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ANOTHER APPROACH TO THE STUDY OF SINGULAR STOCHASTIC LEONTIEFF TYPE EQUATIONS WITH IMPULSE ACTIONS

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ABSTRACT. By a Leontieff-type singular stochastic equation we understand a special class of stochastic differential equations in Ito form that have real rectangular matrices on the left-hand and right-hand sides that form a singular pencil. In addition, on the right-hand side there is a deterministic term, which depends only on time, as well as impulse effects. We assume that the diffusion coefficient of this system is given by a matrix that depends only on time. We apply the Kronecker transform, constructed as a generalization of the real Schur form to the case of a singular pencil of matrices, to the system investigated in this work, and so reduce it to the canonical form. To study the obtained canonical equations, it is necessary to consider derivatives of sufficiently high orders of free terms, including the Wiener process. In this regard, to differentiate the Wiener process, we use the machinery of the Nelson mean derivatives of random processes that allows us not to use the the theory of generalized functions in the study of the equation. As a result, we obtain analytical formulae for the solutions of equations in terms of mean derivatives of random processes.

Introduction

We study the system of Ito stochastic differential equations of the form

$$d\tilde{L}\xi(t) = \tilde{M}\xi(t)dt + f(t)dt + d\tilde{Q}\zeta(t) + P(t)dw(t),\ \ 0\ leqt\ leqT,$$

where $\lambda\tilde{L} + \tilde{M}$ is a singular pencil of real constant matrices of size $n \times m$, moreover, in the case of square matrices $\tilde{L}$ is degenerate, $P(t)$ is the diffusion coefficient, which is a sufficiently smooth $n \times m$-matrix, $f(t)$ is a sufficiently smooth deterministic vector time-dependent function, $tildeQ$ is a numeric $n \times n$-matrix, $\zeta(t)$ is an n-dimensional jump process, $w(t)$ is the Wiener process, $\xi(t)$ is the random process we are looking for. The following titles are used in the literature for such systems: algebraic differentialsystems, descriptor ones and Leontieff type systems. Such equations are in use in mathematical modeling in economic, technical, chemical and other problems, as well as in control theory (see [1])

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The cornerstone for this work is the article [2], in which such equation is studied using the canonical Kronecker form, constructed as a generalization of the Weierstrass form to the case of a singular beam. Since, in the general case, we cannot stably calculate the Weierstrass form (see [3]), it becomes necessary to study this system using the Kronecker transform, which is constructed as a generalization of the real Schur form for the case of singular pencila. iT is known (see [3]) that the Schur canonical form is computed in a stable way.

To study this class of equations, it is necessary to consider higher-order derivatives of the free terms [1], [3], citegant4, i.e., in this case, the deterministic term and the Wiener process or white noise. It is known that derivatives of the Wiener process exist only in the sense of generalized functions and so it is extremely difficult to apply it to the study of specific equations. This circumstance makes direct investigation of our system difficult.

Following [2], in which this class of equations was studied using the Kronecker-Weierstrass transform, we use the machinery of Nelson's mean derivatives of random processes that allows us to describe the solutions without use of generalized functions. Namely, we use the symmetric mean derivatives (current velocities) of the Wiener process. Current velocities, in accordance with the general ideology of mean derivatives, are natural analogues of of the physical velocities of deterministic processes. As a result, for the system under consideration, we obtain physically meaningful formulas for solutions in terms of symmetric mean derivatives of random processes.

1. Mean Derivatives

Consider the stochastic process $\xi(t)$ in R^n, $t \in [0, l]$, defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is an L_1-random variable for all t. It is known that every such a process generates the family of $\mathcal{N}_t^\xi$ σ-subalgebras of $\mathcal{F}$, that is the minimal σ such that $\xi(t)$ is measurable with respect to iut. Such σ-subalgebra is denoted as $\mathcal{N}^{\xi_t}$ and called the "present" of $\xi(t)$, which we assume to be complete, that is, filled with all probability sets of zero probability.

For convenience, we denote the conditional expectation $E(\cdot|\ calN_t^\xi)$ with respect to the "present" $\mathcal{N}_t^\xi$ for $\xi(t)$ by E_t^ξ. Ordinary ("unconditional") mathematical expectation is denoted by the symbol E.

Generally speaking, almost all sample trajectories of the process $\ xi(t)$ are not differentiable, so its derivatives exist only in the sense of generalized functions. To avoid using generalized functions, according to Nelson [5, **?**] we give the following definition:

Definition 1.1 ([8])**.** (i) The forward mean derivative $D\xi(t)$ of the process $\xi(t)$ at the time instant t is an L_1 -random variable of the form

$$D\xi(t) = \lim_{\triangle t \to +0} E_t^\xi \left(\frac{\xi(t + \triangle t) - \xi(t)}{\triangle t}\right),$$

where the limit is assumed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\triangle t \to +0$ means that $\triangle t$ tends to 0 and $\triangle t > 0$. (ii) The backward mean derivative $D_*\xi(t)$ of the process

$\xi(t)$ at time t is an L_1 random variable

$$D_*\xi(t) = \lim_{\triangle t \to +0} E_t^{xi}(\frac{\xi(t) - \xi(t - \triangle t)}{\triangle t}),$$

where (as in (i)) the limit is assumed to exist in $L_1(\ Omega, \mathcal{F}, \mathsf{P})$ and $\triangle t \to +0$ means that $\triangle t$ tends to 0 and $\triangle t > 0$.

It should be noted that, generally speaking, $D\xi(t) \neq D_*\xi(t)$, but if, for example, $\xi(t)$ almost surely has smooth sample trajectories, these derivatives obviously coincide.

From the properties of conditional expectation (see [9]) it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as superpositions of $\xi(t)$ and Borel vector fields (regressions)

$$Y^0 t, x) = \lim_{\triangle t \ \ to +0} E_t^{\xi}(\frac{\xi(t + \triangle t) - \xi(t)}{\triangle t}|\xi(t) = x)$$

$$Y_*^0(t, x) = \lim_{\triangle t \to +0} E_t^{\xi}\frac{\xi(t) - \xi(t - \triangle t)}{\triangle t}|\xi(t) = x)$$

on R^n, that is, $D\xi(t) = Y^0(t, \xi(t))$ and $D_*\xi(t) = Y_*^0(t, \xi(t))$.

Definition 1.2. ([8]) The derivative $D_S = \frac{1}{2}(D + D_*)$ is called the symmetric mean derivative. The derivative $D_A = \frac{1}{2}(D - D_*)$ is called the antisymmetric mean derivative.

Consider the vector fields $v^\xi(t, x) = \frac{1}{2}(Y^0(t, x) + Y_*^0(t, x))$ and $u^\xi(t, x) = \frac{1}{2}(Y^0(t, x) - Y_*^0(t, x))$.

Definition 1.3 ([8])**.** $v^\xi(t) = v^\xi(t, \xi(t)) = D_S\xi(t)$ is called the current velocity of $\xi(t)$; $u^{\ xi}(t) = u^\xi(t, \xi(t)) = D_A\xi(t)$ is called the osmotic velocity of the process $\xi(t)$.

The current velocity for random processes is a direct analogue of the usual physical velocity of deterministic processes (see [8]). The osmotic velocity measures how quickly the "randomness" of the process increases.

The Wiener process plays the determining role in our constructions ([8]). We denote it by the symbol $w(t)$.

Lemma 1.4 ([10])**.** *Let $w(t)$ be an n-dimensional Wiener process, $P(t)$ be a sufficiently smooth $k \ timesn$ -matrix, $t \ in(0, T)$. Then for any t we have the formula*

$$D_S^w \int_0^t P(s)dw(s) = P(t)\frac{w(t)}{2t}.$$

Lemma 1.5 ([8], [11])**.** *For $t \in (0, T)$, the following equalities hold:*

$$Dw(t) = 0, D_*w(t) = \frac{w(t)}{t}, D_Sw(t) = \frac{w(t)}{2t}$$

For all $k \geq 2$

$$D_S^k w(t) = (-1)^{k-1}\frac{\prod_{i=1}^{k-1}(2i-1)}{2^k}\frac{w(t)}{t^k}.$$

2. Canonical form of a singular pencil of constant matrices

We give the necessary information from the theory of real constant matrices, a detailed exposition of which is available in the books [3, 4].

Definition 2.1. If A and B are matrices of size $n \times m$, then the matrix $\lambda A + B$ is called a matrix pencil, or simply a pencil. Here λ is a parameter, not a specific number.

Definition 2.2. If A and B are square matrices and $\det(\lambda A + B)$ is not identically equal to zero, then the pencil $\lambda A + B$ is called regular. Otherwise, the pencil is called singular.

Theorem 2.3. (Generalized real Schur form) *For a regular pencil $\lambda A + B$, there are real orthogonal matrices Q_L and Q_R such that the matrix $Q_L A Q_R$ is the upper quasi-triangular (i.e., the upper block-triangular matrix with diagonal blocks of size 1×1 and 2×2; blocks of size 1×1 correspond to real eigenvalues, and blocks of size 2×2 correspond to conjugate pairs of complex eigenvalues), and the matrix $Q_L B Q_R$ is upper triangular.*

Theorem 2.4. *For a singular matrix pencil $\lambda A + B$ of size $n \times m$, for which rows and columns are not linearly dependent on constant coefficients, there exists the Kronecker transform (described by a pair of non-degenerate matrices (operators) P_L and P_R of sizes $n \times n$ and $m \times m$ respectively), in which the matrix $P_L B P_R + \lambda P_L A P_R$ has a quasi-diagonal form*

$$\begin{pmatrix} \lambda A_0 + B_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{\varepsilon_1} & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & L_{\varepsilon_p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L^T_{\nu_1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & L^T_{nu_q} \end{pmatrix}, \tag{2.1}$$

$$(0 < \varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_p, 0 < \nu_1 \leq \nu_2 \leq \ldots \leq \ nu_q)$$

where

$$L_\varepsilon = \begin{pmatrix} 1 & lambda & 0 & ldots & 0 & 0 \\ 0 & 1 & lambda & ldots & 0 & 0 \\ vdots & vdots & vdots & ddots & vdots & vdots \\ 0 & 0 & 0 & ldots & lambda & 0 \\ 0 & 0 & 0 & ldots & 1 & lambda \end{pmatrix}$$

is a canonical Kronecker singular cell of size $\varepsilon \times (\varepsilon + 1)$, L^T_ν is a matrix transposed to L_ν and $\lambda A_0 + B_0$ is a regular $\delta \times \delta$-pencil of matrices that has a canonical generalized real Schur form.

3. Main result

As it is stated in the introduction, a Leontieff-type singular stochastic equation with impulse actions is a stochastic differential equation in R^n of the form

$$d\tilde{L}\xi(t) = \tilde{M}\xi(t)dt + f(t)dt + d\tilde{Q}\zeta(t) + P(t)dw(t), 0 \le t \le T$$

which in integral form has the form

$$\tilde{L}\xi(t) = \int_0^t \tilde{M}\xi(s)ds + \int_0^t f(s)ds + \tilde{Q}\zeta(t) + \int_0^t P(s)dw(s), \ \ 0 \le t \le T, \tag{3.1}$$

where all the objects in this equation are described in the introduction. For simplicity, we assume that the rows and columns of the pencil $\tilde{M} + \lambda\tilde{L}$ are not connected by linear dependencies with constant coefficients. In addition, we set the jump process $\zeta(t) = \zeta(t,\omega)$ as follows

$$\zeta(t,\omega) = \sum_{k=1}^{N} \tilde{\zeta}_k(\omega)\chi(t - t_k), \ \ 0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = T,$$

where $\chi(t)$ is the Heaviside function, equal to zero for negative values ??of the argument and one for non-negative; $\tilde{\zeta}_k(\omega)$ are random variables with values ??in R^n.

From the form of (3.1) it is clear that (for simplicity) the initial condition for solutions of (3.1) is assumed to be as follows

$$\xi(0,\omega) = 0. \tag{3.2}$$

We should point out that for the solutions we constructed below, this condition is not fulfilled. Therefore, we approximate solutions by processes that satisfy this initial condition, but become solutions only from some (predetermined arbitrarily small) time instant $t_0 > 0$ (see below).

We will look for formulas for the solutions of problem (3.1), (3.2) (as well as in cite b2) among the random processes $\xi(t,\omega)$ that satisfy (in the sense described below) the differential equations

$$\begin{aligned}
\tilde{L}\xi(t) - \tilde{L}\xi(0) &= \tilde{M}\int_0^t \xi(s)ds + \int_0^t f(s)ds + \int_0^t P(s)dw(s), \ 0 \le t \le t_1,\\
\tilde{L}\xi(t) - \tilde{L}\xi(t_r) &= \tilde{M}\int_{t_r}^t \xi(s)ds + \int_{t_r}^t f(s)ds + \int_{t_r}^t P(s)dw(s), \ t_r \le t \le t_{r+1},\\
\tilde{L}\xi(t) - \tilde{L}\xi(t_N) &= \tilde{M}\int_{t_N}^t \xi(s)ds + \int_{t_N}^t f(s)ds + \int_{t_N}^t P(s)dw(s), \ t_N \le t \le T,
\end{aligned}$$

for all $r = 1, 2, \ldots, N-1$, at the points t_r they satisfy the equalities

$$\tilde{L}\xi(t_r + 0,\omega) - \tilde{L}\xi(t_r - 0,\omega) = \tilde{Q}\tilde{\zeta}_r(\omega), r = 1, 2, \ldots, N,$$

and at the initial moment of time $t = 0$ satisfy initial condition (3.2).

So, the process $\xi(t)$ for solving the problem (3.1), (3.2) is determined sequentially for $r=0,1,\ldots,N$ via random processes $\xi_r(t)$, which satisfy the equations

$$\tilde{L}\xi_0(t)-\tilde{L}\xi_0(0)=\tilde{M}\int_0^t \xi_0(s)ds+\int_0^t f(s)ds+\int_0^t P(s)dw(s),\ 0\le t\le t_1,\ (r=0),$$

$$\tilde{L}\xi_r(t)-\tilde{L}\xi_r(t_r)=\tilde{M}\int_{t_r}^t \xi_r(s)ds+\int_{t_r}^t f(s)ds+\int_{t_r}^t P(s)dw(s),\ t_r\le t\le t_{r+1},$$

$$\tilde{L}\xi_N(t)-\tilde{L}\xi_N(t_N)=\tilde{M}\int_{t_N}^t \xi_N(s)ds+\int_{t_N}^t f(s)ds+\int_{t_N}^t P(s)dw(s),\ t_N\le t\le T,$$

$r=1,2,\ldots,N-1$, where

$$\xi_0(0)=0,\ \tilde{L}\xi_r(t_r)=\tilde{L}\xi_{r-1}(t_r,\omega)+\tilde{Q}\tilde{\zeta}_r(\omega),\ r=1,\ldots,N.$$

It is easy to see, that equation (3.1) in the general form is inconvenient for study, so we will bring it to a certain canonical form. We apply the Kronecker transformation, described in the previous section, to the matrix pencil $\tilde{M}+\lambda\tilde{L}$. Then the equation (3.1) is transformed as follows

$$P_L\tilde{L}P_RP_R^{-1}\xi(t)=\int_0^t P_L\tilde{M}P_RP_R^{-1}\xi(\tau)d\tau+ \\ +\int_0^t P_Lf(\tau)d\tau+P_L\tilde{Q}\zeta(t)+\int_0^t P_LP(\tau)dw(\tau)$$

The regular component of the pencil $P_L\tilde{M}P_R+\lambda P_L\tilde{L}P_R$ is denoted by $\lambda A+B$, let $L=P_L\tilde{L}P_R$, $M=P_L\tilde{M}P_R$. Given the corresponding numbering of the basis vectors, we obtain that in $M+\lambda L$ along the main diagonal there is the pencil $\lambda A+B$ and canonical singular Kronecker cells L_ε, L_ε^T in order specified in eqref oks. The elements $\lambda A+B$ are arranged in this way: in A, first there are blocks of size 2×2 along the main diagonal, then non-degenerate blocks of size 1×1, and then degenerate blocks of size 1×1.

Denote $\eta(t)=P_R^{-1}\xi(t)$, $C(t)=P_LP(t)$, $U=P_L\tilde{Q}$. In the new notation, (3.1), (3.2) takes the form

$$L\eta(t)=\int_0^t M\eta(s)ds+\int_0^t P_Lf(s)ds+U\zeta(t)+\int_0^t C(s)dw(s),\tag{3.3}$$

$$\eta(0)=0,\tag{3.4}$$

Then, taking into account the above, we o9btain that the formulas for solutions $\eta(t)$ of problem (3.3), (3.4) are determined sequentially for $r=0,1,\ldots,N$ via random processes $\eta_r(t)$ that satisfy the equations

$$L\eta_0(t)-L\eta_0(0)=\int_0^t M\eta_0(s)ds+\int_0^t P_Lf(s)ds+\int_0^t C(s)dw(s),\ 0\le t\le t_1,\tag{3.5}$$

$$L\eta_r(t) - L\eta_r(t_r) = \int_{t_r}^{t} M\eta_r(s)ds + \int_{t_r}^{t} P_L f(s)ds + \int_{t_r}^{t} C(s)dw(s),\ t_r \le t \le t_{r+1}, \tag{3.6}$$

$$L\eta_N(t) - L\eta_N(t_N) = \int_{t_N}^{t} M\eta_N(s)ds + \int_{t_N}^{t} P_L f(s)ds + \int_{t_N}^{t} C(s)dw(s), t_N \le t \le T, \tag{3.7}$$

$r = 1, \dots, N-1$, where

$$\eta_0(0) = 0,\ \text{Ł}\eta_r(t_r) = L\eta_{r-1}(t_r,\omega) + U\tilde{\zeta}_r(\omega), r = 1, \dots, N. \tag{3.8}$$

Remark 3.1. As noted above, to construct a process describing the model given by the equations (3.5), (3.6) and (3.7), it is necessary to consider higher-order derivatives of free terms (including the Wiener process). Derivatives of the Wiener process exist only in the sense of generalized functions. Therefore, to avoid the use of generalized functions, we will use the symmetric derivatives on average (current speeds) D_S^w for random to construct a process that describes the model given (3.5), (3.6) and (3.7) processes. In this paper, we will use the σ -algebra "real" the Wiener process, to calculate higher-order symmetric derivatives. Note that to calculate derivatives on average, you can use any other σ -algebra, but then the formulas for calculating higher-order symmetric derivatives from the Wiener process change.

It is easy to see, taking into account the quasi-diagonal structure of $M + \lambda L$, that problems (3.3), (3.8) (3.5), (3.6), (3.7) and (3.8) (an equation of special type corresponds to each component in $M + \lambda L$). Denote by $\tilde{\eta}(t)$, $\check{\eta}(t)$, $\hat{\eta}(t)$ the components of vector $\eta(t)$, corresponding the bolcks $\lambda A + B$, L_ε, L_ν^T, by $\tilde{g}(t)$, $\check{g}(t)$, $\hat{g}(t)$ we denote the corresponding components of vector $P_L f(t)$, by $\tilde{U}$, $\check{U}$, $\hat{U}$ – the corresponding blocks of matrix U,and by $\tilde{C}(t)$, $\check{C}(t)$, $\hat{C}(t)$ – the corresponding blocks of matrix $C(t)$. Dernote by $\lambda F + K$, $\lambda G + H$ the blocks containing all singular cells of L_ε, L_ν^T type, respectively. WSe shall investigate every type of these equations.

Let the pencil $\lambda A + B$ be $q \times q$ size. In accordance with canonical Schur form, the equation corresponding to this regular pencil, splits into stochastic equations of the following types.

We unite the last $q-p+1$ components of the process $\tilde{\eta}_r$, corresponding to rows of A with degenerate diagonal blocks of 1×1 size, into unique matrix equation

$$\begin{pmatrix} 0 & a_{p+1}^p & a_{p+2}^p & \dots & a_q^p \\ 0 & 0 & a_{p+2}^{p+1} & \dots & a_q^{p+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \tilde{\eta}_r^p(t) \\ \tilde{\eta}_r^{p+1}(t) \\ \vdots \\ \tilde{\eta}_r^q(t) \end{pmatrix}$$

$$
-\begin{pmatrix} 0 & a^p_{p+1} & a^p_{p+2} & \dots & a^p_q \\ 0 & 0 & a^{p+1}_{p+2} & \dots & a^{p+1}_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \tilde{\eta}^p_r(t_r) \\ \tilde{\eta}^{p+1}_r(t_r) \\ \vdots \\ \tilde{\eta}^q_r(t_r) \end{pmatrix} =
$$

$$
= \int_{t_r}^t \begin{pmatrix} b^p_p & b^p_{p+1} & \dots & b^p_q \\ 0 & b^{p+1}_{p+1} & \dots & b^{p+1}_q \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & b^q_q \end{pmatrix} \begin{pmatrix} \tilde{\eta}^p_r(s) \\ \tilde{\eta}^{p+1}_r(s) \\ \vdots \\ \tilde{\eta}^q_r(s) \end{pmatrix} ds + \int_{t_r}^t \begin{pmatrix} g^p(s) \\ g^{p+1}(s) \\ \vdots \\ g^q(s) \end{pmatrix} ds +
$$

$$
+ \int_{t_r}^t \begin{pmatrix} \tilde{c}^p_1(s) & \tilde{c}^p_2(s) & \dots & \tilde{c}^p_{m-1}(s) & \tilde{c}^p_m(s) \\ \tilde{c}^{p+1}_1(s) & \tilde{c}^{p+1}_2(s) & \dots & \tilde{c}^{p+1}_{m-1}(s) & \tilde{c}^{p+1}_m(s) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \tilde{c}^{q-1}_1(s) & \tilde{c}^{q-1}_2(s) & \dots & \tilde{c}^{q-1}_{m-1}(s) & \tilde{c}^{q-1}_m(s) \\ \tilde{c}^q_1(s) & \tilde{c}^q_2(s) & \dots & \tilde{c}^q_{m-1}(s) & \tilde{c}^q_m(s) \end{pmatrix} d \begin{pmatrix} w^1(s) \\ w^2(s) \\ \vdots \\ w^{m-1}(s) \\ w^m(s) \end{pmatrix}, \tag{3.9}
$$

$$
t_r \le t \le t_{r+1},\ r = 1, 2, ..., N-1.
$$

From the last equation of system (3.9) we obtain that

$$
b^q_q \int_{t_r}^t \tilde{\eta}^q_r(s) ds = -\int_{t_r}^t \tilde{g}^q(s) ds - \sum_{j=1}^m \int_{t_r}^t \tilde{c}^q_j(s) dw^j(s)
$$

Since it is the current velocity (the symmetric mean derivative) that corresponds to the physical velocity, from this equation we find $\tilde{\eta}^q_r(t)$ by applying the derivative D^w_S to both sides of the equation, see Remark 3.1. It is easy to see that applying the mean derivatives D^w and D^w_* (and therefore D^w_S) to the Riemann integrals on the left-hand and right-hand sides gives the same results for $\tilde{\eta}^q_r(t)$ and $g^q(t)$. Thus, using Lemma 1.4 we get that

$$
\eta^q_r(t) = -\frac{1}{b^q_q} \tilde{g}^q(t) - \frac{1}{b^q_q} \sum_{j=1}^m \tilde{c}^q_j(t) \frac{w^j}{2t}, \tag{3.10}
$$

$r = 1, 2, ..., N-1$. From the penultimate equation of system (3.9) we get that

$$
a^{q-1}_q \tilde{\eta}^q_r(t) - a^{q-1}_q \tilde{\eta}^q_r(t_r) = \int_{t_r}^t \left(b^{q-1}_{q-1} \tilde{\eta}^{q-1}(s) + b^{q-1}_q \tilde{\eta}^q(s) \right) ds +
$$

$$
+ \int_{t_r}^t \tilde{g}^{q-1}(s) ds + \sum_{j=1}^m \int_{t_r}^t \tilde{c}^{q-1}_j(s) dw^j(s)
$$

whence, having carried out the reasoning similarly to the above, using Lemma 1.5 we have

$$\tilde{\eta}_r^{q-1}(t) = -\frac{a_n^{q-1}}{b_{q-1}^{q-1}\cdot b_q^q}\cdot\frac{d\tilde{g}^q(t)}{dt} + \frac{b_q^{q-1}}{b_{q-1}^{q-1}\cdot b_q^q}\cdot\tilde{g}^q(t) - \frac{1}{b_{q-1}^{q-1}}\cdot\tilde{g}^{q-1}(t)+$$
$$+\frac{a_n^{q-1}}{b_{q-1}^{q-1}\cdot b_q^q}\cdot\sum_{j=1}^m\tilde{c}_j^q(t)\cdot\frac{w^j}{4t^2} - \frac{a_q^{q-1}}{b_{q-1}^{q-1}\cdot b_q^q}\cdot\sum_{j=1}^m\frac{w^j}{2t}\cdot\frac{d\tilde{c}_j^q(t)}{dt}+$$
$$+\frac{b_q^{q-1}}{b_{q-1}^{q-1}\cdot b_q^q}\cdot\sum_{j=1}^m\tilde{c}_j^q(t)\cdot\frac{w^j}{2t} - \frac{1}{b_{q-1}^{q-1}}\cdot\sum_{j=1}^m\tilde{c}_j^{q-1}(t)\cdot\frac{w^j}{2t} \qquad (3.11)$$

In exactly the same way, for $p \le i \le q-1$ we get the recurrence formula

$$D_S\left(\sum_{j=i+1}^q a_j^i\cdot\tilde{\eta}_r^j(t)\right) = \sum_{j=i}^q b_j^i\cdot\tilde{\eta}_r^j(t) + \tilde{g}^i(t) + \sum_{j=1}^m\tilde{c}_j^i(t)\cdot\frac{w^j}{2t} \qquad (3.12)$$

Note that for systems of form (3.9) definitioned on the intervals $[0,t_1]$ and $[t_N,T]$, we have for $0 < t \le t_1$ and $t_N \le t < T$ respectively, similar formulas for solutions. Moreover, the found processes satisfy conditions (3.8) in the case where the components of the random variable $\tilde{U}\tilde{\zeta}_r(\omega)$ corresponding to zero 1×1 blocks along the main diagonal in A are equal to zero, i.e. $\left((\tilde{U}\tilde{\zeta}_r(\omega))^j\right)_{j=p}^q = 0$.

So, in view of the above, for $0 < t < T$ we obtain the formulas for $\tilde{\eta}^i(t)$:

$$\tilde{\eta}^q(t) = -\frac{1}{b_q^q}\tilde{g}^q(t) - \frac{1}{b_q^q}\sum_{j=1}^m\tilde{c}_j^q(t)\frac{w^j}{2t}, \qquad (3.13)$$

$$D_S\left(\sum_{j=i+1}^q a_j^i\cdot\tilde{\eta}^j(t)\right) = \sum_{j=i}^q b_j^i\cdot\tilde{\eta}^j(t) + \tilde{g}^i(t) + \sum_{j=1}^m\tilde{c}_j^i(t)\cdot\frac{w^j}{2t}, \qquad (3.14)$$
$$p \le i \le q-1$$

For A strings with non-degenerate blocks of size 1×1, we obtain the equations

$$a_j^j\tilde{\eta}_r^j(t) + a_{j+1}^j\tilde{\eta}^{j+1}(t) + \ldots + a_q^j\tilde{\eta}^q(t) - a_j^j\tilde{\eta}^j(t_r) - a_{j+1}^j\tilde{\eta}^{j+1}(t_r) - \ldots - a_q^j\tilde{\eta}^q(t_r) =$$
$$= \int_{t_r}^t (b_j^j\tilde{\eta}_r^j(s) + b_{j+1}^j\tilde{\eta}^{j+1}(s) + \ldots + b_q^j\tilde{\eta}^q(s))ds+$$
$$+\int_{t_r}^t\tilde{g}^j(s)ds + \int_{t_r}^t\tilde{c}_1^j(s)dw^1(s) + \int_{t_r}^t\tilde{c}_2^j(s)dw^2(s) + ... + \int_{t_r}^t\tilde{c}_m^j(s)dw^m(s), \qquad (3.15)$$
$$t_r \le t \le t_{r+1},\ r = 1,\ldots,N-1.$$

For this type of equation, there is an analytical formula for solutions (see [13])

$$\tilde{\eta}_r^j(t) = e^{\frac{b_j^j}{a_j^j}(t-t_r)}\tilde{\eta}_r^j(t_r) + \frac{a_{j+1}^j}{a_j^j}e^{\frac{b_j^j}{a_j^j}(t-t_r)}\tilde{\eta}^{j+1}(t_r) + ... + \frac{a_q^j}{a_j^j}e^{\frac{b_j^j}{a_j^j}(t-t_r)}\tilde{\eta}^q(t_r)+$$

$$+\int_{t_r}^t e^{\frac{b_j^j}{a_j^j}(t-u)}\frac{\tilde{c}_1^j(u)}{a_j^j}dw_u^1 + \int_{t_r}^t e^{\frac{b_j^j}{a_j^j}(t-u)}\frac{\tilde{c}_2^j(u)}{a_j^j}dw_u^2 + ... + \int_{t_r}^t e^{\frac{b_j^j}{a_j^j}(t-u)}\frac{\tilde{c}_m^j(u)}{a_j^j}dw_u^m+$$

$$+\int_{t_r}^t e^{\frac{b_j^j}{a_j^j}(t-u)}[\frac{1}{a_j^j}\tilde{g}^j(u) + \frac{b_{j+1}^j}{a_j^j}\tilde{\eta}^{j+1}(u)+$$

$$+\ldots+\frac{b_q^j}{a_j^j}\tilde{\eta}^q(u) - \frac{b_j^j}{(a_j^j)^2}(a_{j+1}^j\tilde{\eta}^{j+1}(u) + \ldots + a_q^j\tilde{\eta}^q(u))]du-$$

$$-\frac{a_{j+1}^j}{a_j^j}\tilde{\eta}^{j+1} - \ldots - \frac{a_q^j}{a_j^j}\eta^q.$$

Note that for equations of form (3.15) given on the intervals $[0, t_1]$ and $[t_N, T]$, similar formulas for the solutions hold. Considering all $\tilde{\eta}_r^j(t)$, we get the expression for $\tilde{\eta}^j(t)$

$$\tilde{\eta}^j(t) = \sum_{r=1}^N e^{\frac{b_j^j}{a_j^j}(t-t_r)} \cdot \left(\frac{\tilde{u}_1^j\tilde{\zeta}_r^1}{a_j^j} + ... + \frac{\tilde{u}_n^j\tilde{\zeta}_r^n}{a_j^j}\right) \cdot \chi(t-t_r)+$$

$$+\int_0^t e^{\frac{b_j^j}{a_j^j}(t-u)}\frac{c_1^j(u)}{a_j^j}dw_u^1 + \int_0^t e^{\frac{b_j^j}{a_j^j}(t-u)}\frac{\tilde{c}_2^j(u)}{a_j^j}dw_u^2 + ... + \int_0^t e^{\frac{b_j^j}{a_j^j}(t-u)}\frac{\tilde{c}_m^j(u)}{a_j^j}dw_u^m+$$

$$+\int_0^t e^{\frac{b_j^j}{a_j^j}(t-u)}[\frac{1}{a_j^j}\tilde{g}^j(u) + \frac{b_{j+1}^j}{a_j^j}\tilde{\eta}^{j+1}(u)+$$

$$+\ldots+\frac{b_q^j}{a_j^j}\tilde{\eta}^q(u) - \frac{b_j^j}{(a_j^j)^2}(a_{j+1}^j\tilde{\eta}^{j+1}(u) + \ldots + a_q^j\tilde{\eta}^q(u))]du-$$

$$-\frac{a_{j+1}^j}{a_j^j}\tilde{\eta}^{j+1} - \ldots - \frac{a_q^j}{a_j^j}\tilde{\eta}^q. \tag{3.16}$$

For strings A with blocks of size 2×2, we obtain subsystems with the pair of equations

$$\begin{gathered}
a_i^i \tilde{\eta}_r^i(t) + a_{i+1}^i \tilde{\eta}_r^{i+1}(t) + a_{i+2}^i \tilde{\eta}^{i+2}(t) + \ldots + a_q^i \tilde{\eta}^q(t) - \\
- a_i^i \tilde{\eta}_r^i(t_r) - a_{i+1}^i \tilde{\eta}_r^{i+1}(t_r) - a_{i+2}^i \tilde{\eta}^{i+2}(t_r) - \ldots - a_q^i \tilde{\eta}^q(t_r) = \\
= \int_{t_r}^t (b_i^i \tilde{\eta}_r^i(s) + b_{i+1}^i \tilde{\eta}_r^{i+1}(s) + b_{i+2}^i \tilde{\eta}^{i+2}(s) + \ldots + b_q^i \tilde{\eta}^q(s)) ds + \\
+ \int_{t_r}^t \tilde{g}^i(s) ds + \int_{t_r}^t \tilde{c}_1^i(s) dw^1(s) + \int_{t_r}^t \tilde{c}_2^i(s) dw^2(s) + ... + \int_{t_r}^t \tilde{c}_m^i(s) dw^m(s), \\
a_i^{i+1} \tilde{\eta}_r^i(t) + a_{i+1}^{i+1} \tilde{\eta}_r^{i+1}(t) + a_{i+2}^{i+1} \tilde{\eta}^{i+2}(t) + \ldots + a_q^{i+1} \tilde{\eta}^q(t) - \\
- a_i^{i+1} \tilde{\eta}_r^i(t_r) - a_{i+1}^{i+1} \tilde{\eta}_r^{i+1}(t_r) - a_{i+2}^{i+1} \tilde{\eta}^{i+2}(t_r) - \ldots - a_q^{i+1} \tilde{\eta}^q(t_r) = \\
= \int_{t_r}^t (b_{i+1}^{i+1} \tilde{\eta}_r^{i+1}(s) + b_{i+2}^{i+1} \tilde{\eta}^{i+2}(s) + \ldots + b_q^{i+1} \tilde{\eta}^q(s)) ds + \\
+ \int_{t_r}^t \tilde{g}^{i+1}(s) ds + \int_{t_r}^t \tilde{c}_1^{i+1}(s) dw^1(s) + \int_{t_r}^t \tilde{c}_2^{i+1}(s) dw^2(s) + ... + \int_{t_r}^t \tilde{c}_m^{i+1}(s) dw^m(s), \\
t_r \le t \le t_{r+1}, \;\; r = 1, \ldots, N-1.
\end{gathered}$$

In matrix form in the new notation, this subsystem of equations takes the form

$$\begin{gathered}
\bar{\tilde{\eta}}_r(t) - \bar{\tilde{\eta}}_r(t_r) + \vartheta(t) - \vartheta(t_r) = \\
= \int_{t_r}^t K \bar{\tilde{\eta}}_r(s) ds + \int_{t_r}^t \theta(s) ds + \int_{t_r}^t \bar{\tilde{g}}(s) ds + \int_{t_r}^t \Lambda \Theta(s) dw(s)
\end{gathered} \tag{3.17}$$

where

$\bar{\tilde{\eta}}_r = \begin{pmatrix} \tilde{\eta}_r^i \\ \tilde{\eta}_r^{i+1} \end{pmatrix}$, $\Xi = \begin{pmatrix} b_i^i & b_{i+1}^i \\ 0 & b_{i+1}^{i+1} \end{pmatrix}$, $\Lambda = \begin{pmatrix} a_i^i & a_{i+1}^i \\ a_i^{i+1} & a_{i+1}^{i+1} \end{pmatrix}^{-1}$,

$\vartheta(t) = \Lambda \begin{pmatrix} a_{i+2}^i & \cdots & a_q^i \\ a_{i+2}^{i+1} & \cdots & a_q^{i+1} \end{pmatrix} \left(\tilde{\eta}^{i+2} \ldots \tilde{\eta}^q \right)^T$,

$\theta(t) = \Lambda \begin{pmatrix} b_{i+2}^i & \cdots & b_q^i \\ b_{i+2}^{i+1} & \cdots & b_q^{i+1} \end{pmatrix} \left(\tilde{\eta}^{i+2} \ldots \tilde{\eta}^q \right)^T$,

$\bar{\tilde{g}} = \Lambda \begin{pmatrix} \tilde{g}^i \\ \tilde{g}^{i+1} \end{pmatrix}$, $K = \Lambda \Xi$, $\Theta(t) = \begin{pmatrix} \tilde{c}_1^i(t) & \cdots & \tilde{c}_m^i(t) \\ \tilde{c}_1^{i+1}(t) & \cdots & \tilde{c}_m^{i+1}(t) \end{pmatrix}$

Fjr this subsystem there is the analytical formula for solutions (see [13])

$$\begin{gathered}
\bar{\tilde{\eta}}_r(t) = e^{K(t-t_r)} \bar{\tilde{\eta}}_r(t_r) + e^{K(t-t_r)} \vartheta(t_r) + \int_{t_r}^t e^{K(t-\tau)} \Lambda \Theta(\tau) dw_\tau + \\
+ \int_{t_r}^t e^{K(t-\tau)} \left(\theta(\tau) + \bar{\tilde{g}}(\tau) - K \vartheta(\tau) \right) d\tau - \vartheta(t).
\end{gathered}$$

Note that for equations of (3.17) form, given on the intervals $[0, t_1]$ $[t_N, T]$, analogous formulae exist as well. Taking into account all $\bar{\tilde{\eta}}_r(t)$, we obtain the expression for $\bar{\bar{\eta}}(t)$

$$\bar{\bar{\eta}}(t)=\sum_{r=1}^{N}e^{K(t-t_r)}\Lambda\left((\tilde{U}\tilde{\zeta}_r(\omega))^j\right)_{j=i}^{i+1}\cdot\chi(t-t_r)+$$

$$+\int_0^t e^{K(t-\tau)}\left(\theta(\tau)+\bar{\bar{g}}(\tau)-K\vartheta(\tau)\right)d\tau+\int_0^t e^{K(t-\tau)}\Lambda\Theta(\tau)dw(\tau)-\vartheta(t), \quad (3.18)$$

where $\left((\tilde{U}\tilde{\zeta}_r(\omega))^j\right)_{j=i}^{i+1}$ is a 2-dimensional vector, made up from the i-th and the $(i+1)$-th coordinates of vector $\tilde{U}\tilde{\zeta}_r(\omega)$.

Now investogate the subsystem, corresponding to all singular cells of L_ε type

$$F\check{\eta}_r(t)-F\check{\eta}_r(t_r)=\int_{t_r}^{t}K\check{\eta}_r(s)ds+\int_{t_r}^{t}\check{g}(s)ds+\int_{t_r}^{t}\check{C}(s)dw(s), \quad (3.19)$$
$$t_r\le t\le t_{r+1},\ \ r=1,2,...,N-1.$$

We consider this system as the equation, corresponding to one singular cell of size $l\times(l+1)$, that is located in the left upper corner of $\lambda F+K$. In this case the coordinate expression of the first equation of (3.19) takes the form

$$\begin{pmatrix}0&1&0&\dots&0&0\\0&0&1&\dots&0&0\\\vdots&\vdots&\vdots&\dots&\vdots&\vdots\\0&0&0&\dots&1&0\\0&0&0&\dots&0&1\end{pmatrix}\begin{pmatrix}\check{\eta}_r^1(t)\\\check{\eta}_r^2(t)\\\vdots\\\check{\eta}_r^l(t)\\\check{\eta}_r^{l+1}(t)\end{pmatrix}-\begin{pmatrix}0&1&0&\dots&0&0\\0&0&1&\dots&0&0\\\vdots&\vdots&\vdots&\dots&\vdots&\vdots\\0&0&0&\dots&1&0\\0&0&0&\dots&0&1\end{pmatrix}\begin{pmatrix}\check{\eta}_r^1(t_r)\\\check{\eta}_r^2(t_r)\\\vdots\\\check{\eta}_r^l(t_r)\\\check{\eta}_r^{l+1}(t_r)\end{pmatrix}=$$

$$=\int_{t_r}^{t}\begin{pmatrix}1&0&\dots&0&0&0\\0&1&\dots&0&0&0\\\vdots&\vdots&\dots&\vdots&\vdots&\\0&0&\dots&1&0&0\\0&0&\dots&0&1&0\end{pmatrix}\begin{pmatrix}\check{\eta}_r^1(s)\\\check{\eta}_r^2(s)\\\vdots\\\check{\eta}_r^l(s)\\\check{\eta}_r^{l+1}(s)\end{pmatrix}ds+$$

$$+\int_{t_k}^{t}\begin{pmatrix}\check{g}^1(s)\\\check{g}^2(s)\\\vdots\\\check{g}^{l-1}(s)\\\check{g}^l(s)\end{pmatrix}ds+\int_{t_r}^{t}\begin{pmatrix}\check{c}_1^1(s)&\check{c}_2^1(s)&\dots&\check{c}_m^1(s)\\\check{c}_1^2(s)&\check{c}_2^2(t)&\dots&\check{c}_m^2(s)\\\vdots&\vdots&\dots&\vdots\\\check{c}_1^{l-1}(s)&\check{c}_2^{l-1}(s)&\dots&\check{c}_m^{l-1}(s)\\\check{c}_1^l(s)&\check{c}_2^l(s)&\dots&\check{c}_m^l(s)\end{pmatrix}d\begin{pmatrix}w^1(s)\\w^2(s)\\\vdots\\w^{m-1}(s)\\w^m(s)\end{pmatrix},$$

..,

$$\check{\eta}_r^2(t)-\check{\eta}_r^2(t_r)=\int_{t_r}^{t}(\check{\eta}_r^1(s)+\check{g}^1(s))ds+\sum_{j=1}^{m}\int_{t_r}^{t}\check{c}_j^1(s)dw^j(s),$$

$$\check{\eta}_r^3(t)-\check{\eta}_r^3(t_r)=\int_{t_r}^{t}(\check{\eta}_r^2(s)+\check{g}^2(s))ds+\sum_{j=1}^{m}\int_{t_r}^{t}\check{c}_j^2(s)dw^j(s),$$

$$\check{\eta}_r^{l+1}(t)-\check{\eta}_r^{l+1}(t_r)=\int_{t_r}^{t}(\check{\eta}_r^l(s)+\check{g}^l(s))ds+\sum_{j=1}^{m}\int_{t_r}^{t}\check{c}_j^l(s)dw^j(s).$$

This means that we can take as $\breve{\eta}_r^{l+1}$ any random process that satisfies limitations (3.8) and for which you can find a symmetric derivative of order l, and then recursively obtain all other components of the process $\breve{\eta}_r$. This is because the number of unknowns in the system is one greater than the number of equations. Similarly to the case of system (3.9), the following formulae hold:

$$\breve{\eta}_r^l(t) = D_S^w \breve{\eta}_r^{l+1} - \sum_{j=1}^m \breve{c}_j^l \frac{w^j}{2t} - \breve{g}^l(t); \tag{3.20}$$

$$\breve{\eta}_r^l(t) = \int_{t_r}^t (\breve{\eta}_r^{l-1}(s) + \breve{g}^{l-1}(s))ds + \sum_{j=1}^m \int_{t_r}^t \breve{c}_j^{l-1}(s)dw^j(s);$$

$$\breve{\eta}_r^{l-1}(t) = D_S^2 \breve{\eta}_r^{l+1} - \sum_{j=1}^m \frac{d\breve{c}_j^l}{dt}\frac{w^j}{2t} + \sum_{j=1}^m \breve{c}_j^l \frac{w^j}{4t^2} - \sum_{j=1}^m \breve{c}_j^{l-1}\frac{w^j}{2t} - \frac{d\breve{g}^l}{dt} - \breve{g}^{l-1}, \tag{3.21}$$

Similarly, for $1 \le i \le l$ we get

$$\breve{\eta}_r^i(t) = D_S^w \breve{\eta}_r^{i+1} - D_S^w \sum_{j=1}^m \int_{t_r}^t \breve{c}_j^i(s)dw^j(s) - \breve{g}^i(t). \tag{3.22}$$

Using Lemmas 1.4 and 1.5 by formula (3.22) we obtain the explicit expression for any $\breve{\eta}_r^i(t)$, $r = 1, 2, ..., N-1$:

$$\begin{aligned} \breve{\eta}_r^i = -\sum_{k=i}^{l-1} \frac{d^{k-i+1}\breve{g}^{k+1}}{dt^{k-i+1}} - \breve{g}^i - \sum_{j=1}^m \frac{d\breve{c}_j^{i+1}}{dt}\frac{w^j}{2t} + \\ + \sum_{j=1}^m \breve{c}_j^{i+1}\frac{w^j}{4t^2} - \sum_{s=i+1}^{l-1}\sum_{j=1}^m \{ \frac{d^{s-i+1}\breve{c}_j^{s+1}}{dt^{s-i+1}}\frac{w^j}{2t} + \\ + \breve{c}_j^{s+1}(-1)^{s-i+1}\frac{\prod_{r=1}^{s-i+1}(2r-1)}{2^{s-i+2}}\frac{w^j(t)}{t^{s-i+2}} + \\ + \sum_{k=1}^{s-i} C_{s-i+1}^k \frac{d^{s-i+1-k}\breve{c}_j^{s+1}}{dt^{s-i+1-k}}(-1)^k \frac{\prod_{r=1}^k (2r-1)}{2^{k+1}}\frac{w^j(t)}{t^{k+1}} \} - \sum_{j=1}^m \breve{c}_j^i \frac{w^j}{2t} + D_S^{l+1-i}\breve{\eta}^{l+1}, \end{aligned} \tag{3.23}$$

$$1 \le i \le l-2, \;\; C_{n_1}^{k_1} = \frac{n_1!}{k_1!(n_1-k_1)!}$$

It is easy to see that for equations of form (3.19) definitioned on the intervals $[0, t_1]$ and $[t_N, T]$, for $0 < t \le t_1$ and $t_N \le t < T$ respectively similar formulas for solutions hold. Also, as $\breve{\eta}_r^{l+1}$ we take an arbitrary random process satisfying the limitations of (3.8) and for which we can find the symmetric derivative of order l. Moreover, the found processes satisfy limitations of (3.8) where the components of the random variable $\breve{U}\tilde{\zeta}_r(\omega)$ corresponding to the singular cell under consideration are equal to zero. Thus, to find $\check{\eta}(t)$ for $0 < t < T$, the formulae

$$\check{\eta}^l(t) = D_S^w \check{\eta}_r^{l+1} - \sum_{j=1}^m \check{c}_j^l \frac{w^j}{2t} - \check{g}^l(t); \tag{3.24}$$

$$\check{\eta}^{l-1}(t) = D_S^2 \check{\eta}_r^{l+1} - \sum_{j=1}^m \frac{d\check{c}_j^l}{dt} \frac{w^j}{2t} + \sum_{j=1}^m \check{c}_j^l \frac{w^j}{4t^2} - \sum_{j=1}^m \check{c}_j^{l-1} \frac{w^j}{2t} - \frac{d\check{g}^l}{dt} - \check{g}^{l-1}, \tag{3.25}$$

$$\begin{gathered}\check{\eta}^i = -\sum_{k=i}^{l-1} \frac{d^{k-i+1}\check{g}^{k+1}}{dt^{k-i+1}} - \check{g}^i - \sum_{j=1}^m \frac{d\check{c}_j^{i+1}}{dt} \frac{w^j}{2t} + \\ + \sum_{j=1}^m \check{c}_j^{i+1} \frac{w^j}{4t^2} - \sum_{s=i+1}^{l-1} \sum_{j=1}^m \{ \frac{d^{s-i+1}\check{c}_j^{s+1}}{dt^{s-i+1}} \frac{w^j}{2t} + \\ + \check{c}_j^{s+1} (-1)^{s-i+1} \frac{\prod_{r=1}^{s-i+1}(2r-1)}{2^{s-i+2}} \frac{w^j(t)}{t^{s-i+2}} + \\ + \sum_{k=1}^{s-i} C_{s-i+1}^k \frac{d^{s-i+1-k}\check{c}_j^{s+1}}{dt^{s-i+1-k}} (-1)^k \frac{\prod_{r=1}^k (2r-1)}{2^{k+1}} \frac{w^j(t)}{t^{k+1}} \} - \sum_{j=1}^m \check{c}_j^i \frac{w^j}{2t} + D_S^{l+1-i} \check{\eta}^{l+1},\end{gathered} \tag{3.26}$$

$$1 \le i \le l-2$$

take place.

Now consider the subsystem corresponding to Kronecker singular cells of type L_ν^T

$$G\hat{\eta}_r(t) - G\hat{\eta}_r(t_r) = \int_{t_r}^t H\hat{\eta}_r(s)ds + \int_{t_r}^t \hat{g}(s)ds + \int_{t_r}^t \hat{C}(s)dw(s), \tag{3.27}$$
$$t_r \le t \le t_{r+1}, \quad r = 1, 2, ..., N-1.$$

We will also investigate this system using an example of one equation corresponding to one singular cell of size $(d+1)\times d$ located in the upper left corner of $\lambda G+H$. In coordinate form, the first equation (3.27) has the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\eta}_r^1(t) \\ \hat{\eta}_r^2(t) \\ \vdots \\ \hat{\eta}_r^{d-1}(t) \\ \hat{\eta}_r^d(t) \end{pmatrix} - \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\eta}_r^1(t_r) \\ \hat{\eta}_r^2(t_r) \\ \vdots \\ \hat{\eta}_r^{d-1}(t_r) \\ \hat{\eta}_r^d(t_r) \end{pmatrix} =$$

$$\int_{t_r}^t \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\eta}_r^1(s) \\ \hat{\eta}_r^2(s) \\ \vdots \\ \hat{\eta}_r^{d-1}(s) \\ \hat{\eta}_r^d(s) \end{pmatrix} ds + \int_{t_r}^t \begin{pmatrix} \hat{g}^1(s) \\ \hat{g}^2(s) \\ \vdots \\ \hat{g}^d(s) \\ \hat{g}^{d+1}(s) \end{pmatrix} ds +$$

$$+\int_{t_r}^{t}\begin{pmatrix}\hat{c}_1^1(s) & \hat{c}_2^1(s) & \dots & \hat{c}_m^1(s)\\ \hat{c}_1^2(s) & \hat{c}_2^2(t) & \dots & \hat{c}_m^2(s)\\ \vdots & \vdots & \dots & \vdots\\ \hat{c}_1^d(s) & \hat{c}_2^d(s) & \dots & \hat{c}_m^d(s)\\ \hat{c}_1^{d+1}(s) & \hat{c}_2^{d+1}(s) & \dots & \hat{c}_m^{d+1}(s)\end{pmatrix} d\begin{pmatrix}w^1(s)\\ w^2(s)\\ \vdots\\ w^{m-1}(s)\\ w^m(s)\end{pmatrix}.$$

or

$$0=\int_{t_r}^{t}(\hat{\eta}_r^1(s)+\hat{g}^1(s))ds+\sum_{j=1}^{m}\int_{t_r}^{t}\hat{c}_j^1(s)dw^j(s),$$

$$\hat{\eta}_r^1(t)-\hat{\eta}_r^1(t_r)=\int_{t_r}^{t}(\hat{\eta}_r^2(s)+\hat{g}^2(s))ds+\sum_{j=1}^{m}\int_{t_r}^{t}\hat{c}_j^2(s)dw^j(s),$$

$$\hat{\eta}_r^{d-1}(t)-\hat{\eta}_r^{d-1}(t_r)=\int_{t_r}^{t}(\hat{\eta}_r^d(s)+\hat{g}^d(s))ds+\sum_{j=1}^{m}\int_{t_r}^{t}\hat{c}_j^d(s)dw^j(s),$$

$$\hat{\eta}_r^d(t)-\hat{\eta}_r^d(t_r)=\int_{t_r}^{t}\hat{g}^{d+1}(s)ds+\sum_{j=1}^{m}\int_{t_r}^{t}\hat{c}_j^{d+1}(s)dw^j(s).$$

Starting from the first equation, we successively obtain

$$\hat{\eta}_r^1(t)=-\hat{g}^1(t)-D_S^w\sum_{j=1}^{m}\int_{t_r}^{t}\hat{c}_j^1(s)dw^j(s)=-\hat{g}^1-\sum_{j=1}^{m}\hat{c}_j^1\frac{w^j(t)}{2t}, \tag{3.28}$$

$$\hat{\eta}_r^2(t)=-\hat{g}^2(t)+D_S^w\hat{\eta}_r^1-D_S^w\sum_{j=1}^{m}\int_{t_r}^{t}\hat{c}_j^2(s)dw^j(s)=$$

$$=-\hat{g}^2(t)-\frac{d\hat{g}^1(t)}{dt}-\sum_{j=1}^{m}\frac{d\hat{c}_j^1}{dt}\frac{w^j}{2t}+\sum_{j=1}^{m}\hat{c}_j^1\frac{w^j(t)}{4t^2}-\sum_{j=1}^{m}\hat{c}_j^2\frac{w^j(t)}{2t}, \tag{3.29}$$

As before, the recursion formula holds for $2\le i\le d$

$$\hat{\eta}_r^i(t)=-\hat{g}^i(t)+D_S^w\hat{\eta}_r^{i-1}-D_S^w\sum_{j=1}^{m}\int_{t_r}^{t}\hat{c}_j^i(s)dw^j(s).$$

Then

$$\hat{\eta}_r^i(t)=-\hat{g}^i(t)-\frac{d\hat{g}^{i-1}(t)}{dt}-...-\frac{d^{i-1}\hat{g}^1(t)}{dt^{i-1}}-$$

$$-D_S\sum_{j=1}^{m}\int_{t_r}^{t}\hat{c}_j^i(s)dw^j(s)-D_S^2\sum_{j=1}^{m}\int_{t_r}^{t}\hat{c}_j^{i-1}(s)dw^j(s)-...-D_S^i\sum_{j=1}^{m}\int_{t_r}^{t}\hat{c}_j^1(s)dw^j(s)=$$

$$=-\sum_{j=1}^{m}\hat{c}_j^i\frac{w^j}{2t}-\sum_{j=1}^{m}\sum_{p=1}^{i-1}D_S^p(\hat{c}_j^{i-p}\frac{w^j}{2t}).$$

■

Hence, applying Lemmas 1.4 and 1.5, for $i \geq 3$ we obtain the following formulae

$$\hat{\eta}_r^i(t) = -\sum_{p=1}^{i-1} \frac{d^{i-p}\hat{g}^p}{dt^{i-p}} - \hat{g}^i - \sum_{j=1}^{m} \hat{c}_j^i \frac{w^j}{2t} - \sum_{j=1}^{m} \frac{d\hat{c}_j^{i-1}}{dt}\frac{w^j}{2t} + \\ + \sum_{j=1}^{m} \hat{c}_j^{i-1}\frac{w^j}{4t^2} - \sum_{j=1}^{m}\sum_{p=2}^{i-1}\{\frac{d^p\hat{c}_j^{i-p}}{dt^p}\frac{w^j}{2t} + \\ + \sum_{l=1}^{p-1} C_p^l \frac{d^{p-l}\hat{c}_j^{i-p}}{dt^{p-l}}(-1)^l \frac{\prod_{r=1}^{l}(2r-1)}{2^{l+1}}\frac{w^j}{t^{l+1}} + \hat{c}_j^{i-p}(-1)^p\frac{\prod_{r=1}^{p}(2r-1)}{2^{p+1}}\frac{w^j}{t^{p+1}}\}, \tag{3.30}$$

as well as the condition for concordance

$$\hat{\eta}_r^d(t_r) + \int_{t_r}^{t} \hat{g}^{d+1}(s)ds + \sum_{j=1}^{m}\int_{t_r}^{t} \hat{c}_j^{d+1}(s)dw^j(s) = \hat{\eta}_r^d(t).$$

If the components z^i and w^i do not satisfy this condition, then the system has no solutions. Here the number of equations is one more than the number of unknowns, i.e. This subsystem is overridden.

It is easy to see that for equations of form (3.27) definitioned on the intervals $[0, t_1]$ and $[t_N, T]$, for $0 < t \leq t_1$ and $t_N \leq t < T$ respectively, similar concordance conditions and formulae for solutions hold. Moreover, the found processes satisfy conditions (3.8) if the components of the random variable $\hat{U}\tilde{\zeta}_r(\omega)$ corresponding to the singular cell under consideration are equal to zero.

Therefore, for $0 < t < T$, under the concordance conditions

$$\int_0^t \hat{g}^{d+1}(s)ds + \sum_{j=1}^{m}\int_0^t \hat{c}_j^{d+1}(s)dw^j(s) = \hat{\eta}^d(t) \tag{3.31}$$

relations for determining the components $\hat{\eta}^i(t)$

$$\hat{\eta}^1 = -\hat{g}^1 - \sum_{j=1}^{m} \hat{c}_j^1 \frac{w^j(t)}{2t}, \tag{3.32}$$

$$\hat{\eta}^2(t) = -\hat{g}^2(t) - \frac{d\hat{g}^1(t)}{dt} - \sum_{j=1}^{m}\frac{d\hat{c}_j^1}{dt}\frac{w^j}{2t} + \sum_{j=1}^{m}\hat{c}_j^1\frac{w^j(t)}{4t^2} - \sum_{j=1}^{m}\hat{c}_j^2\frac{w^j(t)}{2t}, \tag{3.33}$$

$$\hat{\eta}^i(t) = -\sum_{p=1}^{i-1} \frac{d^{i-p}\hat{g}^p}{dt^{i-p}} - \hat{g}^i - \sum_{j=1}^{m} \hat{c}_j^i \frac{w^j}{2t} - \sum_{j=1}^{m} \frac{d\hat{c}_j^{i-1}}{dt}\frac{w^j}{2t} + \\ + \sum_{j=1}^{m} \hat{c}_j^{i-1}\frac{w^j}{4t^2} - \sum_{j=1}^{m}\sum_{p=2}^{i-1}\{\frac{d^p\hat{c}_j^{i-p}}{dt^p}\frac{w^j}{2t} + \\ + \sum_{l=1}^{p-1} C_p^l \frac{d^{p-l}\hat{c}_j^{i-p}}{dt^{p-l}}(-1)^l \frac{\prod_{r=1}^{l}(2r-1)}{2^{l+1}}\frac{w^j}{t^{l+1}} + \hat{c}_j^{i-p}(-1)^p\frac{\prod_{r=1}^{p}(2r-1)}{2^{p+1}}\frac{w^j}{t^{p+1}}\}, \tag{3.34}$$

hold.

Now we turn to the question of zero initial conditions for the solutions of systems (3.9), (3.19), (3.27) (for $r = 0$). Taking into account the definitionition of symmetric mean derivatives, it is easy to see that they are well-definitioned only on open time intervals, since their construction uses both time increments to the right and to the left. Then from formulae (3.10), (3.11), (3.12), (3.20), (3.21), (3.23), (3.28), (3.29) and (3.30) one can easily see that the solutions $\eta^l(t)$ are described as sums, in which each term contains a factor of the form $\frac{w^j(t)}{t^k}$, $k \geq 1$. Hence, the solutions tend to infinity as $t \to 0$, i.e. the values ??of the solutions for $t = 0$ do not exist. One of the possibilities for resolving this situation (as in [11]) is as follows. We specify an arbitrarily small time instant $t_0 \in (0, T)$ and definitione the function $t_0(t)$ by the formula

$$t_0(t) = \begin{cases} t_0, \text{ if } 0 \leq t \leq t_0; \\ t, \text{ if } t_0 \leq t. \end{cases} \tag{3.35}$$

In formulae (3.10), (3.11), in processes obtained recursively using the formulae (3.10), (3.12), as well as in the formulae (3.20), (3.21), (3.23), (3.28), (3.29) and (3.30), and therefore in processes obtained using (3.13), (3.14), as well as in the formulas (3.24), (3.25), (3.26), (3.32), eqrefs22 and (3.34), We replace the elements $\frac{w^j(t)}{t^k}$ by $\frac{w^j(t)}{(t_0(t))^k}$. The resulting processes at time $t = 0$ take zero values, however those processes become solutions only for $t_0 \leq t < T$. Note that for two different instants of time $t_0^{(1)}$ and $t_0^{(2)}$ for $t \geq \max(t_0^{(1)}, t_0^{(2)})$ values of the corresponding processes a.s. coincide.

Thus, summing up the above, we have proved the following statement

Theorem 3.2. *Let $\tilde{M} + \lambda\tilde{L}$ be a singular matrix pencil of size $n \times m$, in which rows and columns are not connected by linear dependencies with constant coefficients, $\tilde{Q}$ - - $n \times n$ -matrix, $f(t)$ - fairly smooth n -dimensional vector function, $0 \leq t \leq T$; let $0 < t_1 < \cdots < t_N < T$; P_L and P_R are non-degenerate matrices of sizes $n \times n$ and $m \times m$ respectively, leading the sheaf $\lambda\tilde{L} + \tilde{M}$ to the Kronecker canonical form (t .e. to quasidiagonal form), $L = P_L\tilde{L}P_R$ and $M = P_L\tilde{M}P_R$, $U = P_L\tilde{Q}$; let $\tilde{\zeta}_r(\omega)$, $r = 1, 2, \ldots, N$ be random variables with values ??in R^n such that the components of the random variable $\tilde{U}\tilde{\zeta}_r(\omega)$, corresponding to the degenerate blocks 1×1 of the regular component, reduced to the Schur form, the pencil $\lambda L + M$, and the singular cells along the main diagonal in L, are equal to zero, i.e. $\check{U}\tilde{\zeta}_r(\omega) = O$, $\hat{U}\tilde{\zeta}_r($ omega$) = O$; let $\zeta(t, \omega) = \sum_{r=1}^{N} \tilde{\zeta}_r(\omega)\chi(t - t_r)$, where χ - Heaviside function equal to zero for negative values ??of the argument and one for non-negative. Then:*

1) the equation (3.1) is transformed to the canonical equation (3.3), which decomposes into equations and subsystems of equations;

2) for the subsystem corresponding to the degenerate blocks 1×1 of the regular component of the pencil $\lambda L + M$, recurrence formulas for the solutions (3.13), (3.14);

3) for the equations corresponding to the non-degenerate blocks 1×1 of the regular component of the pencil $\lambda L + M$, there is a formula for solutions of the form (3.16);

4) for equations corresponding to non-degenerate blocks 2×2 *of the regular component of the pencil* $\lambda L + M$*, there is a formula for solutions of the form* (3.18)*;*
5) for subsystems corresponding to singular cells L_ε *of* $M + \lambda L$ *of size* $l \times (l+1)$*, for* $0 < t < T$*, formulas for finding solutions of the form* (3.24)*,* (3.25) *and* (3.26)*;*
6) for subsystems corresponding to the singular cells L_ν^T *from* $M + \lambda L$ *of size* $(d+1) \times d$*, for* $0 < t < T$*, they occur under the conditions matching* (3.31) *relations for finding solutions of the form* (3.32)*,* (3.33) *and* (3.34)*;*
7) fixing an arbitrarily small time instant $t_0 > 0$*, in the denominators of processes satisfying the relations given in clauses 2), 5) -6), we replace* t *with* $t_0(t)$ *by the formula* (3.35) *and we get processes that for* $t = 0$ *take zero values, but become solutions only for* $t_0 \leq t < T$*.*

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