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SOME REMARKS ON COMPLETELY α-IRRESOLUTE FUNCTIONS

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Abstract

Chae *et al.* [4] (resp. Navalagi G. B. [14]) have studied the concept of NA-continuous (resp. completely α-irresolute) functions. Now, the aim of this paper we note that NA-continuous functions and completely α-irresolute functions are the same definitions. Also, we investigate several new properties of completely α-irresolute functions are obtained. It is shown that, if f_1 and f_2 are completely α-irresolute functions of a space X into an α-Hausdorff space Y, then the set $\{x \in X: f_1(x) = f_2(x)\}$ is δ-closed in X.

1. INTRODUCTION

Njastad O. [15] defined an α-set in a space as a set S such that $S \subset Int(Cl(Int(S)))$. Maheshwari S. N. [11] defined a feebly open set as a set S such that there exists an open set U such that $U \subset S \subset sCl(U)$, where sCl(U) denotes the semi-closure operator. It was shown in [7] that α-sets and feebly open sets are the same sets in any space. Recently, Chae *et al.* [4] (resp. Navalagi G. B.[14]) have studied the concept of NA-continuous (resp. completely α-irresolute) functions. Now, in the present paper we note that NA-continuous functions and completely α-irresolute functions are the same definitions. It is known in Chae *et al.* (1986) that the type of NA-continuous functions is stronger than the class of super-continuous functions due to Munshi [13], and weaker than the class of strongly continuous functions due to Arya S. P.[1].

The purpose of the present paper is to investigate further properties of completely α-irresolute functions.

2. PRELIMINARIES

Throughout the present paper, spaces always mean topological spaces on which no separation axiom is assumed unless explicitly stated. Let S be a subset of a

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space X. The closure of S and the interior of S are denoted by Cl(S) and Int(S), respectively. A subset S is said to be α-open [15] (resp. θ-open [19]) if $S \subset \text{Int}(\text{Cl}(\text{Int}(S)))$(resp. if for each $x \in S$, there exists an open set U in X such that $x \in U \subset \text{Cl}(U) \subset S$ [17]). It is well-known that for a space (X, τ), X can be retopologized by the family τ^α of all α-open sets of X[10] and also the family τ^θ of all θ-open set of X[19], that is, τ^θ (called θ-topology) and τ^α (called an α-topology) are topologies on X, and it is obvious that $\tau^\theta \subset \tau \subset \tau^\alpha$.

A subset S of a space X is called regular open (resp. regular closed) set if S = Int (Cl(S)) (resp. S= Cl(Int(S)). A subset S of a space X is called δ-open [19] for each $x \in S$, there exists an open set U in X such that $x \in U \subset \text{Int}(\text{Cl}(U)) \subset S$. The family of all α-open (resp. regular open, θ-open and δ-open) sets of X is denoted by $\alpha O(X)$(resp. RO(X), $\theta O(X)$ and $\delta O(X)$). The complement of an α-open (resp. regular open, θ-open and δ-open) sets of X is called α-closed (resp. regular closed, θ-closed and δ-closed) set.

A function $f: X \rightarrow Y$ is said to be α-strongly θ-continuous [5] if for each $x \in X$ and each α-open set H of Y containing $f(x)$, there exists an open set U of X containing x such that $f(\text{Cl}(U)) \subset H$. A function $f: X \rightarrow Y$ is said to be strongly α-irresolute[6] (resp. NA-continuous [4]) if and only if for each α-open (resp. feebly open) subset H of Y, $f^{-1}(H)$ is open (resp. δ-open) in X. A space X is said to be an extremely disconnected [18, p.32] if the closure of each open set of X is open in X. A space X is said to be semi-regular if the family of regularly open sets forms a base for the topology of X. A subset S of a space X is said to be N-closed [16] relative to X if each cover $\{G_i : i \in I\}$ of S by open sets of X, there exists a finite subset I_0 of I such that $S \subset \cup \{\text{Int}(\text{Cl}(G_i)) : i \in I_0\}$.

3. MAIN RESULTS

DEFINITION 3.1[14]: A function $f: X \rightarrow Y$ is said to be completely α-irresolute if the inverse image of each α-open set of Y is regular open in X.

THEOREM 3.1: Let $f: X \rightarrow Y$ be a function. Let $\mathcal{B}$ be any basis for σ^α in Y. Then f is completely α-irresolute functions if and only if for each $B \in \mathcal{B}$, $f^{-1}(B)$ is a regular open subset of X.

LEMMA 3.1[20]: Let $R \in RO(A)$ and $A \in RO(X)$, then $R \in RO(X)$.

THEOREM 3.2: Let $f: X \rightarrow Y$ be any function. If for each $x \in X$, there exists a regular open set R containing x such that $f \mid R$ is completely α-irresolute function, then f is completely α-irresolute function.

PROOF: Let $x \in X$ and let H be any α-open subset containing $f(x)$. Then, there exists a regular open set R containing x such that $f \mid R$ is completely α-irresolute function. Therefore, by [14, Theorem 3.3], there exists a regular open set W in R containing x such that $f|_R(W) \subset H$. Since R is regular open. Therefore, by Lemma 3.1, W is regular open in X and hence $f(W) \subset H$. Thus, f is completely α-irresolute function.

LEMMA 3.2: If $f : X \to Y$ is completely α-irresolute function, then $f^{-1}(V)$ is regular closed for any nowhere dense subset V of Y.

PROOF: Let V be any nowhere dense in Y. Then $Int(Cl(V)) = X \backslash Int(X \backslash V)$. Thus, we have $X = Int(Cl(Int((X \backslash V)))$, for $Int(Cl(V)) = \phi$. Thus, $Y \backslash V$ is α-open in Y. Hence $f^{-1}(V)$ is regular closed in X since $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is regular open and f is completely α-irresolute function.

THEOREM 3.3: (Restricting the range)

If $f : X \to Y$ is completely α-irresolute function and $f(X)$ is taken with the subspsace topology, then $f : X \to f(X)$ is completely α-irresolute function.

PROOF: $f : X \to Y$ is completely α-irresolute function implies $f^{-1}(H)$ is regular open, where H is some α-open subset of Y. Now $f^{-1}[H \cap f(X)] = f^{-1}(H) \cap f^{-1}[f(X)] = f^{-1}(H) \cap X = f^{-1}(H)$ is regular open. Therefore, $f : X \to f(X)$ is completely α-irresolute function.

THEOREM 3.4: Let X be an extremely disconnected. If $f : X \to Y$ is completely α-irresolute function, then it is α-strongly θ-continuous function.

PROOF: Suppose that X is an extremely disconnected and f is completely α-irresolute function. Let H be any α-open set of Y. Since f is completely α-irresolute function. Therefore, $f^{-1}(H)$ is regular open in X. But X is an extremely disconnected. Then, by [3, Lemma 2.18], $f^{-1}(H)$ is θ-open. Thus, by [5, Theorem 2], f is α-strongly θ-continuous.

DEFINITION 3.2: A space X is said to be r-disconnected if there exists two regular open sets R and W such that $X = R \cup W$ and $R \cap W = \phi$, otherwise X is called r-connected.

THEOREM 3.5: If X is r-connected space and $f : X \to Y$ is completely α-irresolute surjection, then Y is α-connected.

PROOF: Suppose Y is not α-connected. Then, there exist non empty α-open sets H_1 and H_2 in Y such that $H_1 \cap H_2 = \phi$ and $H_1 \cup H_2 = Y$ and since f is completely

α-irresolute functions, then we have $f^{-1}(H_1) \cap f^{-1}(H_2) = \phi$ and $f^{-1}(H_1) \cup f^{-1}(H_2) = X$. Since f is surjection, then $f^{-1}(H_j) \neq \phi$ and $f^{-1}(H_j) \in RO(X)$, for j = 1, 2. This indicated that X is not r-connected. This is a contradiction.

COROLLARY 3.1: Let A be r-connected subset of a topological space X, and let f be a completely a-irresolute function of X into a topological space Y. Then $f(A)$ is α-connected.

THEOREM 3.6: For a topological space X to be *r*-disconnected it is necessary and sufficient that there exists a surjection completely α-irresolute function of X onto a discrete space containing more than one point.

PROOF: The condition is sufficient by Theorem 3.5.

Conversely, if X is r-disconnected, there exist two non empty disjoint regular open subsets R and W whose union is X, and the function f of X onto a discrete space of two elements {a, b}, defined by $f(A) = \{a\}$ and $f(B) = \{b\}$, is completely α-irresolute function.

THEOREM 3.7: Let $f: X \rightarrow Y$ be a strongly α-irresolute function from a semi-regular space X into Y. Then f is completely α-irresolute

PROOF: Let $x \in X$ and H be an α-open set containing $f(x)$. Then , $f^{-1}(H)$ is open in X since f is strongly α-irresolute. Therefore, there is an open subset U of x such that $x \in U \in Int(Cl(U)) \subset f^{-1}(H)$, since X is semi-regular. Hence f is completely α-irresolute function.

REMARK 3.1: Every open set in a T_3-space can be written as the union of regular open sets.

COROLLARY 3.2: Let X be a T_3-topological space and let $f: X \rightarrow Y$ be strongly α-irresolute, then f is completely α-irresolute function.

PROOF: Every regular (or T_3) space is semi-regular.

DEFINITION 3.3: A space X is said to be α-Hausdorff [6](resp. rT_2 [2]) if for any x, $y \in X$, $x \neq y$, there exist α-open(resp. regular open) sets G and H such that $x \in G$, $y \in H$ and $G \cap H = \phi$.

THEOREM 3.8: Let $f: X \rightarrow Y$ be injective and completely α-irresolute function. If Y is α-Hausdorff space, then X is rT_2.

PROOF: Let x and y be any two distinct points of X . Since f is injective, $f(x) \neq f(y)$. Now, Y being an α-Hausdorff space, there exist two disjoint α-open sets G and H such that $f(x) \in G$, $f(y) \in H$. Since f is completely α-irresolute function, it follows that $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint regular open sets containing x and y, respectively. Hence X is rT_2.

Recall that a space (X, τ), X is called α-compact [8] if every α-open cover of X has a finite subcover.

DEFINITION 3.4: For a space (X, τ), let A be a subset of X. Then A is said to be α-compact relative to X [8] if every cover of A by α-open sets of X has a finite subcover.

THEOREM 3.9: If $f : X \to Y$ is completely α-irresolute function and F is N-closed subspace relative to X, then $f(F)$ is α-compact relative to Y.

PROOF: Let $\{H_i : i \in I\}$ be a cover of $f(F)$ by α-open sets in Y. For each $x \in F$, there exists an $i(x) \in I$ such that $f(x) \in H_{i(x)}$. Since f is completely α-irresolute function, there exists a regular open set R_x of x such that $f(R_x) \subset H_{i(x)}$. The family $\{R_x : x \in F\}$ is a regular open cover of F. For some finite subset F_0 of F, we have $F \subset \cup \{R_x : x \in F_0\}$ and hence $f(F) \subset \cup \{H_{i(x)} : x \in F_0\}$.This shows that $f(F)$ is α-compact relative to Y.

THEOREM 3.10: Let $g : X \to Y_1 \times Y_2$ be completely α-irresolute function, where X, Y_1 and Y_2 are any topological spaces. Let $f_i : X \to Y_i$ defined as follows:

For $x \in X$, $g(x) = (x_1, x_2)$, $f_i(x) = x_i$ for i = 1, 2. Then $f_i : X \to Y_i$ is completely α-irresolute function, for i = 1, 2.

PROOF: Let x be any point in X and H_1 be any α-open set of Y_1 containing

$f_1(x) = x_1$, then $H_1 \times Y_2$ is α-open in $Y_1 \times Y_2$, which contain (x_1, x_2).

Since g is completely α-irresolute function. Therefore, by [14, Theorem 3.3], there exists a regular open set R containing x such that $g(R) \subset H_1 \times Y_2$. Then $f_1(R) \times f_2(R) \subset H_1 \times Y_2$. Therefore, $f_1(R) \subset H_1$. Hence f_1 is completely α-irresolute function. Similar statement for f_2 is completely α-irresolute function.

THEOREM 3.11: If $f : X \to Y$ is completely α-irresolute function, $g : X \to Y$ is continuous and Y is Hausdorff, then the set $\{y \in X : f(y) = g(y)\}$ is δ-closed in X.

PROOF: Let $A = \{y \in X : f(y) = g(y)\}$ and $x \in X \setminus A$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist open (α-open) sets H_1 and H_2 in Y such that $f(x) \in H_1$, $g(x) \in H_2$ and $H_1 \cap H_2 = \phi$. Since f is completely α-irresolute function. Therefore,by [4, Theorem 2.1], there exists a regular open set R containing x such that $f(R) \subset H_1$. Since g is continuous, there exists an open set U in X containing x such that g

(U) $\subset H_2$. Now, put $R^* = R \cap U$, then by [4, Lemma 2.6], R^* is regular open set in the subspace R and hence it is regular open in X containing x and $f(R^*) \cap g(R^*) \subset H_1 \cap H_2 = \phi$. Therefore, we obtain $R^* \cap A = \phi$. This shows that A is δ-closed in X.

THEOREM 3.11: If f_1 and f_2 are completely α-irresolute functions of a space X into an α-Hausdorff space Y, then the set $\{x \in X: f_1(x) = f_2(x)\}$ is δ-closed in X.

PROOF: Let $A = \{x \in X: f_1(x) = f_2(x)\}$. If $x \in X\backslash A$, then we have $f_1(x) \neq f_2(x)$. Since Y is α-Hausdorff, there exist α-open sets H_1 and H_2 in Y such that $f_1(x) \in H_1$, $f_2(x) \in H_2$ and $H_1 \cap H_2 = \phi$. Since f_j is completely α-irresolute functions, there exists a regular open set R_j in X containing x such that $f_j(R_j) \subset H_j$, where j =1, 2. Put $R = R_1 \cap R_2$, then R is a regular open set in X containing x and $f_1(R) \cap f_2(R) \subset R_1 \cap R_2 = \phi$. This implies that $R \cap A = \phi$ and hence A is δ-closed in X.

LEMMA 3.2[12]: Let X_1 and X_2 be topological spaces with topologies τ_1 and τ_2, respectively. Let $\tau_{\delta 1}$ and $\tau_{\delta 2}$ denote the topologies generated by regularly open sets of X_1 and X_2, respectively. If τ denote the product topology of $X_1 \times X_2$ and τ_δ denote the topology generated by the regularly open sets of $X_1 \times X_2$, then $\tau_{\delta 1} \times \tau_{\delta 2} = \tau_\delta$.

THEOREM 3.13: If Y is an α-Hausdorff space and $f: X \to Y$ is completely α-irresolute function, then the set $A=\{(x_1, x_2): f(x_1) = f(x_2)\}$ is δ-closed in the product space $X \times X$.

PROOF: If $(x_1, x_2) \in X \times (X \setminus A)$, then we have $f(x_1) \neq f(x_2)$. Since Y is α-Hausdorff, there exist α-open sets H_1 and H_2 in Y such that $f(x_1) \in H_1$, $f(x_2) \in H_2$ and $H_1 \cap H_2 = \phi$. Since f is completely α-irresolute function. Therefore, by [4, Theorem 2.1], there exists a δ-open set U_j containing x_j such that $f(U_j) \subset H_j$, where j = 1, 2.

Put $U = U_1 \times U_2$, then by Lemma 3.2, that U is a δ-open set in $X \times X$ containing (x_1, x_2) and $A \cap U = \phi$. This shows that A is δ-closed in the product space $X \times X$.

THEOREM 3.14: If $f_i : X_i \to Y_i$ is completely α-irresolute function, for i = 1, 2. Let $f: X_1 \times X_2 \to Y_1 \times Y_2$ be a function defined as follows:

$f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then f is completely α-irresolute function.

PROOF: Let $H_1 \times H_2 \subset Y_1 \times Y_2$, where H_i is α-open in Y_i, for i = 1, 2, then $f^{-1}(H_1 \times H_2) = f_1^{-1}(H_1) \times f_2^{-1}(H_2)$, since f_i is completely α-irresolute function, for i = 1, 2. By, Definition 3.1 and Theorem 3.10 of [9], $f^{-1}(H_1 \times H_2)$ is regular open in $X_1 \times X_2$. Now if H is any α-open subset of $Y_1 \times Y_2$, then $f^{-1}(H) = f^{-1}(\cup H_\alpha)$, where H_α is of the form $H_{\alpha 1} \times H_{\alpha 2}$. Therefore, by Lemma 3.2, $f^{-1}(H) = \cup f^{-1}(H_\alpha)$ is δ-open in $X_1 \times X_2$, which completes the proof.

THEOREM 3.15: Let $f : X \to Y$ be a completely α-irresolute function on X into an α-Hausdorff space Y. If M is an α-compact subset of Y, then $f^{-1}(M)$ is a δ-closed subset of X.

PROOF: Suppose $f^{-1}(M)$ is not δ-closed in X. Then, there exists an $x \in \text{IntCl}(f^{-1}(M))$, but $x \notin f^{-1}(M)$, it follows that $f(x) \neq m$. Now for each $m \in M$, there exist α-open sets $W_m(f(x))$ and $H(m)$ containing $f(x)$ and m, respectively such that $W_m(f(x)) \cap H(m) = \phi$ because Y is α-Hausdorff. By construction, $M \subset \bigcup_{m \in M} H(m)$, and since M is α-compact. Therefore, there exists a finite subfamily $\{H(m_i) : i = 1, 2,\ldots, n\}$ such that $M \subset \bigcup_{i=1}^{n} H(m_i)$. Let $H^* = \bigcup_{i=1}^{n} H(m_i)$ and $W^* = \bigcap_{i=1}^{n} W_{m_i}(f(x))$. Then $M \subset H^*$ and $H^* \cap W^* = \phi$. Since each $W_{m_i}(f(x))$ is an α-open set of $f(x)$, it follows that W^* is an α-open set of $f(x)$. Since f is completely α-irresolute function.Therefore, by [4, Theorem 3.3], there exists a regular open set U containing x such that $f(U) \subset W^*$. But $x \in \text{IntCl}(f^{-1}(M))$. Therefore, $U \cap f^{-1}(M) \neq \phi$. Hence there exists $z \in U \cap f^{-1}(M)$, and so $f(z) \in f(U) \cap M \subset W^* \cap M \subset W^* \cap H^* = \phi$, which is contradiction. Hence $f^{-1}(M)$ is δ-closed.

Since every compactness implies α-compactness, we obtain from Theorem 3.15 the following corollary.

COROLLARY 3.3: For completely α-irresolute functions into α-Hausdorff spaces, the inverse image of each compact set is δ-closed.

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