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EXTENSIONS OF THE HITSUDA–SKOROKHOD INTEGRAL

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Abstract. We present alternative definitions of the stochastic integral introduced by Ayed and Kuo and of the Hitsuda–Skorokhod integral extended to domains in L^p-spaces, $p \geq 1$. Our approach is motivated by the S-transform characterization of the Hitsuda–Skorokhod integral and based on simple processes of stochastic exponential type. We prove that the new stochastic integral extends the mentioned stochastic integrals above and we outline their connection.

1. Introduction

A series of articles starting from the work by Ayed and Kuo [1, 2] establishes a new stochastic integral with respect to a Brownian motion extending the Itô integral to nonadapted processes. Let $B = (B_t)_{t\geq 0}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, where the σ-field $\mathcal{F}$ is generated by the Brownian motion and completed by null sets. The augmented Brownian filtration is denoted by $(\mathcal{F}_t)_{t\geq 0}$.

The construction of the Ayed–Kuo stochastic integral on a compact interval $[a,b] \subset [0,\infty)$ is based on the consideration of the adapted part $(f(t))_{t\in[a,b]}$, and the remaining part $(\varphi(t))_{t\in[a,b]}$, the so-called instantly independent part, of the integrand, which means that the element $\varphi(t)$ is independent of $\mathcal{F}_t$ for every $t \in [a,b]$. This decomposition of the integrand allows a simple construction of this Ayed–Kuo integral for continuous integrands $f\varphi$ via a Riemann-sum approach, see e.g. [1, 2, 10], similar to the Riemann-sum approach for the general Itô integral (cf. [13, Section 5.3]). In particular, the Ayed–Kuo integral allows to integrate nonadapted processes $f\varphi \notin L^2(\Omega \times [a,b])$.

A classical extension of the Itô integral to nonadapted integrands is the Hitsuda–Skorokhod integral, see e.g. [9, 15]. A Riemann-sum approach to the Hitsuda–Skorokhod integral is given in [16]. However, the Hitsuda–Skorokhod integral can be introduced by various simpler approaches, like Wiener chaos expansions [9], as the adjoint of the Malliavin derivative [15] or from the more general white noise integrals [12]. Whereas these approaches mostly depend on the Hilbert space structure of the space containing the integrands, there exist characterizations of the Hitsuda–Skorokhod integral via transforms, see e.g. [11].

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One of these characterizations is as follows. We denote by $I(f)$ the Wiener integral of $f \in L^2([0,\infty))$ and by $\|\cdot\|$ the norm on this space. The process $u \in L^2(\Omega \times [0,\infty))$ is Skorokhod-integrable, if there exists an element $X \in L^2(\Omega)$ such that for all $f \in L^2([0,\infty))$,

$$\mathbb{E}[Xe^{I(f)-\|f\|^2/2}] = \int_0^\infty \mathbb{E}[u_s e^{I(f)-\|f\|^2/2}]f(s)ds.$$

Then $\delta(u) := X$ is the Hitsuda–Skorokhod integral. This characterization is mainly based on the fact that the linear span of the set $\{e^{I(f)-\|f\|^2/2}, f \in L^2([0,\infty))\}$ is dense in $L^2(\Omega)$.

Such transform characterizations of the Hitsuda–Skorokhod integral can be extended beyond integrands in $L^2(\Omega \times [0,\infty))$, see e.g [11, Chapter 16].

Due to the very different introductions, the relation between the Ayed–Kuo integral and the Hitsuda–Skorokhod integral is not clarified yet and stated as an open problem in the survey article on the Ayed–Kuo integral [14].

In this note we prove that the Ayed–Kuo integral equals the Hitsuda–Skorokhod integral on Skorokhod integrable processes in $L^2(\Omega\times[0,\infty))$. Moreover we propose alternative definitions of stochastic integrals which extend both, the Ayed–Kuo integral and the Hitsuda–Skorokhod integral to domains in $L^p(\Omega \times [0,\infty))$ for all $p \geq 1$.

Our approach imitates the characterization of the Hitsuda–Skorokhod integral. Firstly, we define the stochastic integral on the set of simple processes

$$u_s = e^{I(g)-\|g\|^2/2}h(s)$$

for simple deterministic functions $g, h : [0,\infty) \to \mathbb{R}$. Then, the stochastic integral is extended by linearity and density arguments to different domains. As the integrals of these nonadapted processes, defined via

$$e^{I(g)-\|g\|^2/2}\left(I(h) - \int_0^\infty g(s)h(s)ds\right),$$

are elements in $L^1(\Omega \times [0,\infty))$ and extend the Itô integral elementary, the object of interest in the extension of the integral are the topologies involved. We observe that the Ayed–Kuo integral is extended via a version of an almost-sure-probability closedness which is more general than the $L^p(\Omega \times [0,\infty))$-$L^p(\Omega)$-closedness of the general Hitsuda–Skorokhod integral.

The article is organized as follows. In Section 2 we show that already a subclass of elementary integrands of the Ayed–Kuo integral is in the domain of the Hitsuda-Skorokhod integral and both stochastic integrals then coincide.

In Section 3 we propose alternative definitions of the extended Hitsuda-Skorokhod integral and of the Ayed–Kuo integral to domains in $L^p(\Omega \times [0,\infty))$ for $p \geq 1$. In our main result we prove that the new definition of the Ayed–Kuo integral extends the definition in [14, 10] as well as the extended Hitsuda–Skorokhod integral.

2. Starting from the Ayed–Kuo Stochastic Integral

Firstly, we introduce the *elementary Ayed–Kuo integral.* The following definition is taken from [10].

Definition 2.1. Let $[a,b] \subset [0,\infty)$. Suppose an adapted continuous stochastic process $(f(t))_{t\in[a,b]}$ and an instantly independent continuous process $(\varphi(t))_{t\in[a,b]}$, i.e. for all $t \in [a,b]$, $\varphi(t)$ is independent of $\mathcal{F}_t$. Then, provided the limit on the right hand side exists in probability, the *Ayed–Kuo integral* is given by

$$\bar{I}(f\varphi) := \int_a^b f(t)\varphi(t)dB_t = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n f(t_{i-1})\varphi(t_i)\Delta B_i, \tag{2.1}$$

where $\Delta_n = \{a = t_0 < t_1 < \cdots < t_n = b\}$ with $\|\Delta_n\| = \max_{i=1,\ldots,n} |t_i - t_{i-1}|$ and $\Delta B_i := B_{t_i} - B_{t_{i-1}}$.

It is clear that the elementary Ayed–Kuo integral extends the Itô integral to nonadapted integrands and exhibits the zero mean property. Moreover, the integrands are not necessary in $L^2(\Omega \times [a,b], \mathcal{F} \otimes \mathcal{B}([a,b]), P \otimes \lambda_{[a,b]})$.

A classical extension of the Itô integral to nonadapted integrands with the additive noise structure is the Hitsuda–Skorokhod integral. There are several approaches for the introduction of this stochastic integral. Essentially, these are via Wiener-Itô chaos expansion, as the adjoint operator of the Malliavin derivative (cf. [15, 1.3], [11, 7.3]) or via the following transform characterizations motivated by white noise integrals, compare [11, Theorem 16.46, Theorem 16.50].

We denote the Wiener integral of a function $f \in L^2([0,\infty))$ by $I(f)$, which is the continuous linear extension of the mapping $1_{(0,t]} \mapsto B_t$ from $L^2([0,\infty))$ to $L^2(\Omega) := L^2(\Omega, \mathcal{F}, P)$. Moreover, we denote the norm and inner product on $L^2([0,\infty))$ by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ and by $\mathcal{E}$ the set of step functions on left half-open intervals, i.e., functions of the form

$$g(x) = \sum_{j=1}^m a_j 1_{(b_j,c_j]}(x), \quad m \in \mathbb{N}, a_j \in \mathbb{R}, 0 \le b_j < c_j.$$

For every $X \in L^2(\Omega)$ and $f \in L^2([0,\infty))$, the *S-transform* of X at f is defined as

$$(SX)(f) := \mathbb{E}[X \exp\big(I(f) - \|f\|^2/2\big)].$$

As the set of stochastic exponentials, also known as Wick exponentials,

$$\exp^\diamond(I(g)) := \exp\big(I(g) - \|g\|^2/2\big) \ , \quad g \in \mathcal{E} \tag{2.2}$$

is a total set in $L^2(\Omega)$ (see e.g. [11, Corollary 3.40]), every random variable in $L^2(\Omega)$ is uniquely determined by its S-transform on $\mathcal{E}$, i.e. for $X, Y \in L^2(\Omega)$, $\forall g \in \mathcal{E} : (SX)(g) = (SY)(g)$ implies $X = Y$ P-almost surely. The S-transform is a continuous and injective function on $L^2(\Omega)$ (see e.g. [11, Chapter 16] for more details). As an example, for $f, g \in L^2([0,\infty))$, we have $(S \ \exp\big(I(f) - \|f\|^2/2\big))(g) = \exp(\langle f, g\rangle)$. We denote for every $p \ge 1$,

$$L^p(\Omega \times [0,\infty)) := L^p(\Omega \times [0,\infty), \mathcal{F} \otimes \mathcal{B}([0,\infty)), P \otimes \lambda_{[0,\infty)}).$$

The characterization of random variables via the S-transform can be used to introduce the Hitsuda–Skorokhod integral, compare [11, Theorem 16.46, Theorem 16.50]:

Definition 2.2. The process $u = (u_t)_{t\geq 0} \in L^2(\Omega \times [0,\infty))$ is said to belong to the domain $D(\delta)$ of the Hitsuda–Skorokhod integral, if there is an $X \in L^2(\Omega)$ such that for every $g \in \mathcal{E}$ we have

$$(SX)(g) = \int_0^\infty (Su_t)(g)g(t)dt.$$

In this case, X is uniquely determined and $\delta(u) := X$ is called the *Hitsuda–Skorokhod integral of* u.

Moreover, the characterization via S-transform enables us to introduce stochastic integrals with respect to processes beyond semimartingales, like fractional Brownian motion, see e.g. [3].

The S-transform is closely related to a product imitating uncorrelated random variables as $\mathbb{E}[X \diamond Y] = \mathbb{E}[X]\mathbb{E}[Y]$, which is implicitly contained in the Hitsuda–Skorokhod integral and a fundamental tool in stochastic analysis. Due to the injectivity of the S-transform, the *Wick product* can be introduced via

$$\forall f \in L^2([0,\infty)) \ : \ S(X \diamond Y)(f) = (SX)(f)(SY)(f)$$

on a dense subset in $L^2(\Omega) \times L^2(\Omega)$. For more details on Wick product we refer to [9, 11, 12]. In particular, for a Wiener Integral $I(f)$, Hermite polynomials play the role of monomials in standard calculus as $I(f)^{\diamond k} = h^k_{\|f\|^2}(I(f))$ and the Wick exponential can be reformulated as

$$\exp^\diamond(I(f)) = \sum_{k=0}^\infty \frac{1}{k!} I(f)^{\diamond k}. \tag{2.3}$$

Now we prove that the elementary Ayed–Kuo integral in Definition 2.1 extends the Hitsuda–Skorokhod integral in Definition 2.2 in the setting of Definition 2.1.

Theorem 2.3. *Suppose* $[a,b] \subset [0,\infty)$*, an adapted* L^2*-continuous stochastic process* f *and an instantly independent* L^2*-continuous process* φ *such that the sequence*

$$\sum_{i=1}^n f(t_{i-1})\varphi(t_i)\Delta B_i,$$

converges strongly in $L^2(\Omega)$ *as* $\|\Delta_n\|$ *tends to zero (which are stricter assumptions than in Definition 2.1). Then the limit* $\bar{I}(f\varphi)$ *equals the Hitsuda–Skorokhod integral* $\delta(f\varphi)$*.*

The proof is reduced to the following characterization of strong convergence in terms of the S-transform. We will need

$$\mathbb{E}[(e^{\diamond I(g)})^p] = \exp\left(\frac{p^2 - p}{2}\|g\|^2\right) \tag{2.4}$$

for all $p > 0$, see e.g. [11, Cor. 3.38].

Proposition 2.4. *Suppose* $X, X_n \in L^2(\Omega)$ *for every* $n \in \mathbb{N}$ *and* $\mathbb{E}[(X_n)^2] \to \mathbb{E}[X^2]$*. Then the following assertions are equivalent as* n *tends to infinity:*

(i) $X_n \to X$ *strongly in* $L^2(\Omega)$.
(ii) $(SX_n)(g) \to (SX)(g)$ *for every* $g \in \mathcal{E}$.

Proof. *(i)* $\Rightarrow$ *(ii)*: This is a direct consequence of the L^2-continuity of the S-transform by the Cauchy-Schwarz inequality and (2.4),

$$\begin{aligned}|(SX_n)(g) - (SX)(g)| &= |\mathbb{E}[(X_n - X)\exp^{\diamond}(I(g))]| \\ &\leq \mathbb{E}[(X_n - X)^2]^{1/2}\exp(\|g\|^2/2) \to 0. \qquad (2.5)\end{aligned}$$

(ii) $\Rightarrow$ *(i)*: Thanks to the total set of Wick exponentials (2.2) and [17, Theorem V. 1.3], X_n converges weakly to X in the Hilbert space $L^2(\Omega)$. As the norms converge, we conclude strong convergence as well. □

Proof of Theorem 2.3. The Wiener-Itô chaos expansion of the $\mathcal{F}_t$-measurable random variables $f(t)$ must be of the form

$$f(t) = \sum_{k\geq 0} I^k(f^k(t_1, \ldots, t_k, t)),$$

where the integrands are symmetric functions in $L^2([a,b]^{k+1};\mathbb{R})$ with

$$f^k(t_1, \ldots, t_k, t) = 1_{\{t_1\vee\cdots\vee t_k\leq t\}} f^k(t_1, \ldots, t_k, t),$$

(cf. [9, Lemma 2.5.2]). Similarly, due to the instant independence, the Wiener chaos expansion of $\varphi(t)$ is given by

$$\varphi(t) = \sum_{k\geq 0} I^k(f^k_\varphi(t_1, \ldots, t_k, t)),$$

where the integrands are

$$f^k_\varphi(t_1, \ldots, t_k, t) = 1_{\{t_1\wedge\cdots\wedge t_k\geq t\}} f^k_\varphi(t_1, \ldots, t_k, t)$$

(see e.g. the projection as an influence on the integrands in [15, Lemma 1.2.5]). Hence, all chaoses of $f(t)$ and $\varphi(t)$ are based on disjoint increments of the underlying Brownian motion and we conclude by [9, Proposition 2.4.2] for all partitions

$$f(t_{i-1})\varphi(t_i)\Delta B_i = f(t_{i-1}) \diamond \varphi(t_i) \diamond \Delta B_i$$

and for all $t \in [a,b]$,

$$f(t)\varphi(t) = f(t) \diamond \varphi(t). \qquad (2.6)$$

Therefore, for every partition and $g \in \mathcal{E}$, we have

$$\begin{aligned}\left(S\left(\sum_{i=1}^n f(t_{i-1})\varphi(t_i)\Delta B_i\right)\right)(g) &= \sum_{i=1}^n \left(S(f(t_{i-1}) \diamond \varphi(t_i) \diamond \Delta B_i)\right)(g) \\ &= \sum_{i=1}^n (Sf(t_{i-1}))(g)(S\varphi(t_i))(g)(S\Delta B_i)(g) \\ &= \sum_{i=1}^n (Sf(t_{i-1}))(g)(S\varphi(t_i))(g)(g(t_i) - g(t_{i-1})).\end{aligned}$$

The L^2-continuity of the stochastic process f gives via (2.5)

$$|(S(f(x))(g) - (Sf(x_n))(g)| \leq \mathbb{E}[|f(x) - f(x_n)|^2]^{1/2}\exp(\|g\|^2/2) \to 0$$

as $x_n \to x$. Analogously we conclude the continuity of $(S\varphi(\cdot))(g)$ for every fixed g. Therefore, the function $(Sf(\cdot))(g)(S\varphi(\cdot))(g)g(\cdot)$ is piecewise continuous and Riemann-integrable and the Riemann sum $\sum_{i=1}^{n}(Sf(t_{i-1}))(g)(S\varphi(t_i))(g)(g(t_i) - g(t_{i-1}))$ converges for $\|\Delta_n\| \to 0$ to the Riemann integral

$$\int_a^b (Sf(t))(g)(S\varphi(t))(g)g(t)dt.$$

Thus, via (2.6) and Definition 2.2, for all $g \in \mathcal{E}$ we conclude,

$$\begin{aligned}\lim_{n\to\infty}\left(S\ \sum_{i=1}^{n} f(t_{i-1})\varphi(t_i)\Delta B_i\right)(g) &= \int_a^b (Sf(t))(g)(S\varphi(t))(g)g(t)dt \\ &= \int_a^b (Sf(t)\varphi(t))(g)g(t)dt = (S\,\delta(f\varphi))(g).\end{aligned}$$

Thanks to strong convergence in (2.1) and Proposition 2.4, we conclude $\bar{I}(f\varphi) = \delta(f\varphi)$. □

Remark 2.5. A simple example of an elementary Ayed–Kuo integral which is not a Hitsuda–Skorokhod integral is given by the integrand $f(t) = \exp(B_t^2)$, $\varphi(t) = \exp((B_1 - B_t)^2)$, i.e. the Ayed–Kuo integral

$$\bar{I}\left(\exp(2B_t^2 - 2B_tB_1 + B_1^2))_{t\in[0,1]}\right)$$

exists. Thanks to Fubini's theorem applied on a sum of nonnegative terms, $\frac{(2k-1)!!}{k!} = \frac{(2k)!}{2^k(k!)^2} = 2^k\binom{-1/2}{k}$ and Newton's binomial theorem, we have

$$\begin{aligned}\mathbb{E}[\exp(pB_t^2)] &= \sum_{k=0}^{\infty}\frac{p^k}{k!}\mathbb{E}[B_t^{2k}] = \sum_{k=0}^{\infty}\frac{(2k-1)!!}{k!}(pt)^k = \sum_{k=0}^{\infty}\binom{-1/2}{k}(2pt)^k \\ &= \frac{1}{\sqrt{1-2pt}}.\end{aligned}$$

Hence, $\exp(B_t^2) \in L^p(\Omega)$ if and only if $t \in [0, 1/(2p))$. Therefore, the integrand $\left(f(t)\varphi(t)1_{[0,1]}(t)\right)_{t\geq 0}$ is not in $D(\delta) \subset L^2(\Omega \times [0,\infty))$.

3. Extensions of the Stochastic Integrals

The example in Remark 2.5 indicates that the Ayed–Kuo integral extends the Hitsuda–Skorokhod integral due to the missing square-integrability.

However, this disadvantage can be removed by the following extension of the Hitsuda–Skorokhod integral.

For a measure space $(\Omega, \mathcal{A}, \mu)$, we denote by $L^0(\Omega, \mathcal{A}, P)$ the space $(\Omega, \mathcal{A}, \mu)$ with the topology of convergence in measure. Due to [11, Corollary 3.40], for every $p \geq 0$, we have that the set of Wick exponentials (2.2) is total in $L^p(\Omega, \mathcal{F}, P)$.

Moreover, as the set $\mathcal{E}$ is dense in $L^p([0,\infty), \mathcal{B}([0,\infty)), \lambda_{[0,\infty)})$ for every $p > 0$, we conclude via a straightforward argument on products of dense sets the crucial fact:

Proposition 3.1. *For every $p \geq 0$, the linear span*

$$\mathcal{E}^{Exp} := \text{lin}\left\{\exp^{\diamond}(I(g)) \otimes h\ :\ g, h \in \mathcal{E}\right\}.$$

is dense in the space $L^p(\Omega \times [0, \infty))$ *(as a topological vector space).*

Due to the totality of the Wick exponentials (2.2) in $L^p(\Omega, \mathcal{F}, P)$, the S-transform characterization of the Hitsuda–Skorokhod integral in Definition 2.2 can be extended to $L^p(\Omega \times [0, \infty))$ for $p > 1$, see e.g. [11, Section 16.4 and in particular Theorem 16.64]:

Definition 3.2. The process $u = (u_t)_{t \geq 0} \in L^p(\Omega \times [0, \infty))$ is said to belong to the domain $D(\delta)$ of the Hitsuda–Skorokhod integral, if there is an $X \in L^p(\Omega, \mathcal{F}, P)$ such that for every $g \in \mathcal{E}$

$$(SX)(g) = \int_0^\infty (Su_t)(g) g(t) dt.$$

In this case, X is uniquely determined and $\delta(u) := X$ is called the *L^p-Hitsuda–Skorokhod integral of u.*

Here, the condition $p > 1$ ensures that all integrals involved exist, i.e. are finite and characterize the random elements uniquely.

Moreover, this approach can be extended to $L^1(\Omega \times [0, \infty))$ via an further transform (cf. [11, Definition 16.52]).

We observe the straightforward extension of Proposition 2.4 to $p \in (1, \infty)$:

Proposition 3.3. *Suppose* $X, X_n \in L^p(\Omega)$ *for every* $n \in \mathbb{N}$ *and* $\mathbb{E}[(X_n)^p] \to \mathbb{E}[X^p]$. *Then the following assertions are equivalent as* n *tends to infinity:*

(i) $X_n \to X$ *strongly in* $L^p(\Omega)$.
(ii) $(SX_n)(g) \to (SX)(g)$ *for every* $g \in \mathcal{E}$.

Proof. $(i) \Rightarrow (ii)$: It is again a direct consequence of the L^p-continuity of the S-transform via Hölder-inequality and (2.4),

$$|(SX_n)(g) - (SX)(g)| \leq \mathbb{E}[(X_n - X)^p]^{1/p} \exp\left(\frac{\|g\|^2}{2(p-1)}\right) \to 0. \tag{3.1}$$

$(ii) \Rightarrow (i)$: X_n converges weakly to X in the normed space $L^p(\Omega)$ via [17, Theorem V. 1.3]. As the norms converge, we conclude via [6, Corollary 4.7.16] norm convergence in $L^p(\Omega)$ as well. □

Then we conclude similarly for every $p > 1$:

Theorem 3.4. *Suppose* $[a, b] \subset [0, \infty)$, *an adapted* L^p*-continuous stochastic process* f *and an instantly independent* L^p*-continuous process* φ *such that the sequence*

$$\sum_{i=1}^n f(t_{i-1}) \varphi(t_i) \Delta B_i,$$

converges strongly in $L^p(\Omega)$ *as* $\|\Delta_n\|$ *tends to zero. Then the limit* $\bar{I}(f\varphi)$ *equals the Hitsuda–Skorokhod integral* $\delta(f\varphi)$ *in Definition 3.2.*

The proof follows the lines of the proof of Theorem 2.3 making use of Proposition 3.3 and is omitted.

In a series of articles including [10], the elementary Ayed–Kuo integral is extended as follows:

Definition 3.5. Let $[a,b] \subset [0,\infty)$. Suppose a sequence Φ_n in the linear span of the integrands of the elementary Ayed–Kuo integral and a stochastic process $\Phi : \Omega \times [a,b] \to \mathbb{R}$ satisfying the condition, as n tends to infinity,

$$\int_a^b |\Phi_n(t) - \Phi(t)|^2 dt \to 0 \quad a.s.$$

Then, provided the limit on the right hand side exists in probability, the *Ayed–Kuo integral* of Φ is defined as

$$\bar{I}(\Phi) = \lim_{n\to\infty} \bar{I}(\Phi_n).$$

As the characterization via S-transform is essentially based on the elements in $\mathcal{E}^{Exp}$, here we define for every $p \geq 1$:

Definition 3.6. We define the *Hitsuda–Skorokhod integral* on the set $\mathcal{E}^{Exp}$ via

$$\delta(\exp^{\diamond}(I(g)) \otimes h) := \exp^{\diamond}(I(g))\,(I(h) - \langle g,h\rangle)$$

and linearity and extend it by L^p-closedness to the domain $D(\delta) \subset L^p(\Omega \times [0,\infty))$. By the L^p-closedness we obviously mean for a sequence of processes $(u^n)_{n\in\mathbb{N}} \subset L^p(\Omega \times [0,\infty))$ such that $u^n \to u$ strongly in $L^p(\Omega \times [0,\infty))$ and the existing Hitsuda–Skorokhod integrals $\delta(u^n)$ converge strongly towards X in $L^p(\Omega, \mathcal{F}, P)$, we define

$$\delta(u) := X.$$

Remark 3.7.

(i) The existence and uniqueness of the Hitsuda–Skorokhod integral in Definition 3.6 on the domain $D(\delta)$ is a simple consequence of the construction via the set $\mathcal{E}^{Exp}$ and the (P-almost sure)-uniqueness of the L^p-limits $\delta(u) := X$.

(ii) Suppose $g, h, v \in \mathcal{E}$. Via $\exp^{\diamond}(I(g))\exp^{\diamond}(I(v)) = e^{\langle g,v\rangle}\exp^{\diamond}(I(g+v))$ and the Wiener chaos expansion in (2.3), it is

$$\mathbb{E}[e^{\diamond I(g)} I(h) e^{\diamond I(v)}] = e^{\langle g,v\rangle}\mathbb{E}[e^{\diamond I(g+v)} I(h)] = e^{\langle g,v\rangle}\langle g+v, h\rangle$$

and therefore

$$\begin{aligned}\left(S\, e^{\diamond I(g)}\,(I(h) - \langle g,h\rangle)\right)(v) &= e^{\langle g,v\rangle}\,(\langle g+v,h\rangle - \langle g,h\rangle)\\ &= \int_0^\infty (Se^{\diamond I(g)})(v) h(s) v(s) ds. \qquad (3.2)\end{aligned}$$

Thus, Definition 3.6 equals Definition 3.2 on $\mathcal{E}^{Exp}$. Due to L^p-continuity of the S-transform (3.1) and Proposition 3.1, the equality follows on the domain $D(\delta)$ for $p > 1$. Moreover, the product type formula (3.2) yields the representation

$$\delta(e^{\diamond I(g)} \otimes h) = e^{\diamond I(g)} \diamond I(h).$$

(iii) Obviously, the S-transform characterization in Definition 3.2 cannot be applied on random elements in L^p for $p < 1$, where we are beyond Banach spaces.

Let us consider a simple example of a Hitsuda–Skorokhod integral in Definition 3.6. Let the nonadapted constant process

$$u = \exp(B_{1/3}^2) 1_{[0,1/3]}(t).$$

An approximation via elements in $\mathcal{E}^{Exp}$ can be obtained via the expansion $\exp(B_t^2) = \sum_{k=0}^{\infty} B_t^{2k}/k!$ and the approximation of monomials. This follows by the linear expansion of ordinary monomials B_t^{2k} in terms of Hermite polynomials $h_t^l(B_t)$, $l \in \{0, \dots, 2k\}$ and

$$\frac{\partial^l}{\partial w^l}\, e^{\diamond w B_t}\big|_{w=0} = h_t^l(B_t),$$

P-almost surely and in $L^p(\Omega)$, $p \geq 1$ (see e.g. [4]). We omit these technical reformulations and notice that it suffices to consider also the processes $v^n = \sum_{k=0}^{n} \frac{1}{k!} B_{1/3}^{2k} 1_{[0,1/3]}(t)$ instead of the approximating sequence in $\mathcal{E}^{Exp}$. These finite chaos elements exist in $L^1(\Omega \times [0,\infty))$. Thanks to $\delta(B_{1/3}^{2k} 1_{[0,1/3]}(t)) = B_{1/3}^{2k+1} - \frac{2k}{3} B_{1/3}^{2k-1}$ via S-transform (see e.g. the Skorokhod integration by parts formula in [8, Theorem 6.15]), we obtain the short representation

$$\delta\left(v^n\right) = \sum_{k=0}^{n} \frac{1}{k!} B_{1/3}^{2k+1} - \frac{2}{3} \sum_{k=1}^{n} \frac{k}{k!} B_{1/3}^{2k-1} = \frac{1}{3} \sum_{k=0}^{n} \frac{1}{k!} B_{1/3}^{2k+1}.$$

A simple calculution gives for all $t \geq 0$, $k \in \mathbb{N}$,

$$\mathbb{E}[|B_t^{2k+1}|] = \sqrt{2/\pi}(2k)!!t^{k+1/2}.$$

Hence, by the triangle inequality

$$\mathbb{E}[|\delta\left(v^n\right)|] \leq \frac{\sqrt{2/\pi}}{3} \sum_{k=0}^{n} \frac{(2k)!!}{k!} \frac{1}{3^{k+1/2}} = \frac{\sqrt{2/\pi}}{3^{3/2}} \sum_{k=0}^{n} (2/3)^k.$$

Thanks to dominated convergence we conclude $u \in D(\delta)$ with $p = 1$ (according to Definition 3.6) and

$$\delta(u) = \frac{1}{3} \sum_{k=0}^{\infty} \frac{1}{k!} B_{1/3}^{2k+1} = \frac{1}{3} B_{1/3} \exp(B_{1/3}^2).$$

As observed in Remark 2.5, $u \notin L^p(\Omega \times [0,\infty))$ for all $p \geq 3/2$ and therefore $u \notin D(\delta)$ for $p \geq 3/2$. Via Hölder-inequality and Remark 2.5 it follows that $u \in D(\delta)$ for $p \in [1, 3/2)$.

The dense set in Proposition 3.1 and the S-transform can be used to define the Malliavin derivative, see e.g. [5, Proposition 49]. Similarly to Definition 3.6, the following gives the Malliavin derivative on the Sobolev spaces $\mathbb{D}^{1,p}$, $p \geq 1$, in [15, 1.2]:

Definition 3.8. We define the *Malliavin derivative* on the (dense) set $\{e^{\diamond I(g)}, g \in \mathcal{E}\}$ via

$$D_t(e^{\diamond I(g)}) := e^{\diamond I(g)} g(t)$$

and extend it to the closable operator on the domain $\mathbb{D}^{1,p}$.

Following the extension in Definition 3.6, we introduce for every $p \geq 1$:

Definition 3.9. We define the *Ayed–Kuo integral* on the set $\mathcal{E}^{Exp}$ via

$$\bar{I}(\exp^\diamond(I(g)) \otimes h) := \exp^\diamond\left(I(g)\right)\left(I(h) - \langle g, h\rangle\right)$$

and linearity and extend it by the following closedness to the domain $D(\bar{I}) \subset L^p(\Omega \times [0,\infty))$:

Suppose a sequence of processes $(u^n)_{n\in\mathbb{N}} \subset \mathcal{E}^{Exp}$ such that

$$\int_0^\infty |u_s^n - u_s|^p ds \to 0 \quad a.s.,$$

and the existing Ayed–Kuo integrals $\bar{I}(u^n)$ converge in probability, then we define

$$\bar{I}(u) = \lim_{n\to\infty} \bar{I}(u^n).$$

Remark 3.10. (i) The existence and uniqueness of the stochastic integral in Definition 3.9 on the domain $D(\bar{I})$ follows again by the construction on the dense set $\mathcal{E}^{Exp}$ and the P-a.s.-uniqueness of the limits.

(ii) In contrast to Definitions 2.1 and 3.5, the stochastic integral in Definition 3.9 is defined in one step on the time horizon $[0,\infty)$.

Our main result is the following:

Theorem 3.11.

(1) *Definition 3.9 extends Definition 3.6.*
(2) *Definition 3.9 extends Definition 3.5.*

Proof. (1) Suppose u in the domain of the Hitsuda–Skorokhod integral in Definition 3.6, i.e. there exists a $p \geq 1$ and a sequence $(u^n)_{n\in\mathbb{N}} \subset \mathcal{E}^{Exp}$ with $u^n \to u$ in $L^p(\Omega \times [0,\infty))$ and $\delta(u^n) \to X$ in $L^p(\Omega)$. This implies the convergence $u^n \to u$ in measure $P \otimes \lambda$ and therefore a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $u^{n_k} \to u$ $P\otimes\lambda$-almost everywhere as k tends to infinity. Due to $\delta(u^n) = \bar{I}(u^n)$ we conclude $\delta(u^{n_k}) \to \delta(u)$ in probability. Since the limit in probability is unique, this yields that u is contained in the domain of the Ayed–Kuo integral in Definition 3.9 and $\bar{I}(u) = \delta(u)$.

(2) Firstly, for every element $g(x) = \sum_{j=1}^m a_j 1_{(b_j,c_j]}(x)$ in $\mathcal{E}$, we observe the reformulation

$$\begin{aligned}\exp^\diamond(I(g)) &= \exp^\diamond(I(g1_{[0,t)})) \diamond \exp^\diamond(I(g1_{(t,\infty)})) \\ &= \exp^\diamond(I(g1_{[0,t)}))\exp^\diamond(I(g1_{(t,\infty)})).\end{aligned}$$

Here the Wick product equals the ordinary product since the Wick exponentials are based on disjoint increments of the underlying Brownian motion (cf. [9, Lemma 2.5.2]). Hence, for the integrand $e^{\diamond I(g)}h$ we conclude an adapted part

$$\begin{aligned}&\exp^\diamond(I(g1_{[0,t)})h(t) \\ &= \exp\left(\sum_{j=1}^m a_j(B_{c_j\wedge t} - B_{b_j\wedge t}) - \frac{1}{2}\sum_{j=1}^m a_j^2((c_j\wedge t) - (b_j \wedge t))\right)h(t)\end{aligned}$$

and an instantly independent part

$$\exp^\diamond(I(g1_{(t,\infty)})) = \exp\left(\sum_{j=1}^m a_j(B_{c_j\vee t} - B_{b_j\vee t}) - \frac{1}{2}\sum_{j=1}^m a_j^2((c_j\vee t) - (b_j \vee t))\right).$$

Due to a standard density argument in $L^2([0,\infty))$, we obtain that for all $g, h \in \mathcal{E}$ the process $\exp^\diamond(I(g))h(\cdot)$, is contained in the domain of the Ayed–Kuo integral in Definition 3.5. In particular, thanks to linearity of the integral on disjoint time intervals, we conclude by a simple computation on finite sums in (2.1),

$$\bar{I}(\exp^\diamond(I(g)) \otimes h) = \exp\left(I(g) - \frac{1}{2}\int_0^1 g^2(s)ds\right)\left(I(h) - \int_0^1 g(s)h(s)ds\right).$$

Hence, $\mathcal{E}^{Exp}$ is contained in the domain of the stochastic integral in Definition 3.5. Then, via the (a.s)-P-closedness we conclude that Definition 3.9 extends Definition 3.5. □

Remark 3.12. Comparing Definitions 3.6 and 3.9, we observe the difference of the closedness: In the Hitsuda–Skorokhod integral the first assumed convergence of the integrands in L^p-sense can be stronger than the almost sure convergence in Definition 3.9, whereas the postulated convergence in probability of the integrals in Definition 3.9 is weaker than for the Hitsuda–Skorokhod integral. However, we are unable to construct an example of an integrand in $D(\bar{I}) \setminus D(\delta)$ for the same $p \geq 1$. Therefore, we assume that these stochastic integrals coincide in the usual world of stochastic processes.

Remark 3.13. The extension of the stochastic integrals in Definition 3.9 or 3.6 to $0 < p < 1$ seems difficult for the following reasons. Firstly, the spaces $L^p(\Omega \times [0,\infty))$ are indeed topological vector spaces but not Banach spaces anymore. In particular their dual is the trivial space $\{0\}$ (cf. [7]) and therefore a stochastic integral constructed by the set $\mathcal{E}^{Exp}$ and linearity as in Definition 3.9 must be an unbounded operator. This would be extremely impractical.

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