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RIEMANN-STIELTJES INTEGRAL BASED ON GENERALIZED *g*-SEMIRING WITH COMMUTATIVE GENERALIZED GENERATED PSEUDO-ADDITION AND NON-COMMUTATIVE GENERALIZED GENERATED PSEUDO-MULTIPLICATION

Misha.M.S*

Abstract

In this paper, we make a study of the generalization of classical Riemann-Stieltjes integral in the pseudo-analysis framework. For that, we consider the generalized g-semiring with commutative generalized generated pseudo-addition and non-commutative generalized generated pseudo-multiplication.

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1. INTRODUCTION

Pseudo-analysis is the generalization of the classical analysis, where instead of the field of real numbers a semiring is defined on a real interval $[a, b] \subseteq [-\infty, +\infty]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ ***(cf. [2]).*** Pseudo-analysis uses many mathematical tools from different fields as measure theory, functional analysis, functional equations, variational calculus, optimization theory, semiring theory etc. and still is in the developing form ***(cf. [3])***. Ivana Stajner-Papuga,

T. Grbic and M. Dankova introduced the generalization of Riemann Stieltjes integral based on the generalized g-semiring $([a, b], \oplus, \odot)$ in 2006 ***(cf. [4], [5], [6])***. In this paper, we make a study of the generalization of classical Riemann-Stieltjes integral based on the generalized g-semiring $([a, b], \oplus, \odot^{\gamma})$. Here γ is arbitrary but fixed positive real number and g: $[a, b] \rightarrow [0, +\infty]$ is a strictly monotone generating function defined on $[a, b] \subseteq [-\infty, +\infty]$ such that $0 \in \text{Ran}(g)$.

* Senior Research Fellow, Department of Mathematics, University of Kerala, Kariavattom, Kerala, India. *Email: mishaeyeroor@gmail.com*

Using the generating function g, first, we define a set function and call g_γ-set function. Using the g_γ-set function and the generalized generated pseudo-operations from the generalized g-semiring $([a, b], \oplus, \odot^\gamma)$, we define the generalization of Riemann-Stieltjes integral in the pseudo-analysis framework, and call pseudo-Riemann-Stieltjes integral based on the generalized g-semiring $([a, b], \oplus, \odot^\gamma)$. We then prove various properties of the new integral.

2. PRELIMINARY NOTION

In this section, some of the important required concepts necessary to go further this paper are shown. They are taken from ***[1], [4], [5]*** and ***[6]***.

Definition 1.1: Let $[a, b]$ be a closed subinterval of $[-\infty, +\infty]$ (in some cases semi closed subintervals will be considered) and let $\leq$ be a total order on $[a, b]$. The operation $\oplus$ is called a ***pseudo-addition*** if it is a function;

$\oplus$: $[a, b] \times [a, b] \to [a, b]$ which satisfies the following axioms: associativity, non-decreasing, a left neutral or zero element $\mathbf{0}$; that is $\mathbf{0} \oplus x = x$, for all $x \in [a, b]$ and commutativity. The operation $\odot$ is called a ***pseudo-multiplication*** if it is a function $\odot$: $[a, b] \times [a, b] \to [a, b]$ which satisfies the following conditions: associativity, positively non-decreasing: i.e., if $x \leq y$ implies $x \odot z \leq y \odot z$, where $z \in [a, b]_+$ and $[a, b]_+ = \{x/x \in [a, b], \mathbf{0} \leq x\}$, $\mathbf{1}$ is unit element: that is $\mathbf{1} \odot x = x$, for all $x \in [a, b]$ and commutativity. A ***semiring*** is the srtructure $([a, b], \oplus, \odot)$ having the following properties: $\oplus$ is pseudo-addition;

$\odot$ is psuedo-multiplication; $\mathbf{0} \odot x = 0$ and $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$, that is $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$. ***g-semiring*** is a semiring with strict pseudo-operations defined by strictly monotone and continuous generator function g: $[a, b] \to [0, +\infty]$. Here the operations are given by $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x)\, g(y))$, where g^{-1} be the classical inverse function for function g.

Definition 1.2: For ***non-decreasing function*** f: $[a, b] \to [c, d]$, where $[a, b]$ and $[c, d]$ are closed subintervals of extended real line $[-\infty, +\infty]$, ***pseudo - inverse*** is $f^{(-1)}(y) = \sup\,\{x \in [a, b]/f(x) < y\}$. If f is a ***non-increasing function, its pseudo-inverse*** is $f^{(-1)}(y) = \sup\,\{x \in [a, b]/f(x) > y\}$. For strictly monotone function f, $f^{(-1)}|_{\mathrm{Ran}(f)}$ is also strictly monotone and following identities hold: (1) $fof^{(-1)}|_{\mathrm{Ran}(f)} = id\,|_{\mathrm{Ran}(f)}$ and (2) $f^{(-1)}$ of $= id\,|_{[a, b]}$.

Definition 1.3: It is possible to define a ***metric*** using the generating function g. Let d: $[a, b] \times [a, b] \to [0, \infty]$ be a function defined by $d(x, y) = |\,g(x) - g(y)\,|$, where $x, y \in [a, b]$ and g: $[a, b] \to [0, \infty]$ is strictly monotone function defined on $[a, b] \subseteq [-\infty, +\infty]$ such that $0 \in \mathrm{Ran}\,(g)$.

3. GENERALIZED *g*-SEMIRING ($[a, b], \oplus, \odot^{\gamma}$)

Structure essential for this paper is the generalization of the previously mentioned semiring.

Definition 2.1: Let γ be arbitrary but fixed positive real number, $g: [a, b] \to [0, +\infty]$ be a strictly monotone generating function defined on $[a, b] \subseteq [-\infty, +\infty]$ such that $0 \in \text{Ran}\,(g)$. The structure $([a, b], \oplus, \odot^{\gamma})$ is called ***generalized g-semiring*** if operations $\oplus$ and $\odot^{\gamma}$ are given by $x \oplus y = g^{(-1)}(g(x) + g(y))$ and $x \odot^{\gamma} y = g^{(-1)}(g(x)^{\gamma}\, g(y))$, where $g^{(-1)}$ is the pseudo-inverse function for function g.

If the generating function g is continuous or bijection, then the operations are given by $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot^{\gamma} y = g^{-1}(g(x)^{\gamma}\, g(y))$, where g^{-1} is the classical inverse function for function g.

Properties of generalized *g*-semiring

Let $([a, b], \oplus, \odot^{\gamma})$ be generalized g-semiring from the above definition. Then

1. If $g(x) + g(y), g^{\gamma}(z)g(x), g^{\gamma}(z)g(y) \in \text{Ran}(g)$, $\odot^{\gamma}$ is left distributive over $\oplus$, that is, $z \odot^{\gamma}(x \oplus y) = (z \odot^{\gamma} x) \oplus (z \odot^{\gamma} y)$.
2. Neutral element from the left for $\oplus$ is $g^{(-1)}(0)$.
3. If $1 \in \text{Ran}\,(g)$, the neutral element from the left for $\odot^{\gamma}$ is $g^{(-1)}(1)$.
4. $g^{(-1)}(0) \odot^{\gamma} x = x \odot^{\gamma} g^{(-1)}(0) = g^{(-1)}(0)$ for all $x \in [a, b]$.
5. $\oplus$ is non-decreasing function, that is, for $x \le y$ we have $x \oplus z \le y \oplus z$ and $z \oplus x \le z \oplus y$, where $x, y, z \in [a, b]$.
6. $\odot^{\gamma}$ is non-decreasing function, that is, for $x \le y$ we have $x \odot^{\gamma} z \le y \odot^{\gamma} z$ and $z \odot^{\gamma} x \le z \odot^{\gamma} y$, where $x, y, z \in [a, b]$.
7. In the general case, the cancellation law does not hold for $\oplus$.

We can prove the properties of generalized g-semiring directly from the above definition, the properties of the generating function and its pseudo-inverses.

Definition 2.2: Let $\oplus$ and $\odot$ be operations from Definition 2.1 and $\alpha_i \in [a, b]$ where $i \in \{1, 2, \ldots, n\}$. Then $\oplus_{i=1}^{n} \alpha_i = (\ldots((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) \oplus \ldots) \oplus \alpha_n$.

Proposition 2.3: 1. If $g: [a, b] \to [0, +\infty]$ is either strictly increasing left continuous or strictly decreasing right continuous generating function such that $0 \in \text{Ran}(g)$, then the pseudo-sum of n elements and pseudo-sum of pseudo-products satisfy the following:

(i) $\oplus_{i=1}^{n} \alpha_i \leq g^{(-1)}\left(\sum_{i=1}^{n} g(\alpha_i)\right)$

(ii) $\oplus_{i=1}^{n} (\alpha_i \odot^{\gamma} \eta_i) \leq g^{(-1)}\left(\sum_{i=1}^{n} g(\alpha_i)^{\gamma} \cdot g(\eta_i)\right)$.

2. If g: $[a, b] \to [0, +\infty]$ is either strictly decreasing left continuous or strictly increasing right continuous generating function such that $+\infty \in \text{Ran}(g)$, then the pseudo-sum of n elements and pseudo-sum of pseudo-products satisfy the following:

(i) $\oplus_{i=1}^{n} \alpha_i \geq g^{(-1)}\left(\sum_{i=1}^{n} g(\alpha_i)\right)$

(ii) $\oplus_{i=1}^{n}(\alpha_i \odot^{\gamma} \eta_i) \geq g^{(-1)}\left(\sum_{i=1}^{n} g(\alpha_i)^{\gamma} \cdot g(\eta_i)\right)$.

3. If g: $[a, b] \to [0, +\infty]$ is a strictly monotone bijection, then the pseudo-sum of n elements and pseudo-sum of pseudo-products satisfy the following:

(i) $\oplus_{i=1}^{n} \alpha_i = g^{-1}\left(\sum_{i=1}^{n} g(\alpha_i)\right)$

(ii) $\oplus_{i=1}^{n}(\alpha_i \odot^{\gamma} \eta_i) = g^{-1}\left(\sum_{i=1}^{n} g(\alpha_i)^{\gamma} \cdot g(\eta_i)\right)$.

Proof 1: (i) Let g: $[a, b] \to [0, +\infty]$ be strictly increasing left continuous. Since $0 \in \text{Ran}(g)$, $g(a) = 0$ and since g is strictly increasing, we can write $\alpha = g^{(-1)}(x) = \sup\{y \in [a, b]/g(y) < x\}$.Therefore, $\lim_{y \to \alpha-} g(y) \leq x$. Since g is left continuous, $\lim_{y \to \alpha-} g(y) = g(\alpha)$.That is $g(\alpha) \leq x$. That is go $g^{(-1)}(x) \leq x$ for all $x \in [0, +\infty]$. Similarly, we can prove if g is strictly decreasing right continuous, go $g^{(-1)}(x) \leq x$ for all $x \in [0, +\infty]$.

Now, $\oplus_{i=1}^{n} \alpha_i = (\ldots((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) \oplus \ldots) \oplus \alpha_n$ $(\alpha_1 \oplus \alpha_2) = g^{(-1)}(g(\alpha_1) + g(\alpha_2))$

$((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) = g^{(-1)}(g(g^{(-1)}(g(\alpha_1) + g(\alpha_2)) + g(\alpha_3)) \leq g^{(-1)}(g(\alpha_1) + g(\alpha_2 + g(\alpha_3))$

[since go $g^{(-1)}(x) \leq x$]

That is, $((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) \leq g^{(-1)}(g(\alpha_1) + g(\alpha_2)) + g(\alpha_3))$

Proceeding like this, we get $\oplus_{i=1}^{n} \alpha_i \leq g^{(-1)}\left(\sum_{i=1}^{n} g(\alpha_i)\right)$.

(ii) By what we first proved, $\oplus_{i=1}^{n} (\alpha_i \odot^{\gamma} \eta_i) \leq g^{(-1)}\ g^{(-1)}\left(\sum_{i=1}^{n} g\left(\alpha_i \odot^{\gamma} \eta_i\right)\right)$.

$= g^{(-1)}\left(\sum_{i=1}^{n} \text{go } g^{(-1)}\left(\left(g\left(\alpha_1\right)\right)^{\gamma} g\left(\eta_1\right)\right)\right) \leq g^{(-1)}\left(\sum_{i=1}^{n} g\left(\alpha_i\right)^{\gamma} \cdot g\left(\eta_i\right)\right)$.

Thus, we get $\oplus_{i=1}^{n}(\alpha_i \odot^{\gamma} \eta_i) \leq g^{(-1)}\left(\sum_{i=1}^{n} g(\alpha_i)^{\gamma} \cdot g(\eta_i)\right)$.

Similarly, we can prove part (2) and part (3) of the theorem by proving go $g^{(-1)}(x) \geq x$, for all $x \in [0, +\infty]$ and go $g^{-1}(x) = x$, for all $x \in [0,+\infty]$ respectively.

4. g_γ-SET-FUNCTION

Another notion essential for defining pseudo-Riemann Stieltjes integral based on the generalized g-semiring $([a, b], \oplus, \odot^{\gamma})$ is the notion of a set function. Therefore, we shall consider a set function that is given by means of generating function g and an increasing function ϕ defined on a compact interval $[c, d]$ of R.

Definition 3.1: Let $([a, b], \oplus, \odot^{\gamma})$ be the generalized g-semiring and ϕ be an increasing function defined on a compact interval $[c, d]$ of R. Let C be the family of semi-closed subintervals $(x, y]$ of $[c, d]$. We associate a set function m: $C \rightarrow [a, b]$ called g_γ - ***set - function***, which is defined as follows; for x, y in C, $m((x, y]) = g^{(-1)}(\phi(y) - \phi(x))$, where $g^{(-1)}$ is the pseudo inverse of the generating function g.

Using the proposition 2.3, we can easily prove the following properties of the g_γ-set-function.

1. $m(\varnothing) = g^{(-1)}(0)$,
2. Let $\mathcal{P} = \{(x_i, x_i + 1]$: where $i = 0, 1, \ldots, n - 1\}$ be an n-partition of $(c, d]$ such that $c = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n = d$ then the following hold:
   (i) if g is strictly monotone bijection, then
   $$m(\cup_{i=0}^{n-1}(x_i, x_{i+1}]) = \oplus_{i=0}^{n-1} m((x_i, x_{i+1}]).$$
   (ii) if g is either strictly increasing left - continuous or strictly decreasing right - continuous generating function such that $0 \in \mathrm{Ran}(g)$, then
   $$m(\cup_{i=0}^{n-1}(x_i, x_{i+1}]) \geq \oplus_{i=0}^{n-1} m((x_i, x_{i+1}]).$$
   (iii) if g is either strictly decreasing left - continuous or strictly increasing right - continuous generating function such that $+\infty \in \mathrm{Ran}(g)$, then
   $$m(\cup_{i=0}^{n-1}(x_i, x_{i+1}]) \leq \oplus_{i=0}^{n-1} m((x_i, x_{i+1}]).$$

5. GENERALIZATION OF RIEMANN-STIELTJES INTEGRAL BASED ON GENERALIZED *g*-SEMIRING $([a, b], \oplus, \odot^{\gamma})$

In this section, we define the generalization of Riemann-Stieltjes integral based on generalized g-semiring $([a, b], \oplus, \odot^{\gamma})$. For defining it, the generalized generated pseudo-operations from the generalized g-semiring $([a, b], \oplus, \odot^{\gamma})$, the g_{γ}-set function and the metric presented in the definition 1.3 will be used.

Definition 4.1: Let g: $[a, b] \to [0, +\infty]$ be strictly monotone function, where $[a, b]$ is a closed subinterval of $[-\infty, +\infty]$ and $\oplus$ and $\odot^{\gamma}$ be the generalized generated pseudo-operations from the generalized g-semiring $([a, b], \oplus, \odot^{\gamma})$. Let $[c, d]$ be a compact subinterval of R and ϕ be a real valued function defined, bounded and increasing on $[c, d]$. If $\mathcal{P} = \{(\mathcal{W}_i, (x_{i-1}, x_i]), i = 1, \ldots, n\}$ is a tagged partition of $[c, d]$. For a function f from $[c, d]$ to $[a, b]$, ***Riemann-Stieltjes pseudo - sum*** of f with respect to ϕ on $[c, d]$ for the tagged partition $\mathcal{P}$ is denoted by $\oplus(\mathcal{P}, f, \phi)$ and defined as $\oplus(\mathcal{P}, f, \phi) = \oplus_{i=1}^{n} f(\mathcal{W}_i) \odot^{\gamma} m((x_{i-1}, x_i])$, where m is the g_{γ}-set function given by definition 3.1.

Definition 4.2: The function f: $[c, d] \to [a, b]$ is said to be ***pseudo Riemann-Stieltjes integrable based on the generalized g-semiring*** $([a, b], \oplus, \odot^{\gamma})$ with respect to ϕ on $[c, d]$ if there exists a real number I $\in [a, b]$ satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\oplus(\mathcal{P}, f, \phi), \mathrm{I}) < \varepsilon$, for all tagged partitions $\mathcal{P}$ of $[c, d]$ with $\|\mathcal{P}\| < \delta$. It can easily see that the number I, if it exists, is uniquely determined. This number I is called pseudo ***Riemann-Stieltjes integral based on generalized g-semiring*** $([a, b], \oplus, \odot^{\gamma})$ with respect to ϕ on $[c, d]$, and it will be denoted by ***(pRS)*** $\sum_{[c,d]}^{(\oplus, \odot^{\gamma})} f \, d\phi$.

Specially, for $g(x) = x$, $[a, b] = [0, +\infty]$ and $\gamma = 1$ the previous definition will give the classical Riemann-Stieltjes integral (RS) $\int_c^d f \, d\phi$.

Throughout this paper, we use the term pseudo-Riemann-Stieltjes integral instead of pseudo-Riemann-Stieltjes integral based on the generalized g-semiring $([a, b], \oplus, \odot^{\gamma})$.

Theorem 4.3: Let g: $[a, b] \to [0, +\infty]$ be a strictly monotone function and let f: $[c, d] \to [a, b]$ be pseudo – Riemann-Stieltjes integrable with respect to ϕ on $[c, d]$.

(1) If g is either strictly increasing right - continuous or strictly decreasing left - continuous generating function such that $+\infty \in \mathrm{Ran}(g)$, then

$$g\left((p\mathrm{RS})\sum_{[c,d]}^{(\oplus, \odot^{\gamma})} f \, d\phi\right) \geq (\mathrm{RS})\int_c^d (g^{\gamma} \, of) \, d\phi,$$

if the integral on the right hand side exists.

(2) If g is either strictly increasing left - continuous or strictly decreasing right - continuous generating function such that $0 \in \mathrm{Ran}(g)$, then

$$g\left((p\text{RS})\sum_{[c,d]}^{(\oplus,\odot^{\gamma})} f\,d\phi\right) \le (\text{RS})\int_c^d (g^{\gamma}\ of)\,d\phi,$$

if the integral on the right hand side exists.

(3) If g is a strictly monotone bijection, then $g^{\gamma}\ of$ is a Riemann Stieltjes integrable with respect to ϕ on $[c, d]$ and

$$(p\text{RS})\int_{[c,d]}^{(\oplus,\odot^{\gamma})} f\,d\phi = f^{-1}\left((\text{RS})\int_c^d (g^{\gamma}\ of)\,d\phi\right).$$

Proof: (1) If g: $[a, b] \to [0, +\infty]$ is either strictly increasing right - continuous or strictly decreasing left - continuous generating function such that $+\infty \in \text{Ran}(g)$ then $gog^{(-1)}(x) \ge x$ for all $x \in [0, +\infty]$. Since f: $[c, d] \to [a, b]$ is pseudo Riemann-Stieltjes integrable with respect to ϕ on $[c, d]$, then by the definition, there is a real number $\text{I} \in [a, b]$ satisfying the following: for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\oplus(\mathcal{P}, f, \phi), \text{I}) < \varepsilon$, ..., (4.1), for all tagged partitions $\mathcal{P}$ of $[c, d]$ that fulfills $\|\mathcal{P}\| < \delta$. Suppose that $(g^{\gamma}\ of)$ is Riemann Stieltjes integrable with respect to ϕ on $[c, d]$. Then by the definition, there is a real number $(\text{RS})\int_c^d (g^{\gamma}\ of)\,d\phi$ satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\sum_{i=1}^{n}(g^{\gamma}\ of)(w_i)(\phi(x_i)-\phi(x_{i-1})), -(\text{RS})\int_c^d (g^{\gamma}\ of)\,d\phi\right| < \varepsilon \qquad (4.2)$$

for all tagged partitions $\mathcal{P} = \{(w_i, (x_{i-1}, x_i]), i = 1, 2, \ldots, n\}$ of $[c, d]$ that fulfills $\|\mathcal{P}\| < \delta$. Now, $\oplus(\mathcal{P}, f, \phi) = \oplus_{i=1}^{n} f(w_i) \odot^{\gamma} m((x_{i-1}, x_i])$

$$= \oplus_{i=1}^{n} f(w_i) \odot^{\gamma} g^{(-1)}(\phi(x_i) - \phi(x_{i-1}))$$

$$\ge g^{(-1)}\left(\sum_{i=1}^{n}(g(f(w_i)))^{\gamma}\, g(g^{(-1)}(\phi(x_i)-\phi(x_{i-1})))\right).$$

[By part (2) of proposition 2.3]

$$\ge g^{(-1)}\left(\sum_{i=1}^{n}(g(f(w_i)))^{\gamma}(\phi(x_i)-\phi(x_{i-1}))\right).$$

Hence, $g(\oplus(\mathcal{P}, f, \phi)) \ge g\left(g^{(-1)}\left(\sum_{i=1}^{n}(g(f(w_i)))^{\gamma}(\phi(x_i)-\phi(x_{i-1}))\right)\right).$

$$\ge \sum_{i=1}^{n}(g(f(w_i)))^{\gamma}(\phi(x_i)-\phi(x_{i-1})).$$

$$g(\oplus(\mathcal{P}, f, \phi)) - \sum_{i=1}^{n}(g(f(w_i)))^{\gamma}(\phi(x_i)-\phi(x_{i-1})).\ \ge 0 \qquad (4.3)$$

From (4.1) we can write $|\,g(\oplus(\mathcal{P}, f, \phi)) - g(\text{I})\,| < \varepsilon \Rightarrow g(\oplus(\mathcal{P}, f, \phi)) - \varepsilon < g\ (\text{I})$

$$\Rightarrow \qquad g(\oplus\,(\mathcal{P}, f, \phi)) - \varepsilon < g\left((\text{pRS})\int_{[c,\,d]}^{(\oplus,\,\odot^{\gamma})} f\, d\phi \right) \tag{4.4}$$

From (4.2) we get, $-\sum_{i=1}^{n}(g(f(w_i)))^{\gamma}(\phi(x_i) - \phi(x_{i-1})) - \varepsilon$

$$< -(\text{RS})\int_{c}^{d} (g^{\gamma}\ of)\, d\phi \tag{4.5}$$

From (4.3), (4.4) and (4.5), we can write

$$g\left((\text{pRS})\int_{[c,\,d]}^{(\oplus,\,\odot^{\gamma})} f\, d\phi \right) - (\text{RS})\int_{c}^{d} (g^{\gamma}\ of)\, d\phi$$

$$> g(\oplus\,(\mathcal{P}, f, \phi)) - \varepsilon - \sum_{i=1}^{n}(g(f(w_i)))^{\gamma}(\phi(x_i) - \phi(x_{i-1})) - \varepsilon \geq -2\varepsilon.$$

This holds for all $\varepsilon > 0$ and, after allowing $\varepsilon \to 0$, then

$$g\left((\text{pRS})\int_{[c,\,d]}^{(\oplus,\,\odot^{\gamma})} f\, d\phi \right) - (\text{RS})\int_{c}^{d} (g^{\gamma}\ of)\, d\phi \geq 0.$$

That is,

$$g\left((\text{pRS})\int_{[c,\,d]}^{(\oplus,\,\odot^{\gamma})} f\, d\phi \right) \geq (\text{RS})\int_{c}^{d} (g^{\gamma}\ of)\, d\phi.$$

Similarly, we can prove the second part of the theorem by using the part (1) of the proposition 2.3 and the fact that go $g^{(-1)}(x) \leq x$, for all $x \in [0, +\infty]$. In addition, we can prove the third part of the theorem by using the part (3) of the proposition 2.3 and the fact that go $g^{-1}(x) = x$, for all $x \in [0, +\infty]$. ■

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