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Basis Transformation from Third-Kind Chebyshev into Bernstein Bases

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Abstract: The bases conversions between the Bernstein polynomial basis and the third-kind Chebyshev polynomial basis are considered. The matrices of conversion among these bases are studied. The matrix of transformation of the third-kind Chebyshev polynomial basis into the Bernstein polynomial basis is derived.

Keywords: *Basis Transformation; Third-Kind Chebyshev Polynomials; Bernstein Polynomials; Computer Aided Design.*

1. INTRODUCTION

There are different kinds of bases; each basis has its advantages and disadvantages. Monomials, orthogonal polynomials, and Bernstein polynomials are the most used bases that are utilized in scientific calculations. In [3], it is shown that monomials are not stable to be used in scientific calculations. Unlikely, the Bernstein polynomial basis is stable and has very interesting properties but are not orthogonal, and therefore, can not be used in applications like least-squares that needs the strong property of orthogonality. So, there is a need to convert the Bernstein basis into orthogonal polynomial basis. The Chebyshev polynomials have many applications in applied mathematics. Bases transformations among the well-known bases are extensively studied, see [9, 10, 11] and the references therein. In this paper, we deal with the matrices of transformation between Bernstein polynomial basis and the third-kind Chebyshev polynomial basis.

Materials related to the topic of this paper are first introduced. This includes the Bernstein polynomials and the Chebyshev polynomials of third kind.

2. BERNSTEIN POLYNOMIALS

The Bernstein polynomials of degree n on [0,1] are defined by

$$B^n_i(u) \;=\; \binom{n}{i}(1-u)^{n-i}u^i,\ u \in [0,1],\quad i = 0, \ldots, n,$$

where $\binom{n}{i} = \dfrac{n!}{i!\,(n-i)!}$ is the combinatorial.

From the definition of the Bernstein polynomials it is clear that $B_i^n(u) \geq 0, \forall u \in [0,1]$ and $B_i^n(u) = B_{n-i}^n(1-u)$.

The Bernstein polynomials satisfy the following recurrence relation:

$$\begin{aligned} B_i^n(u) &= (1-u)B_i^{n-1}(u) + u\, B_{i-1}^{n-1}(u) \\ n &\geq 1, \\ i &= 0, 1, 2, \ldots, n, \end{aligned}$$

Where

$$B_0^0(u) = 1$$

and

$$B_i^n(u) = 0 \text{ for } i \notin \{0, 1, \ldots, n\}.$$

The Bernstein polynomials play an important role in the development of B´ezier curves and surfaces in Computer Aided Geometric Design. They possess important geometric, analytic, and stability properties, see [2, 5, 6].

3. THIRD-KIND CHEBYSHEV POLYNOMIALS

The third-kind Chebyshev polynomials, $V_n(x)$, are orthogonal polynomials on $[-1,1]$ with respect to the weight function $w(x) = \sqrt{\frac{1+x}{1-x}}$. They are given using the following trigonometric relation:

$$V_n(x) = \frac{\cos\left(\left(n+\frac{1}{2}\right)\cos^{-1}(x)\right)}{\cos\left(\frac{1}{2}\cos^{-1}(x)\right)}, \forall x \in [-1,1] \tag{1}$$

The Chebyshev polynomials of third kind satisfy the following linear homogeneous differential equation of the second order:

$$(1-x^2)y'' + (1-2x)y' + n(n+1)y = 0$$

The final results are given in terms of Bernstein polynomials which are defined on the interval [0,1]; therefore, the interval of the third-kind Chebyshev polynomials is shifted by the map $x = 2u - 1$. Thus, the (shifted) third-kind Chebyshev polynomials $V_n(u)$ of degree n on $[0,1]$ become the orthogonal polynomials on [0,1] with respect to the weight function: $w(u) = \sqrt{\frac{u}{1-u}}$. They are generated by the recurrence relation:

$$\begin{aligned} V_n(u) &= 2(2u-1)\, V_{n-1}(u) - V_{n-2}(u), u \in [0,1], \\ n &= 2,3,4,\ldots, \end{aligned}$$

where

$$\begin{aligned} V_0(u) &= 1, \\ V_1(u) &= 4u - 3. \end{aligned}$$

The third-kind Chebyshev polynomials $V_n(u)$ are special case of the Jacobi polynomials, $P_n^{(\beta)}(u)$, where $\alpha = \frac{-1}{2}, \beta = \frac{1}{2}$ and are given explicitly by the relation, for more, see [7, 12]:

$$V_n(u) = \frac{2^{2n}}{\binom{2n}{n}} P_n^{\left(\frac{-1}{2},\frac{1}{2}\right)}(u). \tag{2}$$

4. PRELIMINARIES

We begin this section by introducing the factorial, double factorial, and some of their properties. The factorial of an integer n is defined as usual as follows:

$$n! \;=\; n(n-1)(n-2)\cdots(2)(1),$$

The double factorial of an integer n is gives as:

$$n!! \;=\; \begin{cases} n(n-2)\ldots(4)(2), & \text{if } n \text{ even} \\ n(n-2)\ldots(3)(1), & \text{if } n \text{ odd} \end{cases} \tag{3}$$

The double factorials are simplified to the relations, see [9] ,

$$n!! \;=\; \begin{cases} \dfrac{n!}{(2)\dfrac{n-1}{2}\left(\dfrac{n-1}{2}\right)!} & 2^{\frac{n}{2}}\left(\dfrac{n}{2}\right)!, \text{ if } n \text{ even} \end{cases} \tag{4}$$

if n odd

The factorial of an integer plus one half is given by:

$$\left(n+\frac{1}{2}\right)! \;=\; \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\ldots\frac{3}{2}\frac{1}{2}=\frac{(2n+1)!!}{2^{n+1}}. \tag{5}$$

Also, the factorial of an integer minus one half is simplified to:

$$\left(n-\frac{1}{2}\right)! \;=\; \frac{(2n-1)!!}{2^{n}}. \tag{6}$$

Similar to [9], we have the following lemmas.

Lemma 1: For every $n > 0$ and $k = 1,\ldots,n,$ we have

$$\frac{\binom{2n-1}{2k-1}}{\binom{n-1}{k-1}} \;=\; \frac{(2n-1)!!}{(2k-1)!!\,(2n-2k-1)!!}$$

Proof: Expanding the combinatorials into factorials yields:

$$\frac{\binom{2n-1}{2k-1}}{\binom{n-1}{k-1}} \;=\; \frac{(2n-1)!\,(n-k)!\,(k-1)!}{(2k-1)!\,(2n-2k)!\,(n-1)!}$$

$$=\; \frac{(2n-1)!!\,(2n-2)!!\,(n-k)!\,(k-1)!}{(2k-1)!!\,(2k-2)!!\,(2n-2k)!!\,(2n-2k-1)!!\,(n-1)!}$$

$$=\; \frac{(2n-1)!!\,2^{n-1}\,(n-1)!\,(n-k)!\,(k-1)!}{(2k-1)!!\,2^{k-1}\,(k-1)!\,2^{n-k}\,(n-k)!\,(2n-2k-1)!!\,(n-1)!}$$

$$=\; \frac{(2n-1)!!}{(2k-1)!!\,(2n-2k-1)!!}$$

Lemma 2 : The combinatorials with an integer plus and minus one half forms satisfy the following relation:

$$\binom{n-\frac{1}{2}}{n-k}\binom{n+\frac{1}{2}}{k} = \frac{\binom{2n-1}{n}\binom{2n+1}{2k}}{2^{2n-1}}$$

Proof: Using Lemma 1 and the properties of the double factorials yields:

$$\binom{n-\frac{1}{2}}{n-k}\binom{n+\frac{1}{2}}{k} = \frac{\left(n-\frac{1}{2}\right)!\left(n+\frac{1}{2}\right)!}{\left(k-\frac{1}{2}\right)!(n-k)!\,k!\left(n-k+\ \frac{1}{2}\right)!}$$

$$= \frac{(2n-1)!!\,(2n\quad 1)\,2^{k}\,2^{n-k\quad 1}}{2^{n}\,2^{n}\quad (2k-1)!!\,(n-k)!\,k!\,(2n-2k\quad 1)!!}$$

$$= \frac{(2n-1)!\,(2n+1)!\,2^{k-1}\,(k-1)!\,2^{n-k}\,(n-k)!}{2^{n}\,2^{n-1}(n-1)!\,2^{n}\,n!\,(2k-1)!\,(n-k)!\,k!\,(2n-2k+1)!}$$

$$= \frac{(2n-1)!\,(2n+1)!\,(k-1)!}{2^{2n}\,(n-1)!\;n!\,(2k-1)!\,k!\,(2n-2k+1)!}$$

$$= \frac{\binom{2n-1}{n}\binom{2n+1}{2k}}{2^{2n-1}}.$$

The following Lemma writes the third-kind Chebyshev polynomial in terms of the Bernstein polynomials.

Lemma 3: The third-kind Chebyshev polynomial $V_n(u)$ of degree n is expressed in the degree n Bernstein basis $B^n_0(u)$, $B^n_1(u)$, ... , $B^n_n(u)$ as follows:

$$V_n(u) = \sum_{k=0}^{n} \frac{(-1)^{n-k}\binom{2n+1}{2k}B^n_k(u)}{\binom{n}{k}}$$

Proof: Using the stated Bernstein properties and Lemmas 2 and 3 with some computations we reach the result as follows:

$$V_n(u) = \frac{2^{2n}}{\binom{2n}{n}} P_n^{\left(\frac{-1}{2},\frac{1}{2}\right)}(u)$$

$$= \frac{2^{2n}}{\binom{2n}{n}} \sum_{k=0}^{n} \binom{n-\frac{1}{2}}{n-k}\binom{n+\frac{1}{2}}{k}(u-1)^{k}\,(u)^{n-k}$$

Using Lemma 2 leads to:

$$\mathrm{V}_n(u) = \frac{2^{2n}}{\binom{2n}{n}} \sum_{k=0}^{n} \frac{\binom{2n-1}{n}\binom{2n+1}{2k}(u-1)^k\ (u)^{n-k}}{2^{2n-1}}$$

$$= \frac{2}{\binom{2n}{n}} \sum_{k=0}^{n} (-1)^k \binom{2n-1}{n}\binom{2n+1}{2k}(1-u)^k\ u^{n-k}$$

$$= \frac{2}{\binom{2n}{n}} \sum_{k=0}^{n} \frac{(-1)^k \binom{2n-1}{n}\binom{2n+1}{2k}}{\binom{n}{k}} \mathrm{B}^n_{n-k}(u)$$

$$= \sum_{k=0}^{n} \frac{(-1)^{n-k}(2n+1)!k!(n-k)!}{(2k)!(2n-2k+1)!n!} \mathrm{B}^n_n\ (u)$$

$$= \sum_{k=0}^{n} \frac{(-1)^{n-k}\binom{2n+1}{2k}\mathrm{B}^n_k\ (u)}{\binom{n}{k}}$$

5. MATRICES OF BASES CONVERSION

In this section, the matrices of bases conversions from the Chebyshev polynomial of third kind into the Bernstein polynomials and vice versa are considered. A polynomial $\mathrm{P}_n(u),\ u \in [0,1]$ is written in both Bernstein basis and in third-kind Chebyshev basis as follows:

$$\mathrm{P}_n(u) = \sum_{j=0}^{n} c_j\ \mathrm{B}^n_j(u) = \mathrm{B}_n c_n \tag{7}$$

and

$$\mathrm{P}_n(u) = \sum_{k=0}^{n} v_k\ \mathrm{V}_k(u)\ = \mathrm{V}_n v_n, \tag{9}$$

where

$$\mathrm{B}_n = (\mathrm{B}^n_0(u), \mathrm{B}^n_1(u), \ldots,\ \mathrm{B}^n_n(u)),\ \ c_n = (c_0, c_1, \ldots c_n)^{\mathrm{T}},$$

$$\mathrm{V}_n = (\mathrm{V}^n_0(u), \mathrm{V}^n_1(u), \ldots,\ \mathrm{V}^n_n(u)),\ \ v_n = (v_0, v_1, \ldots v_n)^{\mathrm{T}}.$$

We are interested in finding the $(n + 1) \times (n + 1)$ transformation matrix M and its inverse M^{-1} that satisfy:

$$c_j = \sum_{k=0}^{n} \mathrm{M}_{jk}\ v_k$$

and

$$v_j = \sum_{k=0}^{n} \mathrm{M}^{-1}_{jk}\ c_k$$

Hence, M and M^{-1} satisfy

$$c_n = \mathrm{M}\ v_n$$

and

$$v_n = \mathrm{M}^{-1} c_n.$$

The matrices M and M^{-1} are called the transformation matrices between the third-kind Chebyshev polynomial basis and the Bernstein polynomial basis.

The third-kind Chebyshev polynomial $V_k(u)$ can be written in terms of the basis of the Bernstein as follows:

$$v_k(u) = \sum_{j=0}^{n} N_{kj} B_j^n(u), \tag{9}$$

where N is the $(n + 1) \times (n + 1)$ basis conversion matrix.

Hence, by multiplying both sides by v_k and taking the summation over k we have

$$\begin{aligned}\sum_{k=0}^{n} \upsilon_k V_k(u) &= \sum_{k=0}^{n} \upsilon_k \sum_{j=0}^{n} N_{kj} B_j^n(u) \\ &= \sum_{j=0}^{n} \sum_{k=0}^{n} \upsilon_k N_{kj} B_j^n(u)\end{aligned}$$

Comparing this relation with the equation (7) yields

$$c_j = \sum_{k=0}^{n} \upsilon_k N_{kj} \tag{10}$$

Since $c_n = M\, v_n$ and $v_n = M^{-1}\, c_n$ then we get

$$\begin{aligned}c_j &= \sum_{k=0}^{n} M_{jk} v_k\,, \\ &= 0,1, \ldots n\end{aligned}$$

and

$$\begin{aligned}v_k &= \sum_{j=0}^{n} M_{jk}^{-1}\, c_j\,, \\ &= 0,1, \ldots n\end{aligned}$$

Comparing with (10) we find that: $M_{jk} = N_{kj}$,

therefore, $M = N^T$.

At first, N_{kj} is found, then we get M_{jk} by transposing N_{kj}, and M_{jk}^{-1} is found by the same steps.

Theorem 1: The elements of the matrix M that satisfies $V_n = B_n M$ which transforms from the third-kind Chebyshev polynomial basis into the Bernstein basis for $0 \le j \le n$, $0 \le k \le n$ are given by:

$$M_{jk} = \frac{(2k+1)!}{k!\binom{n}{j}} \sum_{i=\max(0,\, j+k-n)}^{\min(j,k)} \frac{(-1)^{k-i}\, i!\,(k-i)!\binom{k}{i}\binom{n-k}{j-i}}{(2i)!\,(2k-2i+1)!}$$

Proof: In Lemma 3, the following relation is given:

$$V_n(u) = \sum_{k=0}^{n} \frac{(-1)^{n-k}\binom{2n+1}{2k} B_k^n(u)}{\binom{n}{k}}$$

Degree elevating the Bernstein polynomials of degree k to n, $k < n$, is carried out using the following formula:

$$B_i^k(u) \;=\; \sum_{j=i}^{n-k+i} \frac{\binom{k}{i}\binom{n-k}{j-i}}{\binom{n}{j}} B_j^n(u),\;\; i = 0, 1, ..., k.$$

By substituting the degree elevation for the third-kind Chebyshev polynomials of degree k we get:

$$V_k(u) \;=\; \sum_{i=0}^{k} \frac{(-1)^{k-i}\binom{2k+1}{2i}}{\binom{k}{i}} B_i^k(u),\;\; i = 0, 1, ..., k.$$

$$= \sum_{i=0}^{k} \frac{(-1)^{k-i}\binom{2k+1}{2i}}{\binom{k}{i}} \sum_{j=0}^{n-k+i} \frac{\binom{k}{i}\binom{n-k}{j-i}}{\binom{n}{j}} B_j^n(u)$$

$$= \sum_{j=0}^{n} B_j^n(u) \sum_{i=\max(0,\, j+k-n)}^{\min(j,k)} \frac{(-1)^{k-i}\binom{2k+1}{2i}\binom{n-k}{j-i}}{\binom{n}{j}}.$$

Since $\quad V_k(u) \;=\; \sum_{j=0}^{n} N_{kj} B_j^n(u)$, we get:

$$N_{kj} \;=\; \frac{1}{\binom{n}{j}} \sum_{i=\max(0,\, j+k-n)}^{\min(j,k)} (-1)^{k-i}\binom{2k+1}{2i}\binom{n-k}{j-i}$$

We get M by transposing N_{kj}.

The elements of the Matrix M^{-1} that satisfies $B_n = V_n M^{-1}$ which transforms from the Bernstien polynomial basis into the third-kind Chebyshev polynomial basis for $0 \le j, k \le n$ are given by (see [11]):

$$M_{jk}^{-1} \;=\; \frac{\binom{n}{k}}{4^{n+j}} \sum_{i=0}^{j} \frac{(-1)^{j-i}\binom{2j+1}{2i+1}\binom{2k+2i+1}{k+i+1}\binom{2n+2j-2k-2i}{n+j-k-i}}{\binom{n+j+1}{k+i+1}}$$

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