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CONVEX RISK MEASURES FOR REFLECTED BACKWARD SDES WITH OPTIONAL BARRIER

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Abstract. Convex risk measures were introduced in order to quantify the riskiness of financial positions and to provide a criterion to determine whether the risk was acceptable or not. The objective of this paper is to study dynamic risk measures by means of reflected backward stochastic differential equations. We prove the existence and uniqueness of such equations where the reflecting barrier has a regulated trajectories and the coefficient is assumed to be stochastic Lipschitz.

1. Introduction

It is well known that backward stochastic differential equations (BSDEs in short) were introduced, in the linear case, by Bismut [6]. The non-linear case was studied by Pardoux and Peng [24]. Due to their significant applications in finance [10, 11], stochastic control and stochastic games [6] and partial differential equations [25], the theory of BSDEs have gained a lot of interest since 1990. Subsequently, it was discovered that BSDEs could also be used to represent risk measures. Briefly, we recall the definition and some key properties of the convex risk measures introduced by Föllmer and Schied [12]; A convex risk measure is a functional $\rho : \mathrm{L}^p(\mathcal{F}_T) \longrightarrow \mathbb{R}$ ($p \geq 2$) that satisfies the following properties (in the sense of Föllmer and Schied):

- $\rho(0) = 0$ (Normalization).
- $\rho(\xi + \alpha) = \rho(\xi) - \alpha$ for all $\xi \in \mathrm{L}^p(\mathcal{F}_T)$ and all $\alpha \in \mathbb{R}$ (Translation invariance).
- ρ is nonincreasing with respect to ξ (Monotonicity).
- $\rho(\lambda\xi^1 + (1-\lambda)\xi^2) \leq \lambda\rho(\xi^1) + (1-\lambda)\rho(\xi^2)$ for all $\lambda \in [0,1]$ and all $\xi^1, \xi^2 \in \mathrm{L}^p(\mathcal{F}_T)$ (Convexity).

Many efforts have been made to construct the risk measures from the solutions to BSDEs (see for example [5, 26, 28, 30]). You can also see the recent work of Agram [1] when the author studies dynamic risk measures by means of backward stochastic Volterra integral equations.

As a variant of BSDEs, El Karoui et al. [8] was introduced the notion of reflected BSDEs (RBSDEs in short), which is a BSDEs when the first component of the solution is constrained to remain greater than or equal a given process called barrier (or obstacle). In [8], the authors proved the existence and uniqueness of

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the solution under Lipschitz condition on the coefficient, a continuous barrier and square integrability on the data in the Brownian setting. The case of discontinuous barrier has been studied by several works, we refer for example to [15, 16, 17]. In all these extensions, the right continuity condition of the barrier is made. In 2017, Grigorova et al. [13] present a new extension of RBSDEs when the barrier is not necessarily right-continuous. In this work, the authors gave a new computation on the existence and uniqueness proof under the Lipschitz condition on the coefficient. As a generalization of the work [13], Baadi and Ouknine [3, 4] considered the case of a general filtration. Further, some works have investigated this new extension, especially [2, 14, 18]. As application, Quenez and Selum [29] studied the optimal stopping problem for dynamic risk measures represented by BSDEs with jumps and its relation with RBSDEs. Recently, Marzougue and El Otmani [22] discussed RBSDEs with right upper-semicontinuous (r.u.s.c in short) barrier under the so-called stochastic Lipschitz coefficient introduced by El Karoui and Huang [9] (you can see also the recent works [20, 21, 23]).

We mainly consider in this paper a further extension of the theory of RBSDEs to the case where the barrier is an optional process (r.u.s.c) and the coefficient is stochastic Lipschitz in a general filtration. We adopt the solution of RBSDEs to construct a convex risk measures for the general position process $(\xi_t)_{t\leq T}$.

The paper is structured as follows: Section 2 is devoted to give some notations and assumptions needed in this paper, and we also define our RBSDEs. In Section 3, we focus on providing the existence and uniqueness of solution to our RBSDEs with a stochastic Lipschitz coefficient and a r.u.s.c barrier ξ. In Section 4, we study convex risk measures by means of RBSDEs.

2. Preliminaries

Let $T > 0$ real number and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq T}, \mathbb{P})$ be a filtered probability space where $(\mathcal{F}_t)_{t\leq T}$ is a complete, right continuous and maybe not quasi-left continuous filtration. We will denote by $|.|$ the Euclidian norm on $\mathbb{R}^d$, $\mathcal{T}_{[t,T]}$ the set of all stopping times τ such that $\tau \in [t, T]$, $\mathcal{M}$ the set of all càdlàg martingales and $\mathcal{P}$ (resp. $\mathcal{B}(\mathbb{R}^d)$) be the predictable (resp. Borelian) σ-algebra on $\Omega \times [0, T]$ (resp. on $\mathbb{R}^d$).

Let us introduce the following spaces:

- $\mathcal{M}^2$: the subspace of $\mathcal{M}$ of all real-valued càdlàg martingales $(M_t)_{t\leq T}$ such that

$$\|M\|^2_{\mathcal{M}^2} = \mathbb{E}\int_0^T d[M,M]_t = \mathbb{E}[M,M]_T < +\infty.$$

- $\mathcal{S}^2$: the space of all real-valued and optional processes $(K_t)_{t\leq T}$ such that

$$\|K\|^2_{\mathcal{S}^2} = \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} |K_\tau|^2\right] < +\infty.$$

Throughout this paper, β is a strictly positive real number and $(a_t)_{t\leq T}$ is a non-negative $\mathcal{F}_t$-adapted process. We define an increasing continuous process $(A_t)_{t\leq T}$

by

$$A_t := \int_0^t a_s^2 ds \qquad \forall t \leq T,$$

and let us introduce the following new spaces:

- $\mathcal{S}_\beta^2$: the space of all real-valued and optional processes $(Y_t)_{t\leq T}$ such that

$$\|Y\|^2_{\mathcal{S}^2_\beta} = \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A_\tau}|Y_\tau|^2\right] < +\infty.$$

- $\mathcal{S}_\beta^{2,a}$: the space of all real-valued and optional processes $(Y_t)_{t\leq T}$ such that

$$\|Y\|^2_{\mathcal{S}^{2,a}_\beta} = \mathbb{E}\int_0^T e^{\beta A_t}|a_t Y_t|^2 dt < +\infty.$$

- $\mathcal{M}_\beta^2$: the subspace of $\mathcal{M}^2$ of all càdlàg martingales $(M_t)_{t\leq T}$ such that

$$\|M\|^2_{\mathcal{M}^2_\beta} = \mathbb{E}\int_0^T e^{\beta A_t} d[M,M]_t < +\infty.$$

- $\mathfrak{B}_\beta^2 := \mathcal{S}_\beta^2 \cap \mathcal{S}_\beta^{2,a}$: the Banach space of the processes endowed with the norm

$$\|Y\|^2_{\mathfrak{B}^2_\beta} := \|Y\|^2_{\mathcal{S}^2_\beta} + \|Y\|^2_{\mathcal{S}^{2,a}_\beta}.$$

- $\mathcal{H}_\beta^2$: the space of real-valued random functions $(F(\omega,t))_{t\leq T}$ such that

$$\|F\|^2_{\mathcal{H}^2_\beta} = \mathbb{E}\int_0^T e^{\beta A_t}|F(t)|^2 dt < +\infty.$$

A function f is called a stochastic Lipschitz driver if

- $f : \Omega \times [0,T] \times \mathbb{R} \times \mathcal{M}^2 \longrightarrow \mathbb{R}$, $(\omega,t,y,m) \mapsto f(\omega,t,y,m)$ is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$-measurable.
- There exists two nonnegative $\mathcal{F}_t$-adapted processes $(\mu_t)_{t\leq T}$ and $(\gamma_t)_{t\leq T}$ such that $\forall t \in [0,T]$, $(y,y') \in \mathbb{R}^2$ and all $(m,m') \in \mathcal{M}^2 \times \mathcal{M}^2$,

$$|f(t,y,m) - f(t,y',m')| \leq \mu_t|y-y'| + \gamma_t e^{-\frac{\beta}{2}A_t}\|m-m'\|_{\mathcal{M}^2}.$$

- For every $t \in [0,T]$, $a_t^2 := \mu_t + \gamma_t^2 > 0$.
- For every $t \in [0,T]$,

$$\frac{f(t,0,0)}{a_t} \in \mathcal{H}_\beta^2.$$

For a làdlàg (right-limited with left-limited) process $(Y_t)_{t\leq T}$, we denote by:

- $Y_{t-} = \lim_{s\nearrow t} Y_s$ the left-hand limit of Y at $t \in [0,T]$, $(Y_{0-} = Y_0)$, $Y_- := (Y_{t-})_{t\leq T}$ and $\Delta Y_t := Y_t - Y_{t-}$ the size of the left jump of Y at t.
- $Y_{t+} = \lim_{s\searrow t} Y_s$ the right-hand limit of Y at $t \in [0,T]$, $(Y_{T+} = Y_T)$, $Y_+ := (Y_{t+})_{t\leq T}$ and $\Delta_+ Y_t := Y_{t+} - Y_t$ the size of the right jump of Y at t.

Let $\xi = (\xi_t)_{t\leq T}$ be a barrier which be assumed to left limited with right upper-semicontinuous process belongs to $\mathcal{S}^2_{2\beta}$.

Definition 2.1. Let f be a stochastic Lipschitz driver and ξ be a barrier. The quadruple of processes (Y, M, K, C) is said to be a solution to the RBSDEs associated with parameters (f, ξ) if: $\forall \tau \in \mathcal{T}_{[0,T]}$

$$\begin{cases} (Y, M, K, C) \in \mathfrak{B}^2_\beta \times \mathcal{M}^2_\beta \times \mathcal{S}^2 \times \mathcal{S}^2, \\ Y_\tau = \xi_T + \displaystyle\int_\tau^T f(s, Y_s, M_s)ds - \int_\tau^T dM_s + K_T - K_\tau + C_{T-} - C_{\tau-}, \\ Y_\tau \geq \xi_\tau, \\ \displaystyle\int_0^T \mathbb{1}_{\{Y_t > \xi_t\}} dK^c_t = 0, \ \ (Y_{\tau-} - \xi_{\tau-})\Delta K^d_\tau = 0 \quad \text{and} \quad (Y_\tau - \xi_\tau)\Delta C_\tau = 0 \ a.s. \end{cases} \tag{2.1}$$

where $K = K^c + K^d$ (continuous and purely discontinuous part's) is a nondecreasing right-continuous predictable process with $\mathbb{E}|K_T|^2 < +\infty$, $K_0 = 0$, and C is a nondecreasing right-continuous adapted purely discontinuous process with $\mathbb{E}|C_T|^2 < +\infty$, $C_{0-} = 0$.

Remark 2.2. If (Y, M, K, C) is a solution to RBSDEs (2.1), then $\Delta C_t = Y_t - Y_{t+}$ for all $t \leq T$ outside an evanescent set. It follows that $Y_t \geq Y_{t+}$ for all $t \leq T$, which implies that Y is necessarily right upper-semicontinuous.

Remark 2.3. If (Y, M, K, C) is a solution to RBSDEs (2.1), then the Y is a làdlàg process and $\left(Y_t + \int_0^t f(s, Y_s, M_s)ds\right)_{t \leq T}$ is a strong supermartingale.

3. Existence and uniqueness of the solution to RBSDEs

In this section, we are going to prove the existence and uniqueness result to the RBSDEs (2.1) associated with parameters (f, ξ) by leaning on Marzougue and El Otmani [22] did for the Brownian filtration case. Firstly, we study the result of existence and uniqueness of solution in the special case which is the stochastic Lipschitz driver f does not depend on (y, m). Let

$$f(t, y, m) = g(t).$$

In what follows, we prove an a priori estimate of solution:

Lemma 3.1 (**A priori estimate**). *Let (Y^1, M^1, K^1, C^1) and (Y^2, M^2, K^2, C^2) be a solution of RBSDEs associated with parameters (g^1, ξ^1) and (g^2, ξ^2) respectively. Then there exists a constant κ_β depending on β such that*

$$\|\bar{Y}\|^2_{\mathfrak{B}^2_\beta} + \|\bar{M}\|^2_{\mathcal{M}^2_\beta} \leq \kappa_\beta \left(\|\bar{\xi}\|^2_{\mathcal{S}^2_{2\beta}} + \left\|\frac{\bar{g}}{a}\right\|^2_{\mathcal{H}^2_\beta} \right)$$

where we denote by $\bar{\Re} := \Re^1 - \Re^2$ for $\Re \in \{Y, M, K, C, g, \xi\}$.

Proof. Let $\tau \in \mathcal{T}_{[0,T]}$, one can derive that

$$\bar{Y}_\tau = \bar{\xi}_T + \int_\tau^T \bar{g}(s)ds - \int_\tau^T d\bar{M}_s + \bar{K}_T - \bar{K}_\tau + \bar{C}_{T-} - \bar{C}_{\tau-}.$$

It's observable that $\bar{Y}$ is an optional semimartingale with decomposition $\bar{Y}_\tau = \bar{Y}_0 + N_\tau + V_\tau + W_\tau$ where $N_\tau = \int_0^\tau d\bar{M}_s$, $V_\tau = -\int_0^\tau \bar{g}(s)ds - \bar{K}_\tau$ and $W_\tau = -\bar{C}_{\tau-}$.

So, by using Gal'chouk-Lenglart formula (see Corollary A.9 in [22]), we have

$$\begin{aligned}
& e^{\beta A_t}|\bar{Y}_t|^2 \\
= \quad & |\bar{Y}_0|^2 + \beta\int_0^t e^{\beta A_s}|a_s\bar{Y}_s|^2 ds + \int_0^t 2e^{\beta A_s}\bar{Y}_{s-}d(N+V)_s + \frac{1}{2}\int_0^t 2e^{\beta A_s}d\langle N^c, N^c\rangle_s \\
& + \sum_{0<s\leq t} e^{\beta A_s}\left[\bar{Y}_s^2 - \bar{Y}_{s-}^2 - 2\bar{Y}_{s-}\Delta\bar{Y}_s\right] + \int_0^t 2e^{\beta A_s}\bar{Y}_s dW_{s+} \\
& + \sum_{0\leq s<t} e^{\beta A_s}\left[\bar{Y}_{s+}^2 - \bar{Y}_s^2 - 2\bar{Y}_s\Delta_+\bar{Y}_s\right].
\end{aligned}$$

Then

$$\begin{aligned}
& e^{\beta A_T}|\bar{\xi}_T|^2 \\
= \quad & e^{\beta A_t}|\bar{Y}_t|^2 + \beta\int_t^T e^{\beta A_s}|a_s\bar{Y}_s|^2 ds - 2\int_t^T e^{\beta A_s}\bar{Y}_{s-}\bar{g}(s)ds - 2\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{K}_s \\
& +2\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{M}_s + \int_t^T e^{\beta A_s}d\langle\bar{M}^c, \bar{M}^c\rangle_s + \sum_{t<s\leq T} e^{\beta A_s}(\bar{Y}_s - \bar{Y}_{s-})^2 \\
& -2\int_t^T e^{\beta A_s}\bar{Y}_s d\bar{C}_s + \sum_{t\leq s<T} e^{\beta A_s}(\bar{Y}_{s+} - \bar{Y}_s)^2.
\end{aligned}$$

Consequently

$$\begin{aligned}
& e^{\beta A_t}|\bar{Y}_t|^2 + \beta\int_t^T e^{\beta A_s}|a_s\bar{Y}_s|^2 ds + \int_t^T e^{\beta A_s}d\langle\bar{M}^c, \bar{M}^c\rangle_s \\
\leq \quad & e^{\beta A_T}|\bar{\xi}_T|^2 + 2\int_t^T e^{\beta A_s}\bar{Y}_{s-}\bar{g}(s)ds + 2\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{K}_s - 2\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{M}_s + 2\int_t^T e^{\beta A_s}\bar{Y}_s d\bar{C}_s \\
\leq \quad & e^{\beta A_T}|\bar{\xi}_T|^2 + \frac{\beta}{2}\int_t^T e^{\beta A_s}|a_s\bar{Y}_s|^2 ds + \frac{2}{\beta}\int_t^T e^{\beta A_s}\left|\frac{\bar{g}(s)}{a_s}\right|^2 ds - 2\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{M}_s.
\end{aligned}$$

Here, we have used the fact that $\bar{Y}_{s-}d\bar{K}_s \leq 0$ and $\bar{Y}_s d\bar{C}_s \leq 0$ (for more detail see [22]). Then

$$\begin{aligned}
& e^{\beta A_t}|\bar{Y}_t|^2 + \frac{\beta}{2}\int_t^T e^{\beta A_s}|a_s\bar{Y}_s|^2 ds + \int_t^T e^{\beta A_s}d\langle\bar{M}^c, \bar{M}^c\rangle_s \\
\leq \quad & e^{\beta A_T}|\bar{\xi}_T|^2 + \frac{2}{\beta}\int_t^T e^{\beta A_s}\left|\frac{\bar{g}(s)}{a_s}\right|^2 ds - 2\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{M}_s \\
\leq \quad & \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{2\beta A_\tau}|\bar{\xi}_\tau|^2 + \frac{2}{\beta}\int_t^T e^{\beta A_s}\left|\frac{\bar{g}(s)}{a_s}\right|^2 ds - 2\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{M}_s. \quad (3.1)
\end{aligned}$$

Let us have a look at $\int_t^T e^{\beta A_s}\bar{Y}_{s-}d\bar{M}_s$. We will prove that $\left(\int_0^t e^{\beta A_s}\bar{Y}_{s-}d\bar{M}_s\right)_{t\leq T}$ is a martingale and then has zero expectation. In the same way that the proof of proposition 2.4 in [22], we show that $\mathbb{E}\left[\sqrt{\int_0^T e^{2\beta A_s}|\bar{Y}_{s-}|^2 d[\bar{M}, \bar{M}]_s}\right] < +\infty$ by using the left continuity of trajectory of the process $\bar{Y}_{s-}$. Then, we can write

$$\begin{aligned}\int_0^T e^{2\beta A_s}|\bar{Y}_{s-}|^2 d[\bar{M},\bar{M}]_s &\leq \int_0^T e^{2\beta A_s}\sup_{t\in[0,T]\cap\mathbb{Q}}|\bar{Y}_{t-}|^2 d[\bar{M},\bar{M}]_s\\ &\leq \int_0^T e^{2\beta A_s}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}|\bar{Y}_\tau|^2 d[\bar{M},\bar{M}]_s.\end{aligned}$$

Cauchy Schwarz inequality implies

$$\mathbb{E}\left[\sqrt{\int_0^T e^{2\beta A_s}|\bar{Y}_{s-}|^2 d[\bar{M},\bar{M}]_s}\right] \leq \|\bar{Y}\|^2_{\mathcal{S}^2_\beta} + \|\bar{M}\|^2_{\mathcal{M}^2_\beta} < +\infty.$$

Whence $\mathbb{E}\int_0^t e^{\beta A_s}\bar{Y}_{s-}d\bar{M}_s = 0$. Now, taking expectation on the both sides of the inequality (3.1) with $t=0$, we get

$$\begin{aligned}&\frac{\beta}{2}\mathbb{E}\int_0^T e^{\beta A_s}|a_s\bar{Y}_s|^2 ds + \mathbb{E}\int_0^T e^{\beta A_s}d\langle\bar{M}^c,\bar{M}^c\rangle_s\\ \leq\ &\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}e^{2\beta A_\tau}|\bar{\xi}_\tau|^2 + \frac{2}{\beta}\mathbb{E}\int_0^T e^{\beta A_s}\left|\frac{\bar{g}(s)}{a_s}\right|^2 ds.\end{aligned}$$

Since $\mathbb{E}\int_0^T e^{\beta A_s}d\langle\bar{M}^c,\bar{M}^c\rangle_s = \mathbb{E}\int_0^T e^{\beta A_s}d[\bar{M},\bar{M}]_s$, then, for all $\beta>2$ we obtain

$$\|\bar{Y}\|^2_{\mathcal{S}^{2,a}_\beta} + \|\bar{M}\|^2_{\mathcal{M}^2_\beta} \leq \|\bar{\xi}\|^2_{\mathcal{S}^2_{2\beta}} + \left\|\frac{\bar{g}}{a}\right\|^2_{\mathcal{H}^2_\beta}. \tag{3.2}$$

On the other hand, taking the essential supremum over $\tau\in\mathcal{T}_{[0,T]}$ and then the expectation on both sides of (3.1) we have

$$\begin{aligned}&\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}e^{\beta A_\tau}|\bar{Y}_\tau|^2\\ \leq\ &\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}e^{2\beta A_\tau}|\bar{\xi}_\tau|^2 + \frac{2}{\beta}\mathbb{E}\int_0^T e^{\beta A_s}\left|\frac{\bar{g}(s)}{a_s}\right|^2 ds + 2\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\bar{Y}_{s-}d\bar{M}_s\right|.\end{aligned}$$

By virtue to Remark A.1 in [13] and use the Burkholder-Davis-Gundy's inequality, one can write

$$\begin{aligned}2\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\bar{Y}_{s-}d\bar{M}_s\right| &= 2\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t e^{\beta A_s}\bar{Y}_{s-}d\bar{M}_s\right|\\ &\leq 2c\mathbb{E}\sqrt{\int_0^T e^{2\beta A_s}|\bar{Y}_{s-}|^2 d[\bar{M},\bar{M}]_s}\\ &\leq \frac{1}{2}\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}e^{\beta A_\tau}|\bar{Y}_\tau|^2 + 2c^2\mathbb{E}\int_0^T e^{\beta A_s}d[\bar{M},\bar{M}]_s\end{aligned}$$

where c is a universal positive constant. It follows that

$$\|\bar{Y}\|^2_{\mathcal{S}^2_\beta} \leq 2(2c^2+1)\|\bar{\xi}\|^2_{\mathcal{S}^2_{2\beta}} + 4\left(c^2+\frac{1}{\beta}\right)\left\|\frac{\bar{g}}{a}\right\|^2_{\mathcal{H}^2_\beta}. \tag{3.3}$$

The desired result obtained by the estimates (3.2) and (3.3) with κ_β depending on β and c. $\square$

In the following, we give the existence and uniqueness theorem of RBSDEs associated with data (g, ξ):

Theorem 3.2. *Assuming that* $\frac{g}{a} \in \mathcal{H}^2_\beta$, *then the RBSDEs associated with parameters* (g, ξ) *admits a unique solution* $(Y, M, K, C) \in \mathfrak{B}^2_\beta \times \mathcal{M}^2_\beta \times \mathcal{S}^2 \times \mathcal{S}^2$. *Moreover,* Y *solve (almost surely) the following optimal stopping problem:*

$$Y_\nu = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[\nu,T]}} \mathbb{E}\left[\xi_\tau + \int_\nu^\tau g(t)dt|\mathcal{F}_\nu\right] \qquad \forall \nu \in \mathcal{T}_{[0,T]}.$$

Proof.

- Existence: For all $\nu \in \mathcal{T}_{[0,T]}$, we define the family $\tilde{Y}(\nu)$ by

$$\tilde{Y}(\nu) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[\nu,T]}} \mathbb{E}\left[\xi_\tau + \int_0^\tau g(t)dt|\mathcal{F}_\nu\right] = Y(\nu) + \int_0^\nu g(s)ds$$

where

$$Y(\nu) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[\nu,T]}} \mathbb{E}\left[\xi_\tau + \int_\nu^\tau g(t)dt|\mathcal{F}_\nu\right]$$

From proposition 1.5 in [19], $\tilde{Y}(\nu)$ is a supermartingale family. On the other hand, from theorem 8.2 in [14] ,there exists a làdlàg process $(\tilde{Y}_t)_{t \leq T}$ which aggregates the family $(\tilde{Y}(\nu))_{\nu \in \mathcal{T}_{[0,T]}}$ (i.e. $\tilde{Y}_\nu = \tilde{Y}(\nu)$ a.s. for all $\nu \in \mathcal{T}_{[0,T]}$). Now, by leaning on Marzougue and El Otmani [22] (Lemma 3.3), the candidate Y is a làdlàg process belongs in $\mathcal{S}^2_\beta$ with $Y_\nu = Y(\nu) = \tilde{Y}(\nu) - \int_0^\nu g(s)ds = \tilde{Y}_\nu - \int_0^\nu g(s)ds$. Since the strong optional supermartingale $\tilde{Y}$ is of class(D) (i.e. the set of all random variable $\tilde{Y}_\nu$, for each finite stopping time ν, is uniformly integrable), then, by Mertens decomposition (see Theorem A.4 in [22]), there exists a uniformly integrable, càdlàg martingale M, nondecreasing right-continuous predictable process K (with $K_0 = 0$) such that $\mathbb{E}|K_T|^2 < +\infty$ and nondecreasing right-continuous adapted purely discontinuous process C (with $C_{0-} = 0$) such that $\mathbb{E}|C_T|^2 < +\infty$, gives the following

$$Y_\tau = -\int_0^\tau g(s)ds + M_\tau - K_\tau - C_{\tau-} \qquad \forall \tau \in \mathcal{T}_{[0,T]}. \tag{3.4}$$

Moreover $Y_T = Y(T) = \tilde{Y}(T) - \int_0^T g(s)ds = \xi_T$, then the quadruple of processes (Y, M, K, C) satisfies the equation:

$$Y_\tau = \xi_T + \int_\tau^T g(s)ds - \int_\tau^T dM_s + K_T - K_\tau + C_{T-} - C_{\tau-}. \tag{3.5}$$

From (3.4), we have $\Delta C_\tau = -\Delta_+ Y_\tau$ a.s. Since $\Delta_+ Y_\tau = \mathbb{1}_{\{Y_\tau = \xi_\tau\}} \Delta_+ Y_\tau$ a.s, then $\Delta C_\tau = \mathbb{1}_{\{Y_\tau = \xi_\tau\}} \Delta C_\tau$ a.s. It follows that the process C satisfies the minimality condition. Thanks to lemma 3.5 in [14], for each predictable stopping time τ, we have $\Delta K^d_\tau = \mathbb{1}_{\{Y_{\tau-} = \xi_{\tau-}\}} \Delta K^d_\tau$ a.s., and from lemma 3.6 in [14], $\int_0^T \mathbb{1}_{\{Y_t > \xi_t\}} dK^c_t = 0$ a.s. Then the process K satisfies the Skorokhod condition. Also by definition of Y, we have $Y_\nu \geq \xi_\nu$ a.s for all $\nu \in \mathcal{T}_{[0,T]}$, then $Y_t \geq \xi_t$ for all $t \leq T$ (see Proposition A.7 in [22]). It remains to show that $(Y, M, K, C) \in \mathcal{S}^{2,a}_\beta \times \mathcal{M}^2_\beta \times \mathcal{S}^2 \times \mathcal{S}^2$. For this, we define the Mertens process associated with $\tilde{Y}$ by $\tilde{K}_t := K_t + C_{t-}$. By using the definition of $\tilde{Y}$ and the Corollary A.6 in [22], there exists a positive constant c' such that

$$\begin{aligned}\mathbb{E}|\tilde{K}_T|^2 &\leq c'\mathbb{E}\left|\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}}|\xi_\tau| + \int_0^T |g(t)|dt\right|^2 \\ &\leq 2c'\left(\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}} e^{2\beta A_\tau}|\xi_\tau|^2 + \frac{1}{\beta}\mathbb{E}\int_0^T e^{\beta A_t}\left|\frac{g(t)}{a_t}\right|^2 dt\right).\end{aligned}$$

The nondecreasingness of $\tilde{K}$ implies that

$$\mathbb{E}\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{[0,T]}}|\tilde{K}_\tau|^2 \leq \mathbb{E}|\tilde{K}_T|^2 < +\infty.$$

It follows that $\tilde{K} \in \mathcal{S}^2$, then $(K, C) \in \mathcal{S}^2 \times \mathcal{S}^2$. On the other hand, applying Gal'chouk-Lenglart formula (see Corollary A.9 in [22]) for (3.5), we have

$$\begin{aligned}& e^{\beta A_t}|Y_t|^2 + \beta\int_t^T e^{\beta A_s}|a_sY_s|^2 ds + \int_t^T e^{\beta A_s} d\langle M^c, M^c\rangle_s \\ = \;& e^{\beta A_T}|\xi_T|^2 + 2\int_t^T e^{\beta A_s}Y_{s-}g(s)ds + 2\int_t^T e^{\beta A_s}Y_{s-}dK_s - 2\int_t^T e^{\beta A_s}Y_{s-}dM_s \\ & +2\int_t^T e^{\beta A_s}Y_s dC_s - \sum_{t<s\leq T} e^{\beta A_s}(Y_s - Y_{s-})^2 - \sum_{t\leq s<T} e^{\beta A_s}(Y_{s+} - Y_s)^2 \\ \leq \;& e^{\beta A_T}|\xi_T|^2 + \frac{\beta}{2}\int_0^T e^{\beta A_s}|a_sY_s|^2 ds + \frac{2}{\beta}\int_0^T e^{\beta A_s}\left|\frac{g(s)}{a_s}\right|^2 ds - 2\int_0^T e^{\beta A_s}Y_{s-}dM_s \\ & +2\int_0^T e^{\beta A_s}Y_{s-}dK_s + 2\int_0^T e^{\beta A_s}Y_s dC_s.\end{aligned}$$

By taking the expectation, we get

$$\begin{aligned}& \frac{\beta}{2}\mathbb{E}\int_t^T e^{\beta A_s}|a_sY_s|^2 ds + \mathbb{E}\int_t^T e^{\beta A_s} d[M, M]_s \\ \leq \;& \mathbb{E}e^{\beta A_T}|\xi_T|^2 + \frac{2}{\beta}\mathbb{E}\int_0^T e^{\beta A_s}\left|\frac{g(s)}{a_s}\right|^2 ds + 2\mathbb{E}\int_0^T e^{\beta A_s}Y_{s-}dK_s + 2\mathbb{E}\int_0^T e^{\beta A_s}Y_s dC_s \\ \leq \;& \mathbb{E}e^{\beta A_T}|\xi_T|^2 + \frac{2}{\beta}\mathbb{E}\int_0^T e^{\beta A_s}\left|\frac{g(s)}{a_s}\right|^2 ds + 2\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{2\beta A_\tau}|\xi_\tau|^2 + \mathbb{E}|K_T|^2 + \mathbb{E}|C_T|^2.\end{aligned}$$

For $\beta > 2$, we can write

$$\|Y\|^2_{\mathcal{S}^{2,a}_\beta} + \|M\|^2_{\mathcal{M}^2_\beta} \leq 3\|\xi\|^2_{\mathcal{S}^2_{2\beta}} + \left\|\frac{g}{a}\right\|^2_{\mathcal{H}^2_\beta} + \mathbb{E}|K_T|^2 + \mathbb{E}|C_T|^2.$$

Then$(Y, M) \in \mathcal{S}^{2,a}_\beta \times \mathcal{M}^2_\beta$.

- Uniqueness: It's remarkably that the uniqueness of the solution comes from the uniqueness of Mertens decomposition of strong optional supermartingale and also from lemma 3.1.

□

Lemma 3.3. *Let f be a stochastic Lipschitz driver and ξ be a left-limited with r.u.s.c barrier. If $(y,m) \in \mathfrak{B}^2_\beta \times \mathcal{M}^2_\beta$, then there exists a unique processes $(Y,M,K,C) \in \mathfrak{B}^2_\beta \times \mathcal{M}^2_\beta \times \mathcal{S}^2 \times \mathcal{S}^2$ solution to the following RBSDE:*

$$Y_\tau = \xi_T + \int_\tau^T f(s,y_s,m_s)ds - \int_\tau^T dM_s + K_T - K_\tau + C_{T-} - C_{\tau-}, \quad Y_\tau \geq \xi_\tau$$

with K and C satisfies the Skorokhod and minimality conditions.

Proof. Thanks to theorem 3.2, it is enough to show that $\frac{f(.,y,m)}{a} \in \mathcal{H}^2_\beta$. Indeed, by virtue on the stochastic Lipschitz condition on f, we have

$$\left|\frac{f(s,y_s,m_s)}{a_s}\right|^2 \leq 3\left(a_s^2|y_s|^2 + e^{-\beta A_s}\|m_s\|^2_{\mathcal{M}^2} + \left|\frac{f(s,0,0)}{a_s}\right|^2\right).$$

Then, by using Fubini's property, we get

$$\begin{aligned} &\mathbb{E}\int_0^T e^{\beta A_s}\left|\frac{f(s,y_s,m_s)}{a_s}\right|^2 ds \\ \leq\ & 3\mathbb{E}\int_0^T e^{\beta A_s}a_s^2|y_s|^2 ds + 3T\mathbb{E}\int_0^T e^{\beta A_s}d[m,m]_s + 3\mathbb{E}\int_0^T e^{\beta A_s}\left|\frac{f(s,0,0)}{a_s}\right|^2 ds. \end{aligned}$$

This implies that $\frac{f(.,y,m)}{a} \in \mathcal{H}^2_\beta$ since $(y,m) \in \mathfrak{B}^2_\beta \times \mathcal{M}^2_\beta$. □

Now, with the help of lemma 3.3, we can prove the main result of this paper which is the solvability of the RBSDEs associated with parameters (f,ξ) in the general case whose the stochastic Lipschitz driver f depends on (y,m) by means of Picard's iteration.

Theorem 3.4. *Let f be a stochastic Lipschitz driver and ξ be a left-limited with r.u.s.c barrier. Then, the RBSDEs associated with parameters (f,ξ) has a unique solution $(Y,M,K,C) \in \mathfrak{B}^2_\beta \times \mathcal{M}^2_\beta \times \mathcal{S}^2 \times \mathcal{S}^2$.*

Proof. Define the sequence (Y^n,M^n,K^n,C^n) as follows: $(Y^0,M^0,K^0,C^0) = (0,0,0,0)$,

$$Y_t^{n+1} = \xi_T + \int_t^T f(s,Y_s^n,M_s^n)ds - \int_t^T dM_s^{n+1} + K_T^{n+1} - K_t^{n+1} + C_{T-}^{n+1} - C_{t-}^{n+1}. \tag{3.6}$$

From lemma 3.3, the RBSDE (3.6) has a unique solution $(Y^{n+1},M^{n+1},K^{n+1},C^{n+1}) \in \mathfrak{B}^2_\beta \times \mathcal{M}^2_\beta \times \mathcal{S}^2 \times \mathcal{S}^2$. Next, we shall prove that (Y^n,M^n) is a Cauchy sequence for the Banach space provided with norm

$$\|(Y^n,M^n)\|^2_{\mathfrak{B}^2_\beta \times \mathcal{M}^2_\beta} = \|Y^n\|^2_{\mathfrak{B}^2_\beta} + \|M^n\|^2_{\mathcal{M}^2_\beta}.$$

For $n \geq p \geq 1$, let us put $\Re^{n,p} = \Re^n - \Re^p$ for $\Re \in \{Y, M, K, C\}$. Using Gal'chouk-Lenglart formula (see Corollary A.9 in [22]), we have

$$\begin{aligned}
&e^{\beta A_t}|Y_t^{n+1,p+1}|^2 + \beta \int_t^T e^{\beta A_s} a_s^2 |Y_s^{n+1,p+1}|^2 ds + \int_t^T e^{\beta A_s} d\langle M^{n+1,p+1}, M^{n+1,p+1}\rangle_s \\
\leq\;& 2\int_t^T e^{\beta A_s} Y_{s-}^{n+1,p+1}(f(s, Y_s^n, M_s^n) - f(s, Y_s^p, M_s^p))ds + 2\int_t^T e^{\beta A_s} Y_{s-}^{n+1,p+1} dK_s^{n+1,p+1} \\
&-2\int_t^T e^{\beta A_s} Y_{s-}^{n+1,p+1} dM_s^{n+1,p+1} + 2\int_t^T e^{\beta A_s} Y_s^{n+1,p+1} dC_s^{n+1,p+1}. \qquad (3.7)
\end{aligned}$$

But $Y_{s-}^{n+1,p+1} dK_s^{n+1,p+1} \leq 0$, $Y_s^{n+1,p+1} dC_s^{n+1,p+1} \leq 0$ and

$$\begin{aligned}
&2\int_t^T e^{\beta A_s} Y_s^{n+1,p+1}(f(s, Y_s^n, M_s^n) - f(s, Y_s^p, M_s^p))ds \\
\leq\;& (\beta - 1)\int_t^T e^{\beta A_s} a_s^2 |Y_s^{n+1,p+1}|^2 ds + \frac{1}{\beta - 1}\int_t^T e^{\beta A_s} \left|\frac{f(s, Y_s^n, M_s^n) - f(s, Y_s^p, M_s^p)}{a_s}\right|^2 ds \\
\leq\;& (\beta - 1)\int_t^T e^{\beta A_s} a_s^2 |Y_s^{n+1,p+1}|^2 ds + \frac{2}{\beta - 1}\int_t^T e^{\beta A_s} \left(a_s^2 |Y_s^{n,p}|^2 + e^{-\beta A_s}\|M_s^{n,p}\|_{\mathcal{M}^2}^2\right) ds.
\end{aligned}$$

Then, by taking the expectation in (3.7), we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{\beta A_s}|a_s Y_s^{n+1,p+1}|^2 ds + \mathbb{E}\int_0^T e^{\beta A_s} d[M^{n+1,p+1}, M^{n+1,p+1}]_s \\
\leq\;& \frac{2}{\beta - 1}\left(\mathbb{E}\int_0^T e^{\beta A_s}|a_s Y_s^{n,p}|^2 ds + T\mathbb{E}\int_0^T e^{\beta A_s} d[M^{n,p}, M^{n,p}]_s\right), \quad (3.8)
\end{aligned}$$

where we have used Fubini's property. On the other hand, from (3.7), we have

$$\begin{aligned}
&\mathbb{E}\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}} e^{\beta A_\tau}|Y_\tau^{n+1,p+1}|^2 \\
\leq\;& \frac{2}{\beta - 1}(T \vee 1)\left(\mathbb{E}\int_0^T e^{\beta A_s}|a_s Y_s^{n,p}|^2 ds + \mathbb{E}\int_0^T e^{\beta A_s} d[M^{n,p}, M^{n,p}]_s\right) \\
&+2\mathbb{E}\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s} Y_{s-}^{n+1,p+1} dM_s^{n+1,p+1}\right|.
\end{aligned}$$

By Burkholder-Davis-Gundy's inequality there exists a universal constant c such that

$$\begin{aligned}
&2\mathbb{E}\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s} Y_{s-}^{n+1,p+1} dM_s^{n+1,p+1}\right| \\
\leq\;& \frac{1}{2}\mathbb{E}\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}} e^{\beta A_\tau}|Y_\tau^{n+1,p+1}|^2 + 2c^2\mathbb{E}\int_0^T e^{\beta A_s} d[M^{n+1,p+1}, M^{n+1,p+1}]_s ds
\end{aligned}$$

Consequently

$$\begin{aligned}&\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A_\tau}|Y_\tau^{n+1,m+1}|^2\\ \leq\ & \frac{4}{\beta-1}(T\vee 1)(2c^2+1)\left(\mathbb{E}\int_0^T e^{\beta A_s}|a_s Y_s^{n,p}|^2 ds+\mathbb{E}\int_0^T e^{\beta A_s} d[M^{n,p},M^{n,p}]_s\right).\end{aligned} \tag{3.9}$$

Combining (3.8) with (3.9), and choosing $\beta > 4(T\vee 1)(2c^2+1)+1$, we deduce that $(Y^n, M^n)_{n\geq 0}$ is a Cauchy sequence in $\mathfrak{B}^2_\beta\times\mathcal{M}^2_\beta$, so it converges in $\mathfrak{B}^2_\beta\times\mathcal{M}^2_\beta$ to a limit (Y,M) (i.e. $\lim_{n\to+\infty}\|(Y^n-Y, M^n-M)\|^2_{\mathfrak{B}^2_\beta\times\mathcal{M}^2_\beta}=0$). Now, let us show that (Y,M), with the additional processes (K,C), is a solution of the RBSDE (2.1) associated with parameters (f,ξ). Remark that

$$\begin{aligned}&\mathbb{E}\left|\int_t^T [f(s,Y_s^n,M_s^n)-f(s,Y_s,M_s)]ds\right|^2\\ \leq\ & \frac{1}{\beta}\mathbb{E}\int_t^T e^{\beta A_s}\left|\frac{f(s,Y_s^n,M_s^n)-f(s,Y_s,M_s)}{a_s}\right|^2 ds\\ \leq\ & \frac{2}{\beta}\left(\mathbb{E}\int_0^T e^{\beta A_s}a_s^2|Y_s^n-Y_s|^2 ds+\mathbb{E}\int_0^T\|M_s^n-M_s\|^2_{\mathcal{M}^2}ds\right)\xrightarrow[n\to+\infty]{} 0\end{aligned}$$

and

$$\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t d(M_s^n-M_s)\right|^2\leq c\mathbb{E}\int_0^T e^{\beta A_s}d[M^n-M,M^n-M]_s\xrightarrow[n\to+\infty]{} 0.$$

So, letting n tends to $+\infty$ in (3.6), we deduce that (Y,M,K,C) is the unique solution to RBSDE (2.1) with K and C satisfies the Skorokhod and minimality conditions. □

Remark 3.5 (**The particular case of a right-continuous barrier**). If the barrier ξ is right-continuous, we have $Y_t\geq Y_{t+}\geq\xi_{t+}=\xi_t$. Hence, if t is such that $Y_t=\xi_t$, then $Y_t=Y_{t+}=\xi_t$. If t is such that $Y_t>\xi_t$, then by the minimality condition on C, $Y_t-Y_{t+}=C_t-C_{t-}=0$. Thus, in both cases, $Y_t=Y_{t+}$, so Y is right-continuous. Moreover, the right-continuity of Y combined with the fact that $\Delta C_t=Y_t-Y_{t+}$ give $C_t=C_{t-}$ for all $t\leq T$. As C is right-continuous, purely discontinuous and such that $C_{0-}=0$, we deduce $C=0$. Thus, we recover the usual formulation of RBSDEs with right-continuous barrier.

4. Convex risk measures by means of RBSDEs

I would like to look at the following optimization problem: we are given a stochastic process $\xi=(\xi_t)_{t\leq T}$ which modeling a dynamic financial position. The risk of ξ is assessed by a dynamic risk measure which taking the form

$$\rho(\xi):=Y_0^{-\xi_T}.$$

Here $\rho(.)$ is the operator which maps a given terminal condition ξ_T to the position at $t=0$ of the Y-component of the solution to the BSDEs associated with a driver

f and a terminal value ξ_T. Now, as we have seen in the introduction, a natural way to construct a dynamic risk measure by means of RBSDEs. More precisely, given τ a stopping time before the terminal time T, the gain of the position at τ is equal to ξ_τ and the risk at time t ($t \leq \tau$) is assessed by $Y_t^{-\xi_\tau}$ and we are interested in stopping the process ξ in such a way that the risk be minimal. However, we can formulate this problem as follows

$$\rho(t, \xi_\tau) := Y_t^{-\xi_\tau} \qquad \tau \in \mathcal{T}_{[0,T]} \tag{4.1}$$

where Y is the first component of the solution to the following equation

$$Y_t = \xi_T + \int_t^T f(s, M_s)ds - \int_t^T dM_s + K_T - K_t + C_{T-} - C_{t-}, \quad Y_t \geq \xi_t. \tag{4.2}$$

Let us first give a comparison theorem for RBSDE (4.2):

Theorem 4.1. *Let Y^i be the first component of the solution to RBSDE associated with parameters (f^i, ξ^i), for $i = 1, 2$. If $f^1(., M^2) \leq f^2(., M^2)$ and $\xi^1 \leq \xi^2$, then for all $t \leq T$, $Y_t^1 \leq Y_t^2$ a.s.*
If Y be the first component of the solution to RBSDE associated with parameters $(f, -\xi^i)$ for $i = 1, 2$ such that for all $\tau \in \mathcal{T}_{[0,T]}$ $\xi_\tau^1 \leq \xi_\tau^2$, then for all $t \leq T$, $Y_t^{-\xi_\tau^1} \geq Y_t^{-\xi_\tau^2}$ a.s.

To prove the comparison theorem for RBSDEs, we give a version of Tanaka's formula of a strong optional semimartingales which can be seen as an extension of theorem 66 page 210 in [27].

Proposition 4.2 (**Tanaka's formula**). *Let Y be a one-dimensional optional semimartingale with decomposition $Y = Y_0 + N + V + W$ where N is a (càdlàg) local martingale, V is a right-continuous process of finite variation such that $V_0 = 0$ and W is a left-continuous process of finite variation which is purely discontinuous and such that $W_0 = 0$. Let F be a convex function. Then, $F(Y)$ is a strong optional semimartingale and there exists a nondecreasing adapted process $(\mathcal{A}_t)_{t \leq T}$ such that*

$$F(Y_t) = F(Y_0) + \int_0^t F'(Y_{s-})d(N+V)_s + \int_0^t F'(Y_s)dW_{s+} + \mathcal{A}_t$$

where F' the left-hand derivative of the convex function F.

Proof of Theorem 4.1. It is enough to apply the proposition 4.2 to $e^{\beta A_t}|\bar{Y}_t^+|^2$, one can derive that $Y_t^1 \leq Y_t^2$ a.s. $\forall t \leq T$. The second property is a consequence of the first one. □

We suppose that $m \longmapsto f(t, m)$ is convex, i.e.,

$$f(t, \lambda m^1 + (1-\lambda)m^2) \leq \lambda f(t, m^1) + (1-\lambda) f(t, m^2)$$

for all $(m^1, m^2) \in \mathbb{R}^2$ and all $\lambda \in [0, 1]$. Then we have the following result:

Theorem 4.3. *The map $\rho : [0, T] \times \mathcal{S}_{2\beta}^2 \longrightarrow \mathbb{R}$ defined as (4.1) is a convex dynamic risk measure.*

Proof.
- *Normalization:* it is assumed for convenience.

- *Translation invariance:* Let $\xi \in \mathcal{S}^2_{2\beta}$ and $\alpha \in \mathbb{R}$. We have

$$\rho(t, \xi_\tau + \alpha) = Y_t^{-(\xi_\tau+\alpha)} = Y_t^{-\xi_\tau} - \alpha = \rho(t, \xi_\tau) - \alpha.$$

- *Monotonicity:* Let $\xi^1, \xi^2 \in \mathcal{S}^2_{2\beta}$. From theorem 4.1, if $\xi^1 \leq \xi^2$ then

$$\rho(t, \xi^1_\tau) = Y_t^{-\xi^1_\tau} \geq Y_t^{-\xi^2_\tau} = \rho(t, \xi^2_\tau).$$

- *Convexity:* Let $\xi^1, \xi^2 \in \mathcal{S}^2_{2\beta}$ and $\lambda \in [0,1]$. We want to prove

$$\rho(t, \lambda\xi^1_\tau + (1-\lambda)\xi^2_\tau) \leq \lambda\rho(t, \xi^1_\tau) + (1-\lambda)\rho(t, \xi^2_\tau) \quad \tau \in \mathcal{T}_{[t,T]}.$$

Let $(\breve{Y}, \breve{M}, \breve{K}, \breve{C})$ be a solution of the following RBSDE

$$\breve{Y}_t = -\lambda\xi^1_T - (1-\lambda)\xi^2_T + \int_t^T f(s, \breve{M}_s)ds - \int_t^T d\breve{M}_s + \breve{K}_T - \breve{K}_t + \breve{C}_{T-} - \breve{C}_{t-}.$$

Define

$$\hat{Y}_t := \lambda Y_t^{-\xi^1_\tau} + (1-\lambda)Y_t^{-\xi^2_\tau}; \qquad \hat{M}_t := \lambda M_t^{-\xi^1_\tau} + (1-\lambda)M_t^{-\xi^2_\tau};$$

$$\hat{K}_t := \lambda K_t^{-\xi^1_\tau} + (1-\lambda)K_t^{-\xi^2_\tau}; \qquad \hat{C}_t := \lambda C_t^{-\xi^1_\tau} + (1-\lambda)C_t^{-\xi^2_\tau}.$$

Then

$$\begin{aligned}\hat{Y}_t &= -\lambda\xi^1_T - (1-\lambda)\xi^2_T + \int_t^T \left[\lambda f\left(s, M_s^{-\xi^1_\tau}\right) + (1-\lambda) f\left(s, M_s^{-\xi^2_\tau}\right)\right] ds \\ &\quad + \hat{K}_T - \hat{K}_t - \int_t^T d\hat{M}_s + \hat{C}_{T-} - \hat{C}_{t-}.\end{aligned}$$

By the convexity of f, we get

$$\hat{Y}_t \geq -\lambda\xi^1_T - (1-\lambda)\xi^2_T + \int_t^T f(s, \hat{M}_s)ds + \hat{K}_T - \hat{K}_t - \int_t^T d\hat{M}_s + \hat{C}_{T-} - \hat{C}_{t-}.$$

Hence, from theorem 4.1, we derive that $\hat{Y}_t \geq \breve{Y}_t$ for all $t \leq T$. It follows that

$$\begin{aligned}\rho(t, \lambda\xi^1_\tau + (1-\lambda)\xi^2_\tau) &= \breve{Y}_t \\ &\leq \hat{Y}_t = \lambda Y_t^{-\xi^1_\tau} + (1-\lambda)Y_t^{-\xi^2_\tau} = \lambda\rho(t, \xi^1_\tau) + (1-\lambda)\rho(t, \xi^2_\tau)\end{aligned}$$

□

Remark 4.4. let us denote by $\mathcal{E}^f_{\cdot,\cdot}(\xi)$ the operator of f-conditional expectation (in the sense of [7, 26, 30]). Then

$$\rho(t, \xi_\tau) = -\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathcal{E}^f_{t,\tau}(\xi_\tau).$$

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