FRACTIONAL MINMAX PROBLEMS CONTAINING GENERALIZED $(\Im,\rho,\theta)$-CONVEX n-SET FUNCTIONS

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ABSTRACT: Sufficient optimality conditions and duality theorems are established for generalized fractional minmax programming problems involving generalized $(\Im,\rho,\theta)$-convex n-set functions.

Keywords: Generalized fractional problem, n-set function, $(\Im,\rho,\theta)$-convexity, duality.

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1. Introduction and Preliminiaries

Optimization problems involving set functions have been extensively studied in recent years. These problems arise in various areas and have many interesting applications in Mathematics, for example, in fluid flow [2], electric insulator design [4], plasma confinement [4] and many more. The first theory of optimizing set functions was developed by Morris [8]. Subsequently, several authors [1,5,7] have made significant contributions in developing optimality conditions and duality results for various optimization problems involving n-set functions under different set ups. In [9], Preda introduced $(\Im,\rho)$-convexity for n-set functions defined by using a sublinear functional which satisfies certain convexity type condition. Later, Jo, Kim and Lee [6] extended the concept of $(\Im,\rho)$-convexity to generalized $(\Im,\rho,\theta)$-convexity for n-set functions and established several sufficient optimality conditions for multiobjective programming problem with inequality and equality constraints. Recently, Bhatia and Kumar [3] obtained sufficient optimality conditions and duality results for fractional minmax problem under generalized ρ-convexity conditions.

The purpose of this paper is to establish sufficient optimality conditions and duality results for generalized fractional minmax programming problem involving n-set functions under generalized $(\Im,\rho,\theta)$-convexity assumptions on few of the objective and the constraint functions.

Throughout the paper $(X,\boldsymbol{A},\mu)$ is a finite atomless measure space with $L_1(X,\boldsymbol{A},\mu)$ separable, $\boldsymbol{A}^n$ is the n-fold product of σ-algebra $\boldsymbol{A}$ of subsets of the set X. A pseudometric 'd' of $\boldsymbol{A}^n$ is defined as

$$d(S,T) = \left(\sum_{i=1}^{n}\mu^2(S_i \Delta T_i)\right)^{1/2}$$

where $S = (S_1, S_2, \dots, S_n)$, $T = (T_1, T_2, \dots, T_n) \in \mathcal{A}^n$, and $S_i \Delta T_i$ denotes the symmetric difference of sets S_i and T_i.

For $f \in L_1(X, \mathcal{A}, \mu)$ and $S \in \mathcal{A}$, the integral $\int_S f\, d\mu$ will be denoted by $\langle f, x_S \rangle$, where x_S is the characteristic function of S.

We now give some definitions from Jo, Kim and Lee [6] that are used in the sequel.

Definition 1.1 A functional $\Im$ on $\mathcal{A}^n \times \mathcal{A}^n \times L_1^n(X, \mathcal{A}^n, \mu)$ is said to be sublinear with respect to its third argument if for any S, T $\in \mathcal{A}^n$,

$$\Im(S,T;\eta_1 + \eta_2) \leqq \Im(S,T;\eta_1) + \Im(S,T;\eta_2) \quad \forall\ \eta_1,\ \eta_2 \in L_1^n(X, \mathcal{A}^n, \mu)$$

$$\Im(S,T;\alpha\eta) = \alpha\, \Im(S,T;\eta)\ \ \forall\ \alpha \geqq 0,\ \forall\ \eta \in L_1^n(X, \mathcal{A}^n, \mu).$$

Throughout this paper, unless otherwise stated, we assume that the sublinear functional on $\mathcal{A}^n \times \mathcal{A}^n \times L_1^n(X, \mathcal{A}^n, \mu)$ satisfy the following condition:

(C) For $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in L_1^n(X, \mathcal{A}^n, \mu)$,

and $S^* = (S_1^*, S_2^*, \dots, S_n^*) \in \mathcal{A}^n$,

we have

$$\langle \eta, x_S - x_{S^*} \rangle = (\langle \eta_1, x_S - x_{S^*} \rangle, \langle \eta_2, x_S - x_{S^*} \rangle, \dots, \langle \eta_n, x_S - x_{S^*} \rangle) \geqq 0$$

implies $\Im(S, S^*; \eta) \geqq 0 \quad \forall\ S \in \mathcal{A}^n$.

Definition 1.2 Let $\Im$ be a sublinear functional on $\mathcal{A}^n \times \mathcal{A}^n \times L_1^n(X, \mathcal{A}^n, \mu)$. Let the function $F : \mathcal{A}^n \to \mathbf{R}$ be differentiable, $\theta : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}^n \times \mathcal{A}^n$ with $\theta(S, S^*) \neq 0$ for $S \neq S^*$, and $\rho \in \mathbf{R}$

(i) The function F is said to be $(\Im, \rho, \theta)$-quasiconvex at S^* if for each $S \in \mathcal{A}^n$ such that $F(S) \leqq F(S^*)$, we have

$$\Im(S, S^*; DF_{S^*}) + \rho\, d^2(\theta(S, S^*)) \leqq 0.$$

(ii) The function F is said to be ($\Im,\rho,\theta$)-pseudoconvex at S^* if for each $S \in \mathcal{A}^n$ such that $F(S) < F(S^*)$, we have

$$\Im(S,S^*;DF_{S^*}) + \rho\, d^2(\theta(S,S^*)) < 0.$$

(iii) The function F is said to be strictly ($\Im,\rho,\theta$)-pseudoconvex at S^* if for each $S \in \mathcal{A}^n$, $S \neq S^*$, such that $F(S) \leqq F(S^*)$, we have

$$\Im(S,S^*;DF_{S^*}) + \rho\, d^2(\theta(S,S^*)) < 0.$$

2. OPTIMALITY CONDITIONS

The following generalized fractional minmax programming problem is studied in this paper

(P) $$\min_{S} \max_{1 \leqq i \leqq p} (F_i(S)/H_i(S))$$

subject to

$$G(S) \leqq 0,$$

$$Q(S) = 0$$

$$S = (S_1, S_2, \ldots, S_n) \in \mathcal{A}^n$$

where $F = (F_1, F_2, \ldots, F_p): \mathcal{A}^n \to \mathbf{R}^p$, $H = (H_1, H_2, \ldots, H_p): \mathcal{A}^n \to \mathbf{R}^p$, $G = (G_1, G_2, \ldots, G_m): \mathcal{A}^n \to \mathbf{R}^m$, $Q = (Q_1, Q_2, \ldots, Q_s): \mathcal{A}^n \to \mathbf{R}^s$, are vector-valued differentiable n-set functions defined on $\mathcal{A}^n$.

Let $\Gamma = \{S \in \mathcal{A}^n \mid G(S) \leqq 0,\ Q(S) = 0\}$ be the set of feasible solutions of (P). We assume that $F(s) \geqq 0$ and $H(S) > 0,\ \forall\ S \in \Gamma$.

Using parametric approach, we associate the following problem with (P)

(EP) min q

subject to

$$F_i(S) - q\, H_i(S) \leqq 0\,,\ 1 \leqq i \leqq p, \tag{2.1}$$

$$S \in \Gamma\,,\ q \in \mathbf{R}_+ \tag{2.2}$$

The following Lemma establishes the equivalence between (P) and (EP).

Lemma 2.1 [3]. If S^* is an optimal solution of (P), then (S^*, q^*) with $q^* = \max_{1 \leqq i \leqq p} (F_i(S^*)/H_i(S^*))$ is an optimal solution of (EP).

Conversely, if (S^0, q^0) is optimal for (EP), then S^0 is optimal for (P).

Theorem 2.1 (Necessary Optimality Conditions) [3]. Let S^* be a regular optimal solution of (P) with $q^* = \max_{1 \leqq i \leqq p} (F_i(S^*)/H_i(S^*))$. Then there exist $u^* \in \mathbf{R}^p_+$, $\sum_{i=1}^{p} u_i^* = 1$, $v^* \in \mathbf{R}^m_+$ and $w^* \in \mathbf{R}^s$ such that

$$\langle u^{*t} D_r(F - q^* H)_{S^*} + v^{*t} D_r G_{S^*} + w^{*t} D_r Q_{S^*},\, x_{S_r} - x_{S_r^*} \rangle \geqq 0$$

$$\forall\, S_r \in \mathcal{A},\ 1 \leqq r \leqq n \tag{2.3}$$

$$u_i^* (F_i(S^*) - q^* H_i(S^*)) = 0,\ 1 \leqq i \leqq p \tag{2.4}$$

$$v_j^* G_j(S^*) = 0,\ 1 \leqq j \leqq m \tag{2.5}$$

where the superscript 't' denotes the transpose of a vector.

We now present sufficient optimality conditions for the existence of an optimal solution of problem (P).

Theorem 2.2 Let $S^* \in \mathcal{A}^n$ be a feasible solution of (P) with $q^* = \max_{1 \leqq i \leqq p} (F_i(S^*)/H_i(S^*))$. Assume that there exist $u^* \in \mathbf{R}^p_+$, $\sum_{i=1}^{p} u_i^* = 1$, $v^* \in \mathbf{R}^m_+$ and $w^* \in \mathbf{R}^s$ such that conditions (2.3)-(2.5) are satisfied. Further, if

(i) $F_i - q^* H_i$, $i \in I(S^*)$ are $(\Im, \rho_{1i}, \theta)$-psedoconvex at S^*,

(ii) G_j, $j \in J(S^*)$ are $(\Im, \rho_{2j}, \theta)$-quasiconvex at S^*,

(iii) Q_k, $1 \leqq k \leqq s$ are $(\Im, \rho_{3k}, \theta)$-quasiconvex at S^*,

(iv) $-Q_k$, $1 \leqq k \leqq s$ are $(\Im, \rho_{4k}, \theta)$-quasiconvex at S^* with $\rho_{3k} + \rho_{4k} \geqq 0$ $\forall\, k$, $1 \leqq k \leqq s$.

(v) $\sum_{i\in I(S^*)} u_i^* \rho_{1i} + \sum_{j\in J(S^*)} v_j^* \rho_{2j} + \sum_{k=1}^{s} w_k^* \rho_{3k} \geqq 0$

where $I(S^*) = \{ i \mid F_i(S^*) - q^* H_i(S^*) = 0\}$;

$J(S^*) = \{ j \mid G_j(S^*) = 0\}$.

Then S^* is an optimal solution of (P).

Proof. Suppose S^* is not an optimal solution of (P). Then by Lemma 2.1, (S^*, q^*) is not optimal for (EP). Therefore, there exists (S,q), feasible for (EP), with $S \neq S^*$ and $q < q^*$. This along with (2.1) yields

$$F_i(S)/H_i(S) \leqq q < q^*, \ 1 \leqq i \leqq p,$$

which implies that

$$F_i(S) - q^* H_i(S) < 0 = F_i(S^*) - q^* H_i(S^*), \ i \in I(S^*)$$

Thus, $(\Im, \rho_{1i}, \theta)$-psedoconvexity of $F_i - q^* H_i$ at S^* implies

$$\Im(S, S^*; D(F_i - q^* H_i)_{S^*}) + \rho_{1i}\, d^2(\theta(S, S^*)) < 0, \ i \in I(S^*).$$

From (2.4) it follows that $u_i^* = 0$ for $i \notin I(S^*)$, and therefore, $\sum_{i\in I(S^*)} u_i^* = 1$ ensures the existence of at least one $u_i^* > 0$, $i \in I(S^*)$. Hence, by multiplying each of the above inequalities by u_i^*, $i \in I(S^*)$, summing, and using sublinearity of $\Im$, we get

$$\Im(S, S^*; \sum_{i\in I(S^*)} u_i^* D(F_i - q^* H_i)_{S^*}) + \sum_{i\in I(S^*)} u_i^* \rho_{1i}\, d^2(\theta(S, S^*)) < 0$$

Again, as $u_i^* = 0$, $i \notin I(S^*)$, the above inequality can be rewritten as

$$\Im(S,S^*;\ \sum_{i=1}^{p} u_i^* D(F_i - q^* H_i)_{S^*}) + \sum_{i\in I(S^*)} u_i^* \rho_{1i}\, d^2(\theta(S,S^*)) < 0 \qquad (2.6)$$

Also, we have $G_j(S) \leqq 0 = G_j(S^*),\ j \in J(S^*)$.

Applying $(\Im, \rho_{2j}, \theta)$-quasiconvexity of G_j at S^*, we get

$$\Im(S,S^*; D((G_j)_{S^*}) + \rho_{2j}\, d^2(\theta(S,S^*)) \leqq 0,\ j \in J(S^*).$$

Multiplying each of the above inequalities by $v_j^* \geqq 0,\ j \in J(S^*)$, summing, and using the sublinearity of $\Im$, we obtain

$$\Im(S,S^*;\ \sum_{j\in J(S^*)} v_j^* D(G_j)_{S^*}) + \sum_{j\in J(S^*)} v_j^* \rho_{2j}\, d^2(\theta(S,S^*)) \leqq 0$$

In view of (2.5), $v_j^* = 0,\ j \notin J(S^*)$, thus

$$\Im(S,S^*;\ \sum_{j=1}^{m} v_j^* D(G_j)_{S^*} + \sum_{j\in J(S^*)} v_j^* \rho_{2j}\, d^2(\theta(S,S^*)) \leqq 0 \qquad (2.7)$$

Further, we have

$$Q_k(S) = Q_k(S^*) = 0,\ 1 \leqq k \leqq s$$

So, by $(\Im, \rho_{3k}, \theta)$-quasiconvexity of Q_k and at S^*, $(\Im, \rho_{4k}, \theta)$-quasiconvexity of $-Q_k$ at S^*, we have

$$\Im(S,S^*; D((Q_k)_{S^*}) + \rho_{3k}\, d^2(\theta(S,S^*)) \leqq 0,\ 1 \leqq k \leqq s \qquad (2.8)$$

$$\Im(S,S^*; -D((Q_k)_{S^*}) + \rho_{4k}\, d^2(\theta(S,S^*)) \leqq 0,\ 1 \leqq k \leqq s \qquad (2.9)$$

As $\rho_{3k} + \rho_{4k} \geqq 0\ \forall\, k,\ 1 \leqq k \leqq s$, therefore (2.9) can be rewritten as

$$\Im(S,S^*; -D((Q_k)_{S^*}) - \rho_{3k}\, d^2(\theta(S,S^*)) \leqq 0,\ 1 \leqq k \leqq s \qquad (2.10)$$

Let $w^* = w^1 - w^2;$

$$w^1 = (w_1^1, w_2^1, \ldots, w_s^1),\ w^2 = (w_1^2, w_2^2, \ldots, w_s^2) \geqq 0;\ w^1, w^2 \in \mathbf{R}^s.$$

Multiplying (2.8) by w_k^1, (2.10) by w_k^2, and using sublinearity of $\Im$, we obtain

$$\Im(S, S^*;\ w_k^1 D((Q_k)_{S^*}) + w_k^1 \rho_{3k} d^2(\theta(S, S^*)) \leqq 0,\ 1 \leqq k \leqq s$$

$$\Im(S, S^*; -w_k^2 D((Q_k)_{S^*}) - w_k^2 \rho_{3k} d^2(\theta(S, S^*)) \leqq 0,\ 1 \leqq k \leqq s.$$

Adding the above inequalities and using sublinearity of $\Im$, it follows that

$$\Im(S, S^*;\ \sum_{k=1}^{s} w_k^* D((Q_k)_{S^*}) + \sum_{k=1}^{s} w_k^* \rho_{3k} d^2(\theta(S, S^*)) \leqq 0, \qquad (2.11)$$

Adding (2.6), (2.7) and (2.11); and by sublinearity of $\Im$, we get

$$\Im(S, S^*; u^{*t} D(F - q^* H)_{S^*} + v^{*t} DG_{S^*} + w^{*t} DQ_{S^*})$$

$$+ \left(\sum_{i \in I(S^*)} u_i^* \rho_{1i} + \sum_{j \in J(S^*)} v_j^* \rho_{2j} + \sum_{k=1}^{s} w_k^* \rho_{3k} \right) d^2(\theta(S, S^*)) < 0$$

which in view of assumption (v) implies

$$\Im(S, S^*; u^{*t} D(F - q^* H)_{S^*} + v^{*t} DG_{S^*} + w^{*t} DQ_{S^*}) < 0 \qquad (2.12)$$

Since the sublinear functional $\Im$ satisfy condition **(C)**, it follows from (2.3) that

$$\Im(S, S^*; u^{*t} D(F - q^* H)_{S^*} + v^{*t} DG_{S^*} + w^{*t} DQ_{S^*}) \geqq 0\ \ \forall\, S \in \mathcal{A}^n$$

which contradicts (2.12).

Hence S^* is an optimal solution of (P).

Remark 2.1

Theorem 2.1 also holds under any of the following different sets of assumptions

(i) $\sum_{i \in I(S^*)} u_i^* (F_i - q^* H_i)$ is $(\Im, \rho_1, \theta)$-pseudoconvex at S^*

(ii) $\sum_{j \in J(S^*)} v_j^* G_j$ is $(\Im, \rho_2, \theta)$-quasiconvex at S^*

(iii) $\sum_{k=1}^{s} w_k^* Q_k$ is $(\Im, \rho_3, \theta)$-quasiconvex at S^*

(iv) $\rho_1 + \rho_2 + \rho_3 \geqq 0$

'OR'

(i) $\sum_{i \in I(S^*)} u_i^* (F_i - q^* H_i)$ is $(\Im, \rho_1, \theta)$-quasiconvex at S^*

(ii) $\sum_{j \in J(S^*)} v_j^* G_j$ is strictly $(\Im, \rho_2, \theta)$-pseudoconvex at S^*

(iii) $\sum_{k=1}^{s} w_k^* Q_k$ is $(\Im, \rho_3, \theta)$-quasiconvex at S^*

(iv) $\rho_1 + \rho_2 + \rho_3 \geqq 0$

'OR'

(i) $\sum_{i \in I(S^*)} u_i^* (F_i - q^* H_i)$ is $(\Im, \rho_1, \theta)$-quasiconvex at S^*

(ii) $\sum_{j \in J(S^*)} v_j^* G_j$ is $(\Im, \rho_2, \theta)$-quasiconvex at S^*

(iii) $\sum_{k=1}^{s} w_k^* Q_k$ is strictly $(\Im, \rho_3, \theta)$-pseudoconvex at S^*

(iv) $\rho_1 + \rho_2 + \rho_3 \geqq 0$

'OR'

(i) $\sum_{i \in I(S^*)} u_i^* (F_i - q^* H_i)$ is $(\Im, \rho_1, \theta)$-pseudoconvex at S^*

(ii) $\sum_{j \in J(S^*)} v_j^* G_j$ is $(\Im, \rho_2, \theta)$-quasiconvex at S^*

(iii) $\sum_{k=1}^{s} w_k^* Q_k$ is $(\Im, \rho_3, \theta)$-quasiconvex at S^*

(iv) $\rho_1 + \rho_2 + \rho_3 \geqq 0$

'OR'

(i) $\sum_{i \in I(S^*)} u_i^* (F_i - q^* H_i)$ is $(\Im, \rho_1, \theta)$-pseudoconvex at S^*

(ii) $\sum_{j \in J(S^*)} v_j^* G_j + \sum_{k=1}^{s} w_k^* Q_k$ is $(\Im, \rho_2, \theta)$-quasiconvex at S^*

(iii) $\rho_1 + \rho_2 \geqq 0$

'OR'

(i) $\sum_{i \in I(S^*)} u_i^* (F_i - q^* H_i) + \sum_{j \in J(S^*)} v_j^* G_j + \sum_{k=1}^{s} w_k^* Q_k$ is $(\Im,\rho,\theta)$-pseudoconvex at S^* with $\rho \geqq 0$.

3. DUALITY

In this section, we present duality results between problem (P) and its following dual

(D) Min q

subject to

$$\langle u^t D_r (F - qH)_T + v^t D_r G_T + w^t D_r Q_T,\ x_{S_r} - x_{T_r} \rangle \geqq 0$$

$$\forall\ S_r \in \mathcal{A},\ 1 \leqq r \leqq n \tag{3.1}$$

$$u_i (F_i - qH_i)(T) \geqq 0,\ 1 \leqq i \leqq p \tag{3.2}$$

$$\sum_{j=1}^{m} v_j G_j(T) + \sum_{k=1}^{s} w_k Q_k(T) \geqq 0 \tag{3.3}$$

$$u \in \mathbf{R}_+^p,\ \sum_{i=1}^{p} u_i = 1,\ v \in \mathbf{R}_+^m \text{ and } w \in \mathbf{R}^s,\ q \in \mathbf{R}_+,\ T \in \mathcal{A}^n \tag{3.4}$$

Theorem 3.1 (Weak Duality) Let S and (T,u,v,w,q) be arbitrary feasible solutions of (P) and (D) respectively. Further, assume that

(i) $\sum_{i=1}^{p} u_i (F_i - qH_i)$ is $(\Im, \rho_1, \theta)$-pseudoconvex

(ii) $\sum_{j=1}^{m} v_j G_j + \sum_{k=1}^{s} w_k Q_k$ is $(\Im, \rho_2, \theta)$-quasiconvex

(iii) $\rho_1 + \rho_2 \geqq 0$

Then, $\max_{1 \leqq i \leqq p} (F_i(S)/H_i(S)) \geqq q$.

Proof. Suppose on the contrary $\max_{1 \leqq i \leqq p} (F_i(S)/H_i(S)) < q$

which implies that $F_i(S) - q H_i(S) < 0, \quad \forall\ 1 \leqq i \leqq p$.

From (3.4), we have $u \in \mathbf{R}_+^p$ with $\sum_{i=1}^{p} u_i = 1$, which ensures existence of atleast one $u_i > 0$, $1 \leqq i \leqq p$. Therefore, by multiplying the above inequalities by u_i, $1 \leqq i \leqq p$, summing and using (3.2), we get

$$\sum_{i=1}^{p} u_i (F_i - qH_i)(S) < \sum_{i=1}^{p} u_i (F_i - qH_i)(T)$$

By $(\Im, \rho_1, \theta)$-pseudoconvexity hypothesis, the above inequality implies

$$\Im(S,T; u^t D(F - qH)_T) + \rho_1 d^2(\theta(S,T)) < 0 \tag{3.5}$$

Moreover, feasibility of S for (P) implies

$$G_j(S) \leqq 0, \ 1 \leqq j \leqq m$$

$$Q_k(S) = 0, \ 1 \leqq k \leqq s$$

Since $v_j \geqq 0$, $1 \leqq j \leqq m$, so we obtain

$$\sum_{j=1}^{m} v_j G_j(S) + \sum_{k=1}^{s} w_k Q_k(S) \leqq 0$$

The above inequality along with (3.3) yields

$$\sum_{j=1}^{m} v_j G_j(S) + \sum_{k=1}^{s} w_k Q_k(S) \leqq \sum_{j=1}^{m} v_j G_j(T) + \sum_{k=1}^{s} w_k Q_k(T)$$

Applying $(\Im, \rho_2, \theta)$-quasiconvexity of $\sum_{j=1}^{m} v_j G_j + \sum_{k=1}^{s} w_k Q_k$, we get

$$\Im(S,T; v^t DG_T + w^t DQ_T) + \rho_2 d^2(\theta(S,T)) < 0 \tag{3.6}$$

From (3.5) and (3.6) it follows that

$$\Im(S,T; u^t D(F - qH)_T + v^t DG_T + w^t DQ_T) + (\rho_1 + \rho_2) d^2(\theta(S,T)) < 0$$

which in view of assumption (iii) implies

$$\Im(S,T; u^t D(F - qH)_T + v^t DG_T + w^t DQ_T) < 0 \tag{3.7}$$

From (3.1) and the fact that $\Im$ satisfies condition **(C)**, we have

$$\Im(S,T; u^t D(F - qH)_T + v^t DG_T + w^t DQ_T) \geqq 0 \quad \forall\, S \in \boldsymbol{A}^n$$

which contradicts (3.7).

Hence the result.

Remark 3.1. Theorem 3.1 also holds good under other conditions as stated below.

Theorem 3.2 (Weak Duality) Let S and (T,u,v,w,q) be arbitrary feasible solutions of (P) and (D) respectively. Further, if

(i) $\sum_{i=1}^{p} u_i (F_i - qH_i)$ is $(\Im, \rho_1, \theta)$-quasiconvex,

(ii) $\sum_{j=1}^{m} v_j G_j + \sum_{k=1}^{s} w_k Q_k$ is strictly $(\Im, \rho_2, \theta)$-pseudoconvex and

(iii) $\rho_1 + \rho_2 \geqq 0$

'OR'

(i) $\sum_{i=1}^{p} u_i (F_i - qH_i)$ is $(\Im, \rho_1, \theta)$-quasiconvex,

(ii) $\sum_{j=1}^{m} v_j G_j + \sum_{k=1}^{s} w_k Q_k$ is $(\Im, \rho_2, \theta)$-quasiconvex and

(iii) $\rho_1 + \rho_2 > 0$

'OR'

$\sum_{i=1}^{p} u_i (F_i - q H_i) + \sum_{j=1}^{m} v_j G_j + \sum_{k=1}^{s} w_k Q_k$ is $(\Im, \rho, \theta)$-pseudoconvex with $\rho \geqq 0$.

Then, $\max_{1 \leqq i \leqq p} (F_i(S)/H_i(S)) \geqq q$.

Theorem 3.3 (Strong Duality) Let S^* be a regular optimal solution of (P). Then there exist $u^* \in \mathbf{R}_+^p$, $\sum_{i=1}^{p} u_i^* = 1$, $v^* \in \mathbf{R}_+^m$, $w^* \in \mathbf{R}^s$ and $q^* \in \mathbf{R}_+$ such that $(S^*, u^*, v^*, w^*, q^*)$ is feasible for (D). Further, if the conditions of any one of the Weak Duality theorems hold, then $(S^*, u^*, v^*, w^*, q^*)$ is an optimal solution of (D), and $\max_{1 \leqq i \leqq p} (F_i(S^*)/H_i(S^*)) = q^*$.

Proof. Since S^* is a regular optimal solution of (P), therefore, by Theorem 2.1, there exist $u^* \in \mathbf{R}_+^p$, $v^* \in \mathbf{R}_+^m$, $w^* \in \mathbf{R}^s$, and $q^* = \max_{1 \leqq i \leqq p} (F_i(S^*)/H_i(S^*))$ such that conditions (2.3)-(2.5) hold.

Now, as S^* is feasible for (P), we also have $Q(S^*) = 0$, and therefore,

$$w_k^* Q_k(S^*) = 0, \ 1 \leqq k \leqq s$$

This along with (2.5) gives

$$\sum_{j=1}^{m} v_j^* G_j(S^*) + \sum_{k=1}^{s} w_k^* Q_k(S^*) = 0 \tag{3.8}$$

It follows from (2.3), (2.4) and (3.8) that $(S^*, u^*, v^*, w^*, q^*)$ is feasible for (D).

Optimality of $(S^*, u^*, v^*, w^*, q^*)$ follows from the Weak Duality theorem.

Theorem 3.4 (Strict Converse Duality) Let S^* be optimal for (P) with $q^* = \max\limits_{1 \leqq i \leqq p} (F_i(S^*)/H_i(S^*))$. Let (T,u,v,w,q^*) be optimal for (D). Further, if $u^t(F - q^* H)$ is strictly $(\Im,\rho_1,\theta)$-pseudoconvex, $v^t G + w^t Q$ is $(\Im,\rho_2,\theta)$-quasiconvex with $\rho_1 + \rho_2 \geqq 0$.

Then $S^* = T$.

Proof. We assume that $S^* \neq T$ and exhibit a contradiction.

Since $q^* = \max\limits_{1 \leqq i \leqq p} F_i(S^*)/H_i(S^*)$, therefore, we have

$$F_i(S^*) - q^* H_i(S^*) \leqq 0, \quad 1 \leqq i \leqq p.$$

Multiplying the above inequalities by $u_i \geqq 0$, $1 \leqq i \leqq p$, and adding, we get

$$u^t(F - q^* H)(S^*) \leqq 0$$

which along with (3.2) yields

$$u^t(F - q^* H)(S^*) \leqq u^t(F - q^* H)(T)$$

By strict $(\Im,\rho_1,\theta)$-psedoconvexity of $u^t(F - q^* H)$, it follows that

$$\Im(S^*,T;u^t D(F - q^* H)_T) + \rho_1 d^2(\theta(S^*,T)) < 0 \tag{3.9}$$

Also since S^* is feasible for (P), and $v_j \geqq 0$, $1 \leqq j \leqq m$, we have

$$\sum_{j=1}^{m} v_j G_j(S^*) + \sum_{k=1}^{s} w_k Q_k(S^*) \leqq 0$$

which together with (3.3) imply

$$v^t G(S^*) + w^t Q(S^*) \leqq v^t G(T) + w^t Q(T).$$

Using $(\Im,\rho_2,\theta)$-quasiconvexity of $v^t G + w^t Q$, we get

$$\Im(S^*,T;v^t DG_T + w^t DQ_T) + \rho_2 d^2(\theta(S^*,T)) \leqq 0 \tag{3.10}$$

Adding (3.9) and (3.10), and using sublinearity of $\Im$, we obtain

$$\Im(S^*,T;u^t D(F-q^* H)_T + v^t DG_T + w^t DQ_T) + (\rho_1+\rho_2)d^2(\theta(S^*,T)) < 0$$

Since $\rho_1 + \rho_2 \geqq 0$, the above inequality can be rewritten as

$$\Im(S^*,T;u^t D(F-q^* H)_T + v^t DG_T + w^t DQ_T) < 0 \qquad (3.11)$$

Now, from (3.1) and the fact that $\Im$ satisfy condition **(C)**, we have

$$\Im(S,T;u^t D(F-q^* H)_T + v^t DG_T + w^t DQ_T) \geqq 0 \qquad \forall\, S \in \boldsymbol{A}^n .$$

But this contradicts (3.11) for $S = S^*$.

This completes the proof.

Remark 3.2 Theorem 3.4 also holds good under other conditions as stated below.

Theorem 3.5 (Strict Converse Duality) Let S^* be an optimal solution of (P) with $q^* = \max_{1 \leqq i \leqq p} (F_i(S^*)/H_i(S^*))$. Let (T,u,v,w,q^*) be an optimal solution of (D). Further, if $u^t(F-q^*H)$ is $(\Im,\rho_1,\theta)$-quasiconvex, $v^tG + w^tQ$ is $(\Im,\rho_2,\theta)$-quasiconvex with $\rho_1 + \rho_2 > 0$

'OR'

$u^t(F-q^*H)$ is $(\Im,\rho_1,\theta)$-quasiconvex, $v^tG + w^tQ$ is strictly $(\Im,\rho_2,\theta)$-psedoconvex with $\rho_1 + \rho_2 \geqq 0$.

Then $S^* = T$.

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