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Common Coupled Fixed Point Theorems in V-Fuzzy Metric Spaces

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Abstract : In this paper we establish a coupled common fixed point theorem in V-fuzzy metric spaces. These spaces are further generalization of fuzzy metric spaces in the sense of George and Veeramani. The concept of W-compatibility is utilized here.

1. INTRODUCTION AND PRELIMINARIES

The purpose of the paper is to establish a common coupled fixed point theorem in V- fuzzy metric spaces. In 1965 Zadeh introduced the concept of fuzzy sets in his famous work in [15]. After that many eminent authors established many fixed point results using this concepts. Some references may be noted as [6, 7, 9, 13]. Fuzzy metric spaces are important generalizations of metric spaces. Fuzzy metric spaces have been introduced in different ways by many authors [3, 2, 10]. George and Veeramani [4] introduced the concept of fuzzy metric spaces modifying the work of Kramosil and Michalek [11] to introduce a Hausdroff topology on fuzzy metric space. Recently Gupta and Kanwar introduced V-fuzzy metric spaces in their work [8]. The concept is another generalization of fuzzy metric spaces .

Before we go to our main result, we recall some of the basic concepts and results which are discussed below.

Denition 1.1 (t – norm) [14] A binary operation* : $[0, 1] \times [0, 1] \to [0, 1]$ is a t – norm if it satises the following conditions:

1. $*(1, a) = a$, $*(0, 0) = 0$
2. $*(a, b) = *(b, a)$
3. $*(c, d) \geq (a, b)$ whenever $c \geq a$ and $d \geq b$
4. $*(*(a, b), c) = *(a, *(b, c))$ where $a, b, c, d \in [0, 1]$

Typical examples of t–norms are $a_{*1}\, b = \min\{a, b\}$, $a_{*2} = ab$ and $a_{*3}\, b = \text{Max}\{a + b - 1, 0\}$.

Denition 1.2 (Fuzzy Metric Space – George and Veeramani) [4] The 3-tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t–norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions:

1. $M(x, y, 0) > 0$,
2. $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, z, t + s) \geq (M(x, y, t) * M(y, z, s))$,
5. $M(x, y, .) : (0, \infty) \to [0, 1]$ is left- continuous,where $x, y, z \in X$ and $t, s > 0$.

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Example 1.1 [4] Let X = R. Let $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ $x, y \in X$,

let
$$M(x, y, t) = \frac{t}{t + |x - y|}$$

Then (R, M, *) is a fuzzy metric space.

Denition 1.3 [5] Let $(X, \preceq)$ be a partially ordered set and $F : X \to X$ be a mapping from X to itself. The mapping F is said to be non-decreasing if, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and non-increasing if, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \succeq F(x_2)$.

Denition 1.4 [5] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is non- decreasing in its rst argument and is non-increasing in its second argument, that is, if, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for fixed $y \in X$ and if, for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$, for fixed $x \in X$.

Denition 1.5 [5] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $h : X \to X$ be two mappings. The mapping F is said to have the mixed h-monotone property if F is monotone h-non-decreasing in its first argument and is monotone h-non-increasing in its second argument, that is, if, for all $x_1, x_2 \in X$, $hx_1 \preceq hx_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for all $y \in X$ and if, for all $y_1, y_2 \in X$, $hy_1 \preceq hy_2$ implies $F(x, y_1) \succeq F(x, y_2)$, for any $x \in X$.

Denition 1.6 [5] Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping if
$$F(x, y) = x, F(y, x) = y.$$

Further Lakshmikantham and –Ciric have introduced the concept of coupled coincidence point.

Denition 1.7 [12] Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and if
$$F(x, y) = hx, F(y, x) = hy.$$

If further $x = hx = F(x, y)$ and $y = hy = F(y, x)$ then (x, y) is a common coupled fixed point of h and F.

Denition 1.8 [1] Let X be a nonempty set. Mappings F and h are called W-compatible if
$$g\,(F\,(x, y)) = F\,(gx, gy)$$
whenever
$$g(x) = F(x, y) \text{ and } g(y) = F(y, x) \text{ for some } (x, y) \in X \times X.$$

Example 1.2 Define $P : X \times X \to X$ and $Q : X \to X$ where $X = [-1, 1]$, $P(x, y) = \frac{x^2 + y^2}{2}$, $Q(x) = x$,which satises $Q(P(x, y)) = P(Qx, Qy)$ and $Q(P(y, x)) = P(Qy, Qx)$. For $x = 1$ and $y = 1$,we get $P(x, y) = Q(x)$ and $P(y, x) = Q(y)$. This implies that the mappings $P : X \times X \to X$ and $Q : X \to X$ are W-compatible.

Denition 1.9 (V – Fuzzy Metric Space) [8] Let X be a nonempty set. A triplet (X, V, *) is said to be a V-fuzzy metric space (denoted by VF-space), where * is a continuous t-norm, and V is a fuzzy set on $X^n \times (0, \infty)$ satisfying the following conditions for all $t, s \geq 0$:

1. $V(x, x, x, \dots, x, y, t) \geq 0$ for all $x, y \in X$ with $x \neq y$,
2. $V(x_1, x_1, x_1, \dots, x_1, x_2, t) \geq V(x_1, x_2, x_3, \dots, x_n, t)$ for all $x_1, x_2, x_3, \dots, x_n \in X$ with $x_1 \neq x_2 \neq x_3 \dots \neq x_n$,
3. $V(x_1, x_2, x_3, \dots, x_n, t) = 1$ if and only if $x_1 = x_2 = x_3 = \dots = x_n$,
4. $V(x_1, x_2, x_3, \dots, x_n, t) = V(p(x_1, x_2, x_3, \dots, x_n), t)$ where p is a permutation function,
5. $V(x_1, x_2, x_3, \dots, x_{n-1}, t + s) \geq V(x_1, x_2, x_3, \dots, x_{n-1}\ l, t) * V(l, l, l, \dots, l, x_n, s)$,
6. $\lim_{t \to \infty} V(x_1, x_2, x_3, \dots, x_n, t) = 1$,

7. $V(x_1, x_2, x_3, \ldots, x_n,) : (0, \infty) \to [0, 1]$ is continuous.

Example 1.3 [8] Let (X, A) be an A-metric space. The t-norm $a * b = ab$ or $a * b = \min\{a, b\}$. For all $x_1, x_2, x_3, \ldots, x_n \in X$, $t > 0$, denote

$$V(x_1, x_2, x_3, \ldots, x_n, t) = \frac{t}{t + A(x_1, x_2, x_3, \ldots, x_n)}$$

Then (X, V, *) is a V -fuzzy metric space.

Lemma 1.1 [8] Let (X, V, *) be a V -fuzzy metric space. Then $V(x_1, x_2, x_3, \ldots, x_n, t)$ is nondecreasing with respect to t.

Lemma 1.2 [8] Let (X, V, *) be a V -fuzzy metric space such that

$$V(x_1, x_2, x_3, \ldots, x_n, kt) \geq V(x_1, x_2, x_3, \ldots, x_n, t)$$

with $k \in (0, 1)$. Then $x_1 = x_2 = x_3 = \ldots = x_n$.

Denition 1.10 [8] Let (X, V, *) be a V -fuzzy metric space. A sequence $\{x_r\}$ is said to converge to a point $x \in X$ if $V(x_r, x_r, x_r, \ldots, x_r, x, t) \to 1$ as $r \to \infty$ for all $t > 0$, that is , for each $\varepsilon > 0$, there exists $n \in N$ such that for all $r \geq N$, we have $V(x_r, x_r, x_r, \ldots, x_r, x, t) > 1 - \varepsilon$, and we write $\lim_{n \to \infty} x_r = x$.

Denition 1.11 [8] Let (X, V, *) be a V -fuzzy metric space. A sequence $\{x_r\}$ is said to be a V -Cauchy sequence if $V(x_r, x_r, x_r, \ldots, x_r, x_q, t) \to 1$ as $r, q \to \infty$ for all $t > 0$. In otherwords, for each $\in > 0$, there exists $n_0 \in N$ such that for all $r, q \geq n_0$, we have $V(x_r, x_r, x_r, \ldots, x_r, x_q, t) > 1 - \in$.

Denition 1.12 [8] The V -fuzzy metric space (X, V, *) is said to be V -complete if every Cauchy sequence in X is convergent.

Now we give our main result.

2. MAIN RESULT

Lemma 2.1 Let (X, V, *) be a V -fuzzy metric space. If there exists $k \in (0, 1)$ such that $\min\{V(x_1, x_2, x_3, \ldots, x_n, kt), (y_1, y_2, y_3, \ldots, y_n, kt)\} \geq \min\{V(x_1, x_2, x_3, \ldots, x_n, t), (y_1, y_2, y_3, \ldots, y_n, t)\}$ for all $x_1, x_2, x_3, \ldots, x_n, y_1, y_2, y_3, \ldots, y_n \in X$ and $t > 0$ then $x_1 = x_2 = x_3 = \ldots = x_n$ and $y_1 = y_2 = y_3 = \ldots = y_n$.

Proof : Using lemma 1.2 we can prove this lemma.

2.1. Main Theorem

Let (X, V, *) be a V-fuzzy metric space with $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $F: X \times X \to X$, $g : X \to X$ be mappings satisfying

$$V(F(x, y), F(u, v), F(u, v), \ldots, F(u, v), kt) \geq \min\{V(gx, gu, gu, \ldots, gu, t), V(gy, gv, gv, \ldots, gv, t)\} \quad (2.1)$$

for all $x, y, u, v \in X$, where $0 < k < 1$, $F(X \times X) \subset g(X)$ is a complete subspace of X and the pair (g, F) is W-compatible. Then F and g have unique common fixed point.

Proof : Let $x_0, y_0 \in X$, denote $Z_n = F(x_n, y_n) = gx_{n+1}$ and $p_n = F(y_n, x_n) = gy_{n+1}$, $n = 0, 1, 2, \ldots$

Let $r_n(t) = V(z_n, z_{n+1}, z_{n+1}, \ldots, z_{n+1}, t)$

and $s_n(t) = V(p_n, p_{n+1}, p_{n+1}, \ldots, p_{n+1}, t)$

2Now,
$$\begin{aligned} r_{n+1}(kt) &= V(Z_{n+1}, Z_{n+2}, Z_{n+2}, \ldots, Z_{n+2}, kt) \\ &= V(F(x_{n+1}, y_{n+1}), F(x_{n+2}, y_{n+2}), \ldots, F(x_{n+2}, y_{n+2}), kt) \\ &\geq \min\{V(gx_{n+1}, gx_{n+2}, gx_{n+2}, \ldots, gx_{n+2}, t), \\ &\qquad V(gy_{n+1}, gy_{n+2}, gy_{n+2}, \ldots, gy_{n+2}, t)\} \quad \text{[Using 2. 1]} \\ &= \min\{V(Z_n, Z_{n+1}, Z_{n+1}, \ldots, Z_{n+1}, t), V(p_n, p_{n+1}, p_{n+1}, \ldots, p_{n+1}, t)\} \\ &= \{r_n(t), s_n(t)\} \end{aligned}$$

Therefore,

$$r_{n+1}(kt) \geq \min\{r_n(t), s_n(t)\}. \tag{2.2}$$

Also, $s_{n+1}(kt) = V(p_{n+1}, p_{n+2}, p_{n+2}, \dots, p_{n+2}, kt)$

$$\begin{aligned} &= V(F(y_{n+1}, x_{n+1}), F(y_{n+2}, x_{n+2}), \dots, F(y_{n+2}, x_{n+2}), kt) \\ &\geq \min\{V(gy_{n+1}, gy_{n+2}, gy_{n+2}, \dots, y_{n+2}, t), V(gx_{n+1}, gx_{n+2}, gx_{n+2}, \dots, \\ &\qquad gx_{n+2}, t)\} \text{ [Using 2:1]} \\ &= \min\{V(p_n, p_{n+1}, p_{n+1}, \dots, p_{n+1}, t), V(Z_n, Z_{n+1}, Z_{n+1}, \dots, Z_{n+1}, t)\} \\ &= \{s_n(t), r_n(t)\}: \end{aligned}$$

Therefore,

$$s_{n+1}(kt) \geq \min\{s_n(t), r_n(t)\}. \tag{2.3}$$

Thus, $\min\{r_{n+1}(kt), s_{n+1}(kt)\} \geq \min\{r_n(t), s_n(t)\}$.

Hence,

$$\begin{aligned} \min\{r_n(t), s_n(t)\} &\geq \min\left\{r_{n-1}\left(\frac{1}{k}\right), s_{n-1}\left(\frac{t}{k}\right)\right\} \\ &\geq \min\left\{r_{n-2}\left(\frac{1}{k}\right), s_{n-2}\left(\frac{t}{k}\right)\right\} \\ &\geq \min\left\{r_0\left(\frac{1}{k^n}\right), s_0\left(\frac{t}{k^n}\right)\right\}. \end{aligned}$$

So,

$$\min\{r_n(t), s_n(t)\} \geq \min\left\{V\left(z_0, z_1, z_1, \dots, z_1, \frac{t}{k^n}\right), V\left(p_0, p_1, p_1, \dots, p_1, \frac{t}{k^n}\right)\right\}. \tag{2.4}$$

For any positive integer n and xed positive integer p, we have

$$\begin{aligned} V(z_n, z_{n+p}, z_{n+p}, \dots z_{n+p}, t) &\geq V\left(z_{n+p-1}, z_{n+p}, z_{n+p}, \dots z_{n+p}, \frac{t}{p}\right) \\ &* V\left(z_{n+p-2}, z_{n+p-1}, z_{n+p-1}, \dots z_{n+p-1}, \frac{t}{p}\right) \\ &* \dots * V\left(z_n, z_{n+1}, z_{n+1}, \dots z_{n+1}, \frac{t}{p}\right) \\ &\geq \min\left\{V\left(z_0, z_1, z_1, \dots, z_1, \frac{t}{pk^{n+p-1}}\right), V\left(p_0, p_1, p_1, \dots, p_1, \frac{t}{pk^{n+p-1}}\right)\right\} \\ &* \min\left\{V\left(z_0, z_1, z_1, \dots, z_1, \frac{t}{pk^{n+p-2}}\right), V\left(p_0, p_1, p_1, \dots, p_1, \frac{t}{pk^{n+p-2}}\right)\right\} \\ &* \dots * \min\left\{V\left(z_0, z_1, z_1, \dots, z_1, \frac{t}{pk^n}\right), V\left(p_0, p_1, p_1, \dots, p_1, \frac{t}{pk^n}\right)\right\} \end{aligned} \tag{2.5}$$

Letting $n \to \infty$ and using the axiom (6) from the denition of V-fuzzy metric space we get,

$$\lim_{n\to\infty} V(Z_n, Z_{n+p}, Z_{n+p}, \dots, Z_{n+p}, t) \geq 1 * 1 * 1 \dots * 1 = 1. \tag{2.6}$$

Hence,

$$\lim_{n\to\infty} V(Z_n, Z_{n+p}, Z_{n+p}, \dots, Z_{n+p}, t) = 1:$$

Thus $\{Z_n\}$ is V -Cauchy sequence in X. Similarly we can show that $\{p_n\}$ is also a V - Cauchy sequence in X. Since $g(X)$ is V -complete, $\{Z_n\}$ and $\{p_n\}$ converge to some u and v in $g(X)$ respectively. Hence there exist x and y in X such that $u = gx$ and $v = gy$.

Now,
$$\begin{aligned} & V(Z_n, F(x, y), F(x, y), \ldots, F(x, y), kt) \\ = \; & V(F(x_n, y_n), F(x, y), F(x, y), \ldots F(x, y), kt) \\ \geq \; & \min\{V(gx_n, gx, gx, \ldots, gx, t), V(gy_n, gy, gy, \ldots, gy, t)\} \\ = \; & \min\{V(Z_{n-1}, gx, gx, \ldots, gx, t), V(p_{n-1}, gy, gy, \ldots, gy, t)\} \end{aligned} \tag{2.7}$$

Taking $n \to \infty$, we get
$$V(gx, F(x, y), F(x, y), \ldots, F(x, y), kt) \geq \min\{1, 1\} = 1. \tag{2.8}$$

Hence, $F(x, y) = gx$. Similarly we can show that $F(y, x) = gy$. So,
$$\begin{aligned} gu &= ggx \\ &= g(F(x, y)) \\ &= F(gx, gy) \qquad \text{[as } (g, F) \text{ are W-compatible]} \\ &= F(u, v). \end{aligned}$$
and
$$\begin{aligned} gv &= ggy \\ &= g(F(y, x)) \\ &= F(gy, gx) \qquad \text{[as } (g, F) \text{ are W-compatible]} \\ &= F(v, u). \end{aligned}$$

So, $(u, v) \in X \times X$ is a coupled coincidence point of F and g. To show that (u, v) is also a common coupled xed point of F and g.

Now,
$$\begin{aligned} V(Z_n, gu, gu, \ldots, gu, kt) &= V(F(x_n, y_n), F(u, v), F(u, v), \ldots, F(u, v), kt) \\ &\geq \min\{V(gx_n, gu, gu, \ldots, gu, t), V(gy_n, gv, gv, \ldots, gv, t)\} \end{aligned} \tag{2.9}$$

Taking $n \to \infty$ in above eqn,
$$V(u, gu, gu, \ldots, gu, kt) \geq \min\{V(u, gu, gu, \ldots, gu, t), V(v, gv, gv, \ldots, gv, t)\} \tag{2.10}$$

Similarly we can show that
$$V(v, gv, gv, \ldots, gv, kt) \geq \min\{V(v, gv, gv, \ldots, gv, t), V(u, gu, gu, \ldots, gu, t)\} \tag{2.11}$$

From (2.10) and (2.11)
$$\begin{aligned} & \min\{V(u, gu, gu, \ldots, gu, kt), V(v, gv, gv, \ldots, gv, kt)\} \\ \geq \; & \min\{V(u, gu, gu, \ldots, gu, t), V(v, gv, gv, \ldots, gv, t)\} \end{aligned}$$

So by lemma 2.1 $u = gu$ and $v = gv$. Therefore, $u = gu = F(u, v)$ and $v = gv = F(v, u)$. Hence (u, v) is a common coupled fixed point of F and g. For uniqueness , let (u_1, v_1) be another common coupled xed point of F and g. So,
$$\begin{aligned} V(u, u_1, u_1, \ldots, u_1, kt) &= V(F(u, v), F(u_1, v_1), F(u_1, v_1), \ldots, F(u_1, v_1), kt) \\ &\geq \min\{V(u, u_1, u_1, \ldots, u_1, t), V(v, v_1, v_1, \ldots, v_1, t)\} \end{aligned} \tag{2.12}$$

Similarly,
$$V(v, v_1, v_1, \ldots, v_1, kt) \geq \min\{V(v, v_1, v_1, \ldots, v_1, t), V(u, u_1, u_1, \ldots, u_1, t)\} \tag{2.13}$$

Thus from (2.12) and (2.13),
$$\min\{V(u, u_1, u_1, \ldots, u_1, kt), V(v, v_1, v_1, \ldots, v_1, kt)\} \geq \min\{V(u, u_1, u_1, \ldots, u_1\ t), V(v, v_1, v_1, \ldots, v_1, t)\}$$

From lemma 2.1 $u_1 = u$ and $v_1 = v$.

Hence (u, v) is the unique common coupled fixed point of F and g.

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