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SIGNED DOMINATION NUMBER ZERO, NEGATIVE AND POSITIVE

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ABSTRACT: A two-valued function f defined on the vertices of a graph $G = (V, E)$, $f : V \to \{-1, 1\}$ is a signed dominating function (SDF), if the sum of its function values over any closed neighbourhood is at least one. The weight of a signed dominating function is defined to be $w(f) = \sum f(v)$, over all vertices $v \in V$. The signed domination number of a graph G, denoted by $\gamma_s(G)$ and $\gamma_s(G) = \min \{w(f)\}$ where f is Signed dominating function of G. In this paper, we characterize the class of graphs G with $\gamma_s(G) = n$, where n is any integer and we found both upper and lower bounds on the size of a graph with $\gamma_s(G) = n$, where n is any integer.

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1. INTRODUCTION

Let $G = (V, E)$ be a simple graph. The open neighbourhood of a vertex v in G,denoted $N(v)$, is the set $\{u \mid uv \in E\}$ and the closed neighbourhood of v, denoted $N[v]$, is the set $N(v) \cup \{v\}$. A function $f : V \to \{-1, 1\}$ is a signed dominating function, if for every vertex $v \in V$, $\sum_{u \in N[v]} f(u) \geq 1$ holds. The weight $w(f)$ of f is the sum of the function value of all vertices in G. The signed domination number $\gamma_s(G)$ is defined to be the minimum weight taken over all signed dominating functions of G.

In [1], Dunbar *et al*., introduced this concept and it has been studied by several researchers [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. If we consider the vertices of a graph G as an individuals in a society and edges as the interpersonel relationship between the individuals. The function values +1 or –1 are treated as an individual with positive attitude or negative attitude. The signed dominating function in a graph G assures that positive attitude of a local group, that is, total attitude of an individual and his neighbours accounts into positive attitude. If this is the case, one may feel that the total attitude of the whole society (Ofcourse ! group of individuals) is positive. But, actually it is not so, since there exists graphs with signed domination number $\gamma_s(G)$ is zero and even negative also (see [4]). In this paper,we are interested in the class of graphs with $\gamma_s(G) = n$, where n is any integer. We are succeeded in characterizing the class of graphs with $\gamma_s(G) = n$, where n is any integer and found the lower and upper bounds on the size of graph with given order. Throughout this paper, we consider only finite, undirected simple graphs, we mean, without multiple edges or loops.

2. SOME OBSERVATIONS AND EXISTING RESULTS

The definition of signed dominating function $f : V \to \{-1, 1\}$ which satisfies $\sum_{u \in N[v]} f(u) \geq 1$, for all $v \in V$.

If f is a signed dominating function then f defines a partition of a vertex set V in to two sets M and P given by

$$M = \{v \in V : f(v) = -1\} \quad \text{and} \quad P = \{v \in V : f(v) = 1\}$$

and further, if f is a minimum signed dominating function then,

$$\gamma_s(G) = |P| - |M| = n - 2|M| \ (1)$$

By the very definition of signed dominating function, we have the following observations:

Observation 1: If f is a signed dominating function of a graph G, M and P are the sets of vertices assigned –1 by f and +1 by f respectively, then

(a) $|N(u) \cap M| + 2 \leq |N(u) \cap P|$ for every $u \in M$ and

(b) $|N(u) \cap M| \leq |N(u) \cap P|$ for every $u \in P$.

Observation 2: Let f be a signed dominating function of G, then

(a) If v is an isolated vertex, then $f(v) = 1$.

(b) If u is a pendent vertex and v is adjacent to u (that is, v is a support) then $f(u) = f(v) = 1$.

Proposition 1 [1]: Let G be a graph on n vertices. Then $\gamma_s(G) = n$ if and only if every vertex of G is an end vertex or a support in G.

Proposition 2 [1]: For every k regular graph G,

$$\gamma_s(G) \geq \frac{n}{k+1} .$$

3. NEW RESULTS ON SIGNED DOMINATION NUMBER

Proposition 3: Let G be a (n, m) graph with $\gamma_s(G) = 0$, then $O(G) \geq 6$.

Proposition 4: Let G be a (n, m) graph with $\gamma_s(G) = -k$; $k \geq 1$ then $O(G) \geq 9$.

Proposition 5: Let G be a (n, m) graph with $\gamma_s(G) = k$; $k \geq 1$ then $O(G) \geq 3$.

Theorem 6: For any (n, m) graph G, If $\gamma_s(G) = k$, $k \leq 0$, and then the induced subgraph induced by the vertices of V_1, (a set of vertices assigned by $-$ve sign) is not complete.

Proof: Let G be a (n, m) graph with $\gamma_s(G) = k$, $k \leq 0$ then the vertex set $V(G)$ can be partitioned in to two sets V_1 and V_2 such that $|V_1| = \frac{n-k}{2}$ and $|V_2| = \frac{n+k}{2}$. Assume that $\langle V_1 \rangle$ is complete graph on $\left(\frac{n-k}{2}\right)$-vertices. which implies that $q(V_1) = \frac{\left(\frac{n-k}{2}\right)\left(\frac{n-k}{2}-1\right)}{2}$, where $q(V_1)$ is the total number of edges in $\langle V_1 \rangle$.

$$\Rightarrow \; 2q(V_1) = \left(\frac{n-k}{2}\right)\left(\frac{n-k-2}{2}\right) = \frac{(n-k)\,(n-k-2)}{4} \sum_{i=1}^{\frac{n-k}{2}} \left| N(u_i) \cap V_1 \right|$$

By *Observation 1*, for every $u_i \in V_1$,

$$\sum_{i=1}^{\frac{n-k}{2}} \left\{ \left| N(u_i) \cap V_1 \right| + 2 \right\} \leq \sum_{i=1}^{\frac{n-k}{2}} \left| N(u_i) \cap V_2 \right|$$

$$\Rightarrow \; \frac{(n-k)\,(n-k-2)}{4} + 2\left(\frac{n-k}{2}\right) \leq \left(\frac{n-k}{2}\right)\left(\frac{n+k}{2}\right)$$

$$\Rightarrow \; n^2 - 2nk + k^2 - 2\,(n-k) + 4\,(n-k) \leq n^2 - k^2$$

$$\Rightarrow \; 2n \leq 0, \quad \text{if} \quad k = 0 \quad \text{and} \quad 2k^2 - 2nk + 2n - 2k < 0, \quad \text{if} \quad k < 0$$

which is impossible because $O(G) > 0$. Hence the proof.

Theorem 7: Let G be a (n, m) graph with $\gamma_s(G) = -k$, k is a positive integer.

The vertex set $V(G)$ can be partitioned into two sets such that

$$\left| V_1 \right| = \frac{n+k}{2}, \left| V_2 \right| = \frac{n-k}{2} \quad \text{then}$$

$$\frac{3}{2}(n+k) \leq m \leq \frac{(n-k)^2 - 3n + k}{2}; \quad \text{where}$$

$$n = 2l + 2k + 5, \text{ if } k \text{ is odd integer and } l \geq 1$$

$$n = 2l + 2k + 4, \text{ if } k \text{ is even integer and } l \geq 1.$$

Proof: Let G be a (n, m) graph with $\gamma_s(G) = -k$, and let f be a *SDF* of G with $w(f) = -k$. which makes partition the vertex set $V(G)$ in to two sets say, V_1 and V_2 such that $|V_1| = \frac{n+k}{2}$ and $|V_2| = \frac{n-k}{2}$ where V_1 and V_2 are the set of vertices assigned by $-$ve and $+$ve sign respectively satisfying the condition of the *Observation 1*, Clearly, one can see that

$$m = q(V_1) + q(V_1V_2) + q(V_2) \tag{1}$$

where, $q(V_i)$; i = 1, 2 denotes the number of edges joining the vertices of V_1 and V_2 (or V_2V_1). One can notice that, the minimum (or maximum) number of edges in a graph G depends on the minimum (or maximum) value of $q(V_i)$; i = 1, 2 and $q(V_1V_2)$. As f is a minimum signed dominating function,then minimum weight of f will be achieved, if

$$q(V_1) = 0 \quad \text{and} \quad q(V_1V_2) = 2\left(\frac{n+k}{2}\right) \tag{2}$$

since each vertex of V_1 is adjacent to at least two vertices of V_2. Further, By the second condition of the *Observation 1*, the minimum number of edges in $\langle V_2 \rangle$ will occur only when

$$|N(u) \cap V_2| = |N(u) \cap V_1| \text{ for every vertex } u \text{ in } V_2.$$

Thus,

$$q(V_2) = \frac{1}{2}\sum_{u \in V_2} degu = \frac{1}{2}\sum_{u \in V_2} |N(u) \cap V_2|$$

$$= \frac{1}{2}\sum_{u \in V_2} |N(u) \cap V_1| = \frac{1}{2} q(V_1V_2) \text{ Therefore by (2),}$$

$$q(V_2) = \frac{1}{2}(n+k) \tag{3}$$

Thus, from equation (2) and (3), equation (1) we have,

$$m \geq 0 + 2\left(\frac{n+k}{2}\right) + \frac{1}{2}(n+k) = \frac{3(n+k)}{2} \tag{4}$$

To prove the upper bound, we need maximum number of edges in $q(V_1)$, $q(V_1V_2)$ and $q(V_2)$, since $|V_2| = \frac{n-k}{2}$, complete graph on $\left(\frac{n-k}{2}\right)$-vertices gives $\frac{1}{2}\left(\frac{n-k}{2}\right)\left(\frac{n-k}{2}-1\right)$ number of edges between the vertices of V_2 only.Therefore,

$$q(V_2) = \frac{(n-k)(n-k-2)}{8} \tag{5}$$

since each vertex in V_2 is of degree $\left(\frac{n-k}{2}-1\right)$ between the vertices of V_2. i.e., for every $u \in V_2$, $|N(u) \cap V_2| = \left(\frac{n-k}{2}-1\right)$.

By the second condition of the *Observation 1*,

for every $u \in V_2$, $|N(u) \cap V_2| \geq |N(u) \cap V_1|$

$$q(V_1V_2) = \sum_{u \in V_2} |N(u) \cap V_1| \leq \sum_{u \in V_2} |N(u) \cap V_2| = \left(\frac{n-k}{2}\right)\left(\frac{n-k}{2} - 1\right)$$

$$\Rightarrow \qquad q(V_1V_2) \leq \frac{(n-k)(n-k-2)}{4} \tag{6}$$

Lastly, $2q(V_1) = \sum_{u \in V_1} degu = \sum_{u \in V_1} |N(u) \cap V_1| \leq \sum_{u \in V_1} \{|N(u) \cap V_2| - 2\}$

$$= \sum_{u \in V_1} |N(u) \cap V_2| - 2\left(\frac{n+k}{2}\right) = q(V_1V_2) - 2\left(\frac{n+k}{2}\right)$$

$$2q(V_1) \leq \frac{(n-k)(n-k-2)}{4} - (n+k)$$

$$\Rightarrow q(V_1) \leq \frac{n^2 - 2nk + k^2 - 6n - 2k}{8} \tag{7}$$

From equation (5), (6) and (7), equation (1) becomes,

$$m \leq \frac{n^2 - 2nk + k^2 - 6n - 2k}{8} + \frac{(n-k)(n-k-2)}{4} + \frac{(n-k)(n-k-2)}{8}$$

$$\Rightarrow \qquad \frac{3}{2}(n+k) \leq m \leq \frac{(n-k)^2 - 3n + k}{2} \tag{8}$$

Then from equation, (4) and (8), we have therefore,

$$\frac{3}{2}(n+k) \leq m \leq \frac{(n-k)^2 - 3n + k}{2}$$

Hence the proof.

Corollary 7.1: Put $k = 0$, we get $\gamma_s(G) = 0$ and there vertex set $V(G)$ can be partitioned in to two sets $|V_1| = \frac{n}{2}$ and $|V_2| = \frac{n}{2}$ moreover, $\frac{3n}{2} \leq m \leq \frac{n^2 - 3n}{2}$.

Corollary 7.1.1: Construction of (n, m) graph G, If $\gamma_s(G) = 0$, then $\frac{3n}{2} \leq m \leq \frac{n^2 - 3n}{2}$, $n = 2l + 4$ and $l \geq 1$.

Proof: Let G be a graph of order $n = 2l + 4$, $l \geq 1$. since $\gamma_s(G) = 0$.

So the vertex set $V(G)$ can be partitioned in to two sets say, V_1 and V_2 such that $|V_1| = |V_2| = \frac{2l+4}{2} = l + 2$, $l \geq 1$. Let $V_1 = \{v_1, v_2, ..., v_{l+2}\}$ be the set of vertices assigned by –ve sign and let $V_2 = \{u_1, u_2, ..., u_{l+2}\}$ be the set of vertices assigned by +ve sign. By the definition of *SDF*, each vertex of V_1 is adjacent to at least two vertices of V_2.

For one way of constructing the graph G, first we construct a graph G, to attain its lower bound. For that, necessarily, each vertex of $v_i \in V_1$ is adjacent to two non adjacent vertices say, u_i and u_j. In this way, all the vertices of V_1, have $2(l + 2)$-edges between V_1 to V_2, and $l + 2$-edges within the vertices of V_2. In all totally, $\{2(1 + 2)\} + \{1 + 2\} = n + \frac{n}{2}$ edges.

i.e., $m = \frac{3n}{2}$ edges, it attains a lower bound.

Next, we construct G, to attain its upper bound, it is necessary to draw $\left\{\frac{n^2-3n}{2} - \frac{3n}{2}\right\} = \frac{n^2-6n}{2}$, number of edges without aecting the signed dominating function for each step.

Unless, we create an edge between the vertices of V_2, we can not take edges between the vertices of V_1 or between the vertices of V_1 to V_2. Now the question is, how many possible number of edges can take between the vertices of V_2?

Clearly, there are $\frac{(l+2)(l+2-1)}{2} - (l+2) = \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{2} - \frac{n}{2} = \frac{n(n-2)}{8} - \frac{n}{2} = \frac{n^2-2n-4n}{8} = \frac{n^2-6n}{8}$, number of edges are possible between the vertices of V_2.

Without taking all possible edges in V_2, can we take edges between the vertices of V_1 and V_2, and within the vertices of V_1? Yes, it possible in the following way,

Step 1: Draw the first edge among $\left\{\frac{n^2-6n}{8}\right\}$-number of edges between any two nonadjacent vertices in V_2, say ui and u_j.

Step 2: Second edge is to be draw between any two non-adjacent vertices, from u_i (or u_j) to one of the vertices of V_1, say u_i (or u_j) adjacent to v_i.

Step 3: Third edge is to be draw between two non-adjacent vertices from u_j (or u_i) to one of the vertices of V_1, say u_j (or u_i) adjacent to v_j such that v_i not adjacent to v_j.

Step 4: Fourth edge is to be draw between the vertices v_i and v_j in V_1. Repeat the steps for each 4-edges in the possible (or given) total number of edges. This means, each new edge between the vertices of V_2, allows us to take two new edges between the vertices of V_2 to V_1, and these two new edges allows us to take one more new edge within the vertices of V_1. In this way, it is to be distributed in $q(V_2) = \frac{n^2-6n}{2}$ and $q(V_1V_2) = \frac{n^2-6n}{2}$ and $q(V_1) = \frac{n^2-6n}{8}$ to attain its upper bound.

Hence the proof.

Theorem 8: Let G be a (n, m) graph with $\gamma_s(G) = k$, $k \geq 1$ the vertex set $V(G)$ can be partitioned into two sets such that $|V_1| = \frac{n-k}{2}$, $|V_2| = \frac{n+k}{2}$ then $\frac{3}{2}(n-k) \leq m \leq \frac{n(n-1)}{2}$; $n = 2l + k$ and $l \geq 1$.

Proof: Proof is similar to Theorem 7

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