \tag{2.1.3}$$

In f_1 and f_2 are yet to be determined

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 f_1}{dr^2}+\frac{1}{r}\frac{df_1}{dr}\right)=0 \quad (2.1.4)$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4}{r^2}\right)\left(\frac{d^2 f_2}{dr^2}+\frac{1}{r}\frac{df_2}{dr}-\frac{4f_2}{r^2}\right)=0 \quad (2.1.5)$$

The solution of equation is

$$f_1 = Ar^2 \log r + B \log r + Cr^2 + D \quad (2.1.6)$$

$$f_2 = Er^2 + Fr^4 + \frac{G}{r^2} + H \quad (2.1.7)$$

Where A, B, C, D, E, F, G and H are constant of integration. The stress function is then obtain integrating equation (2.1.6) and (2.1.7) into (2.1.3) by substituting φ into

The stress are found to be

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r} = A(1+2\log r)+2B+C\frac{1}{r}-\left(2E+\frac{6G}{r^4}+\frac{4H}{r^2}\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2} = A(3+2\log r)+2B-\frac{c}{r^2}+\left(2E+6Fr^2+\frac{6G}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) = \left(2E+6Fr^2-\frac{6G}{r^4}-\frac{2H}{r^2}\right)\sin 2\theta \quad (2.1.8)$$

The absence of D indicates that it has no influence on the solution.

According to the boundary condition (2) A = F = 0 in equation (8) because as $r \to \infty$ the stresses must assume finite values. Then, according to the condition (a), the equation (2.1.8) yield

$$2B+\frac{C}{a^2}=0,\ 2E+\frac{6G}{a^4}+\frac{4H}{a^2}=0,\ 2E-\frac{6G}{a^4}+\frac{2H}{a^2}=0 \quad (G)$$

Also, from equation (2.1.2) and (2.1.8)

We have

$$\sigma_r = -4F,\ \sigma_0 = 4B \quad (H)$$

Solving the preceding five expressions, we obtain

$$B=\frac{\sigma_0}{4},\ C=\frac{-a^2\sigma_0}{2},\ E=\frac{-\sigma_0}{4},\ G=\frac{-a^4\sigma_0}{4},\ H=\frac{a^2\sigma_0}{2} \quad (I)$$

The determination of the stress distribution in a large plate containing a small circular hole is completed by substituting these constant into equation (2.1.8)

$$\sigma_r = \frac{1}{2}\sigma_0\left[\left(1-\left(\frac{a}{r}\right)^2\right)+\left(1+3\left(\frac{a}{r}\right)^4-4\left(\frac{a}{r}\right)^2\right)\cos 2\theta\right]$$

$$\sigma_\theta = \frac{1}{2}\sigma_0\left[\left(1+\left(\frac{a}{r}\right)^2\right)-\left(1+3\left(\frac{a}{r}\right)^4\right)\cos 2\theta\right]$$

$$\tau_{r\theta} = -\frac{1}{2}\sigma_0\left[1-3\left(\frac{a}{r}\right)^4+2\left(\frac{a}{r}\right)^2\right]\sin 2\theta \qquad (2.1.9)$$

REFERENCE

[1] S.H.O, C. Hillman, F.F. Lange and Z. Suo."Surface cracking in layers under Biaxial, Residual compressive stress" J.AM.ceram.soc 78[9] 2353-59 (1995).

[2] S.P. Timoshenko and J.N. Goodier,"Theory of Elasticity" 3rd ed.McGrawHill, New York (1970).

[3] S. Woinowshy-krieger and S. Timoshenko "Theory of plates and shell" McGraw-Hill, New York (1970).

[4] Lekhnitskii S.G. "*Theory of Elasticity of an Anisotropic Body*" Mir Publishers, Moscow (1977).

[5] Poulos and Davis "*Elastic Solutions for Soil and Rock Mechanic*s" Wiley, New York (1974).

[6] "Two-dimensional problems in elasticity"-chapter 3

[7] Sokolnikoff, I.S. "Mathematical Theory of Elasticity" (2nd Ed.) N.Y.: McGraw-Hill (1956).

[8] England A.H. "Complex Variable Methods In Elasticity" London, Great Britain: Wiley & Sons Ltd (1971).

[9] Darshini Rao Kavati and Nomura, Seiichi. (2005) "Airy Stress Function for Two-Dimensional Circular Inclusion Problems" UTA Master's Thesis. (2005).

[10] Solution of 2dimensional problem in polar coordinates "solid mechanics "by ES240 (Fall 2007).