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ON AN M/G/1 RETRIAL QUEUEING MODEL

N. Paranjothi and V. Rajagopalan

Abstract

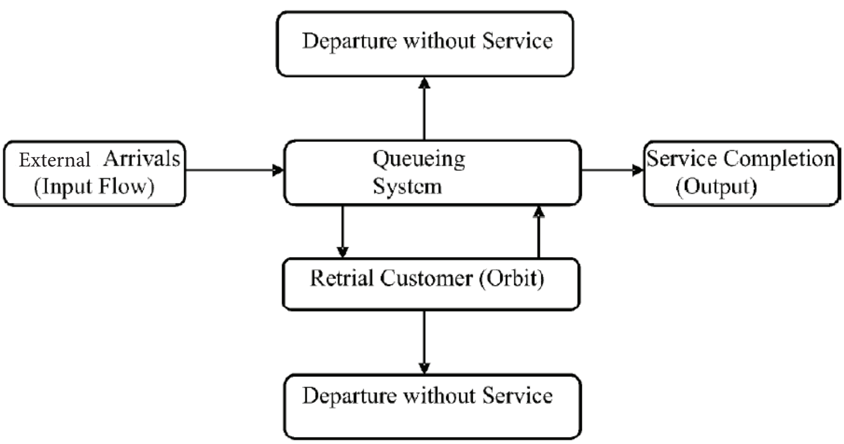
In this paper, a single server retrial queueing model with general service time distribution is studied. The steady state probabilities are computed. The measures relating to retrial, customers in the system and the distribution of the server state are studied.

Keywords: Retrials, Steady state probabilities, Distribution of Server State

1. INTRODUCTION

Queueing systems with retrials arise when a customer, on arriving for service finds the system full and decides to retry after sometimes, i.e., when blocked customers do not leave the system. Instead they may come back to the service facility after a random period of time. Such queueing system arises frequently in practice. A general block diagram representation of a Retrial queueing system is as follows.

In retrial queues inflows are of two types – the inflow of primary customers and that of repeated customers. When a customer arrives for service and finds a free server or a small queue, he waits for service and leaves the system once his service is over, such arrivals are



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In retrial queues inflows are of two types – the inflow of primary customers and that of repeated customers. When a customer arrives for service and finds a free server or a small queue, he waits for service and leaves the system once his service is over, such arrivals are called primary arrivals. On the other hand, if an arriving customer finds the server and waiting positions occupied, he may wait for sometimes and retry for service, such arrivals are called retrials or repeated customers. Between consecutive trials, a customer is said to be in orbit.

The *M/M/1* retrial queue in the steady state was studied by Cohen (1957), *M/M/1* constant retrial queue is investigated by Fayolle (1986). Hanschke (1987) obtained explicit formulas for the characteristics for the M/M/2/2 queue with repeated attempts. Artalejo (1996) studied a stationary analysis of the characteristics of the *M/M/2* queue with constant repeated attempts. Artalejo, Rajagopalan and Sivasamy (2000) discussed the finite Markovian queues with repeated attempts. Stochastic inequalities for the distribution of the number of customers in the *M/G/1* retrial queue and related results were obtained by Falin (1986).

Moreno (2004) studied an *M/G/1* retrial queue with recurrent customer and general retrial times in which the ergodicity condition for the system to be stable and derived the analysed results for the stationary distribution as well as some performance measures of the system. Shang et.al (2006) derived the tail asymptotics for the queue length in an *M/G/1* retrial queue Dudin et. al(2015a) discuss the single server retrial queue with group admission of customers.

2. DESCRIPTION OF THE MODEL

Customers arrive in a Poisson process with rate λ to the service facility with single server. These arriving customers are identified as primary arrivals. If the server is free at the time of primary arrival, it is served immediately and leaves the system after service completion. If the server is busy, the arriving customer becomes a source of repeated arrival. The orbit of sources of repeated arrival may be viewed as a sort of queue. Every such source produces a Poisson process of repeated arrivals with intensity μ. If the server is free at the time of a repeated arrival, it is served and leaves the system.

3. MATHEMATICAL FORMULATION

B(x) be the service time distribution for primary arrivals and repeated arrivals. The input flow of primary arrivals, intervals between repetitions and service times are mutually independent.

The queueing process evolves in the following manner. Suppose that (i – 1)th arrival completes its service at epoch η_{i-1} (the arrivals are numbered in the order of service) and the server becomes free. Even if there are some customers in the system who wants to get service, they cannot occupy the server immediately, because of their ignorance of the server state. Hence the i^{th} arrival is a primary arrival with probability $\lambda/(\lambda + n\mu)$ and is a repeated arrival with probability $n\mu/(\lambda + n\mu)$. At time t, let N(t) be the number of sources of repeated calls and C(t) be the status of the server. The process (C(t), N(t)) which describes the number of customers in the system is the simplest and simultaneously the most important process associated with the above queueing system. If the service time distribution is not exponential then the process is not Markov. In this case, we introduce a supplementary variable; if C(t)=1 we define $\xi(t)$ as the elapsed service time of the call being served. Let $\beta(s) = \int_0^\infty e^{-\lambda x} dB(x)$ be the Laplace Stieltijes transform of the service time distribution function B(x). $\beta_k = (-1)^k \beta^{(k)}(0)$ be the k^{th} moment of the service time about the origin. $\rho = \lambda\beta_1$ be the system load due to the primary arrivals, $b(x) = \frac{B(x)}{1 - B(x)}$ be the instantaneous service intensity given that the elapsed service time is equal to x,

$$k(z) = \beta(\lambda - \lambda z),$$

$$k(z) = \sum_{n=0}^{\alpha} k_n z^n \text{ where } k_n = \int_0^{\alpha} \frac{(\lambda x)^n}{n!} e^{-\lambda x} dB(x),$$

is the distribution of the number of primary arrivals which arrive during the service time of an arrival.

4. STEADY STATE PROBABILITIES

For an M/G/1 retrial queue in the steady state, the joint distribution of the server state and queue length

$$P_{0n} = P\{C(t) = 0, N(t) = n\}$$

$$P_{1n}(x) = \frac{d}{dx} P\{C(t) = 1, \xi(t) < x, N(t) = n\}$$ has the partial generating functions.

$$P_0(z) = \sum_n Z^n P_{1n} = (1-\rho)\exp\left\{\frac{\lambda}{\mu}\int_1^z \frac{1-k(u)}{k(u)-u} du\right\}$$

$$P_1(Z,X) = \sum_{n=0}^{\alpha} Z^n P_{1n} = \lambda \frac{(1-z)}{k(z)-z} P_0(z) \quad [1-B(x)]\, e^{-(\lambda-\lambda z)x}$$

If in the case of C(t) = 1 we neglect the elapsed service time $\xi(t)$, then for the probabilities $P_{1n} = P\{C(t) = 1, N(t) = n\}$ we have

$$P_1(Z) = \sum_{n=0}^{\alpha} Z^n P_{1n} = \frac{1-k(z)}{k(z)-z} P_0(z)$$

5. DISTRIBUTION OF THE NUMBER OF SOURCES OF RETRIALS

The distribution of the number of sources of retrials $q_n = P\{N(t)\}$ has generating function.

$$P(z) = P_0(z) + P_1(z) = \frac{(1-\rho)\ \ (1-z)}{k(z)-z} \exp\left\{\frac{\lambda}{\mu}\int_1^z \frac{1-k(u)}{k(u)-u}du\right\}$$

$$E(N(t)) = P(1) = (P(z))_{z=1} = \frac{d}{dz}\left((1-\rho)\left[\frac{1-z}{k(z)-z}\right]\exp^{\frac{\lambda}{\mu}}\int_1^z \frac{1-k(u)}{k(u)-u}du\right)_{z=1}$$

By simplification we have

$$E(N(t)) = (1-\rho)\left\{\left(\frac{1}{(1-\rho)}\right)\frac{\lambda}{\mu}\frac{\rho}{1-\rho} + \frac{k^n(1)}{2(1-\rho)^2}\right\} = \frac{\lambda}{\mu}\frac{\rho}{1-\rho} + \frac{k^n(1)}{2(1-\rho)}$$

$$= \frac{\lambda^2}{1-\rho}\left(\frac{\beta_1}{\mu} + \frac{\beta_2}{2}\right); \text{where } \beta_1 = \frac{\rho}{\lambda}, \beta_2 = \frac{k^n(t)}{(1-\rho)^2}$$

$$Var\{N(t)\} = \frac{\lambda^3\beta_3}{3(1-\rho)} + \frac{\lambda^3\beta_2}{2\mu(1-\rho)^2} + \frac{\lambda^4\beta_2^2}{4(1-\rho)^2} + \frac{\lambda\rho}{\rho(1-\rho)} + \frac{\lambda^2\beta_2}{2(1-\rho)}$$

6. MEASURES RELATING TO CUSTOMERS IN THE SYSTEM

The distribution of the number of customers in the system $Q_n = P\{K(t) = n\}$ has the generating function.

$$Q(z) = P_0(z) + zP_1(z) = (1-\rho)\frac{(1-z)\ \ k(z)}{k(z)-z}\exp\left\{\frac{\lambda}{\mu}\int_1^z \frac{1-k(u)}{k(u)-u}du\right\}$$

$$E(K(t)) = Q'(1) = \rho + \frac{\lambda^2}{1-\rho}\left(\frac{\beta_1}{\mu} + \frac{\beta_2}{2}\right)$$

$$Var(K(t)) = Q''(1) + |\,Q'(1) - [Q'(1)]^2 =$$

$$\rho - \rho^2 = \frac{\lambda^3\beta_3}{3(1-\rho)} + \frac{\lambda^4\beta_2^2}{4(1-\rho)^2} + \frac{\lambda^2\beta_2}{2(1-\rho)} + \lambda^2\beta_2 + \frac{\lambda^3\beta_2}{2\mu(1-\rho)^2} + \frac{\lambda\rho}{\mu(1-\rho)}$$

7. DISTRIBUTION OF THE SERVER STATE

The stationary distribution of the server state $P_0 = (1 - \rho)$, $P_1 = \rho$ depends on the service time distribution B(x) only through its mean β_1 and does not depend on the rate of retrial μ. This result is obvious if we take into account that for any server queue the probability P_1 is equal to the mean number of busy servers, which in turn, is equal to the intensity of carried traffic. Since the customers are not lost, the carried traffic is equal to the offered traffic, which is obviously equal $\lambda\beta_1 = \rho$.

The stationary distribution of the number of busy servers

$P_0 = (1-\rho)$, $P_1 = \rho$ can be written as

$$P_0 = \frac{1}{1-\rho'}, \quad P_1 = \frac{\rho'}{1-\rho'} \text{ where } \rho' = \rho + \frac{\rho^2}{1-\rho}$$

It is observed that the stationary distribution of the process C(t) coincides with the distribution of the number of busy servers in the M/G/1 Erlang loss model with increased arrival rate $\lambda' = \frac{\lambda+\lambda\rho}{1-\rho}$. The additional intensity equal to $\lim_{\mu\to 0} \mu[E(N(t))]$ is mean rate of flow of retrials. Thus as $\mu \to 0$ the additional load can be thought of as a load formed by sources of retrials.

If we put $\mu \to 0$ in the initial process, the retrial queue can be thought of as an Erlang loss model with the same arrival rate λ.

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N.Paranjothi and V.Rajagopalan
Department of Statistics
Annamalai University
Annamalainagar – 608 002