GENERALIZED EULERIAN INTEGRALS AND FRACTIONAL INTEGRAL FORMULAS

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ABSTRACT: *The main object of this paper is to derive two generalized Eulerian integrals with integrands involving the product of a Fox's H-function, the generalized polynomial set and the H-function of several variables with general arguments. Being of very general nature, due to the presence of these functions and polynomial, our results provides interesting generalization and extension of a large number of new and known results obtained by several authors earlier and hitherto lying scattered in the literature. For the sake of illustration, we have evaluated here two new integrals as special cases of our main integrals. As an application, the main integral formulas have also been expressed as fractional integrals, which would provide useful generalization of known results in the theory of fractional calculus.*

Keywords and phrases: *Eulerian integrals, Generalized polynomial set, multivariable H-function, Fox's H-function, Generalized Lauricella function, Gould and Hopper polynomial.*

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1. INTRODUCTION

The Fractional integral operator is defined and represented as follows:

$$ {}_cD_y^{-v}\{f(y)\} = \frac{1}{\Gamma(\upsilon)}\int_c^y (y-t)^{v-1} f(t)\,dt,\ \mathrm{Re}(\upsilon) > 0 \cdot \tag{1.1}$$

The fractional integral operator (1.1) defines the classical Riemann-Liouville fractional integral operator of order υ when $c = 0$ and when $c \to \infty$ it may be identified with the definition of the Weyl fractional integral operator of order υ.

The generalized polynomial set is defined by the following Rodrigues type formula [13, p. 64, Eq. (2.1.8)]:

$$S_n^{\alpha,\beta,\tau}[x; r, s, q, A, B, m, k, l] = S_n^{\alpha,\beta,\tau}[x]$$

$$= (Ax+B)^{-\alpha}(1-\tau x^r)^{-\frac{\beta}{\tau}} T_{k,l}^{m+n}\left[(Ax+B)^{\alpha+qn}(1-\tau x^r)^{\frac{\beta}{\tau}+sn}\right], \tag{1.2}$$

with the differential operator defined by

$$T_{k,l} \equiv x^l \left(k + x \frac{d}{dx} \right). \tag{1.3}$$

The explicit form of this generalized polynomial set [13, Eq. (2. 3. 4), p. 71] is

$$S_n^{\alpha,\beta,\tau}[x] = \sum_{u=0}^{m+n} \sum_{\upsilon=0}^{u} \sum_{e=0}^{m+n} \sum_{p=0}^{e} \frac{B^{qn}(-1)^e(-e)_p(\alpha)_e(-u)_\upsilon(-\alpha-qn)_p\left(-\frac{\beta}{\tau}-sn\right)_u}{u!\,\upsilon!\,e!\,p!(1-\alpha-e)_p}$$

$$\times\, l^{m+n}(-\tau)^u \left(\frac{p+k+r\upsilon}{l}\right)_{m+n} \left(\frac{A}{B}\right)^e x^{l(m+n)+ru+e}\left(1-\tau x^r\right)^{sn-u}, \tag{1.4}$$

It may be pointed out here that the polynomial set defined by (1.2) is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers such as Chatterjea [11], Dhillon [12], Gould and Hopper [1], Krall and Frink [2], Singh and Srivastava [10] etc.

We shall also require the following important special case obtained on taking $A = 1$, $B = 0$ and $\tau \to 0$ in (1.4),

$$S_n^{\alpha,\beta,\tau}[x, r, s, q, 1, 0, m, k, l] = \sum_{u=0}^{m+n} \sum_{\upsilon=0}^{u} \frac{(-u)_\upsilon}{u!\,\upsilon!} \left(\frac{\alpha+qn+k+r\upsilon}{l}\right)_{m+n} l^{m+n}\beta^u x^{l(m+n)+ru+qn}. \tag{1.5}$$

The multivariable H-function introduced and studied by Srivastava and Panda [6], occurring in this paper will be defined and represented as follows [4, p.251, Eqs.(C.1)-(C.3)]:

$$H[z_1, \ldots, z_t] = H^{0,N:M_1,N_1;\ldots;M_t,N_t}_{p,Q:P_1,Q_1;\ldots;P_t,Q_t}\left[z_1, \ldots, z_t \left| \begin{matrix} \left(a_j:\alpha_j^{(1)},\ldots,\alpha_j^{(t)}\right)_{1,P} : \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1};\ldots;\left(c_j^{(t)},\gamma_j^{(t)}\right)_{1,P_t} \\ \left(b_j:\beta_j^{(1)},\ldots,\beta_j^{(t)}\right)_{1,Q} : \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1};\ldots;\left(d_j^{(t)},\delta_j^{(t)}\right)_{1,Q_t} \end{matrix} \right. \right]$$

$$= \frac{1}{(2\pi\omega)^t} \int_{L_1} \cdots \int_{L_t} \psi(\xi_1, \ldots, \xi_t)\phi_1(\xi_1)\ldots\phi_t(\xi_t) z_1^{\xi_1}\ldots z_t^{\xi_t}\, d\xi_1 \ldots d\xi_t, \tag{1.6}$$

where $\omega = -\sqrt{-1}$,

$$\psi(\xi_1, \ldots, \xi_t) = \frac{\prod_{j=1}^{N}\Gamma\left(1-a_j+\sum_{i=1}^{t}\alpha_j^{(i)}\xi_i\right)}{\prod_{j=N+1}^{P}\Gamma\left(a_j-\sum_{i=1}^{t}\alpha_j^{(i)}\xi_i\right)\prod_{j=1}^{Q}\Gamma\left(1-b_j+\sum_{i=1}^{t}\beta_j^{(i)}\xi_i\right)} \tag{1.7}$$

and

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{M_i} \Gamma\left(d_j^{(i)} - \delta_j^{(i)}\xi_i\right) \prod_{j=1}^{N_i} \Gamma\left(1 - c_j^{(i)} + \gamma_j^{(i)}\xi_i\right)}{\prod_{j=M_i+1}^{Q_i} \Gamma\left(1 - d_j^{(i)} + \delta_j^{(i)}\xi_i\right) \prod_{j=N_i+1}^{P_i} \Gamma\left(c_j^{(i)} - \gamma_j^{(i)}\xi_i\right)}, (i = 1, ..., t) \tag{1.8}$$

For the sake of brevity,

$$T_i = -\sum_{j=N+1}^{P} \alpha_j^{(i)} + \sum_{j=1}^{N_i} \gamma_j^{(i)} - \sum_{j=N_i+1}^{P_i} \gamma_j^{(i)} - \sum_{j=1}^{Q} \beta_j^{(i)} + \sum_{j=1}^{M_i} \delta_j^{(i)} - \sum_{j=M_i+1}^{Q_i} \delta_j^{(i)} > 0, \ (\forall\ i = 1, ..., t) \tag{1.9}$$

For the convergence and existence conditions of multivariable H-function, we refer to the book [4, pp. 252-253, Eqs. (C·4)–(C·8)]. Throughout the paper it is assumed that this function satisfies the above cited conditions.

The series representation of Fox's H-function is defined as follows [7]:

$$H_{p',q'}^{m',n'}\left[x \left| \begin{matrix} (e_{p'}, E_{p'}) \\ (f_{q'}, F_{q'}) \end{matrix} \right.\right] = \sum_{g'=1}^{m'} \sum_{G=0}^{\infty} \frac{(-1)^G \phi(\eta_G) x^{\eta_G}}{G!\, F_{g'}}, \tag{1.10}$$

where

$$\phi(\eta_G) = \frac{\prod_{j=1,\, j\neq g'}^{m'} \Gamma\left(f_j - F_j\eta_G\right) \prod_{j=1}^{n'} \Gamma\left(1 - e_j + E_j\eta_G\right)}{\prod_{j=m'+1}^{q'} \Gamma\left(1 - f_j + F_j\eta_G\right) \prod_{j=n'+1}^{p'} \Gamma\left(e_j - E_j\eta_G\right)} \tag{1.11}$$

and

$$\eta_G = \frac{\left(f_{g'} + G\right)}{F_{g'}}. \tag{1.12}$$

Also

$$T = \sum_{i=1}^{n'} E_i - \sum_{i=n'+1}^{p'} E_i + \sum_{i=1}^{m'} F_i - \sum_{i=m'+1}^{q'} F_i > 0. \tag{1.13}$$

2. EULERIAN INTEGRALS

First Integral

$$\int_a^b \frac{(x-a)^{\rho-1}}{(cx+d)^{\eta}} \frac{(b+x)^{\sigma-1}}{(gx+f)^{\omega}} H^{m',n'}_{p',q'}\left[\frac{z(x-a)^{\zeta}(b-x)^{\xi}}{(cx+d)^{\lambda}(gx+f)^{\delta}}\middle| \begin{matrix}(e_{p'},E_{p'})\\(f_{q'},F_{q'})\end{matrix}\right]$$

$$\times S_n^{\alpha,\beta,\tau}\left[\frac{h(x-a)^{\mu}(b-x)^{\upsilon}}{(cx+d)^{\gamma}(gx+f)^{\theta}}\right] H\left[\frac{z_1(x-a)^{p_1}(b-x)^{q_1}}{(cx+d)^{r_1}(gx+f)^{s_1}},\dots,\frac{z_t(x-a)^{p_t}(b-x)^{q_t}}{(cx+d)^{r_t}(gx+f)^{s_t}}\right]dx$$

$$=\sum_{g'=1}^{m'}\sum_{G=0}^{\infty}\sum_{u=0}^{m+n}\sum_{\upsilon=0}^{u}\sum_{e=0}^{m+n}\sum_{p=0}^{e}\frac{(-1)^G\phi(\eta_G)z^{\eta_G}B^{qn}(-1)^e(-e)_p(\alpha)_e(-u)_{\upsilon}}{G!F_{g'}\,u!\upsilon!e!p!}$$

$$\times\frac{(-\alpha-qn)_p}{(1-\alpha-e)_p}\left(-\frac{\beta}{\tau}-sn\right)_u l^{m+n}(-\tau)^u\left(\frac{p+k+r\upsilon}{l}\right)_{m+n}\left(\frac{A}{B}\right)^e\frac{h^R}{\Gamma(u-sn)}$$

$$\times(ac+d)^{-\eta-\lambda}\eta_G^{-\gamma R}(bg+f)^{-\omega-\delta}\eta_G^{-\theta R}(b-a)^{\rho+\sigma+(\zeta+\xi)}\eta_G^{+(\mu+\upsilon)R-1}$$

$$\times H^{0,\,N+4:M_1,\,N_1;\dots;M_t,\,N_t;1,1;1,0;1,0}_{P+4,\,Q+3:P_1,\,Q_1;\dots;P_t,\,Q_t;1,1;0,1;0,1}\left[\frac{z_1(b-a)^{p_1+q_1}}{(ac+d)^{r_1}(bg+f)^{s_1}},\dots,\frac{z_t(b-a)^{p_t+q_t}}{(ac+d)^{r_t}(bg+f)^{s_t}},\right.$$

$$\frac{-\tau h^r(b-a)^{(\mu+\upsilon)r}}{(ac+d)^{\gamma r}(bg+f)^{\theta r}},\frac{c(b-a)}{(ac+d)},\frac{-g(b-a)}{(bg+f)}\left|\begin{matrix}(1-\eta-\lambda\eta_G-\gamma R:r_1,\dots,r_t,\gamma r,1,0),\\(1-\eta-\lambda\eta_G-\gamma R:r_1,\dots,r_t,\gamma r,0,0),\end{matrix}\right.$$

$$(1-\omega-\delta\eta_G-\theta R:s_1,\dots,s_t,\theta r,0,1),(1-\rho-\zeta\eta_G-\mu R:p_1,\dots,p_t,\mu r,1,0),$$

$$(1-\omega-\delta\eta_G-\theta R:s_1,\dots,s_t,\theta r,0,0),(1-\rho-\sigma-(\zeta+\xi)\eta_G-(\mu+\upsilon)R:$$

$$(1-\sigma-\xi\eta_G-\upsilon r:q_1,\dots,q_t,\upsilon r,0,1),\left(a_j:\alpha_j^{(1)},\dots,\alpha_j^{(t)},0,0,0\right)_{1,P}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1}$$

$$p_1+q_1,\dots,p_t+q_t,(\mu+\upsilon)r,1,1),\left(b_j:\beta_j^{(1)},\dots,\beta_j^{(t)},0,0,0\right)_{1,Q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1}$$

$$\left.\begin{matrix};\dots;\left(c_j^{(t)},\gamma_j^{(t)}\right)_{1,P_t};(1-u+sn,1);-;-\\;\dots;\left(d_j^{(t)},\delta_j^{(t)}\right)_{1,Q_t};(0,1);(0,1);(0,1)\end{matrix}\right], \tag{2.1}$$

where $R = l\,(m+n) + ru + e$.

The above result is valid under the following conditions:

$$\mathrm{Re}(\rho, \sigma, r) > 0,$$

$$\min(\mu, \upsilon, \gamma, \theta, p_i, q_i, r_i, s_i) \geq 0 \ (i = 1, \ldots, t), \text{(not all zero simultaneously)}$$

$$\max\left\{\left|\frac{(b-a)c}{(ac+d)}\right|, \left|\frac{(b-a)g}{(bg+f)}\right|\right\} < 1;\ b \neq a,$$

$$\mathrm{Re}\left[\rho + \zeta\left(\frac{f_{j'}}{F_{j'}}\right) + \mu\{l(m+n) + e + rsn\} + \sum_{i=1}^{t} p_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0,$$

$$\mathrm{Re}\left[\sigma + \xi\left(\frac{f_{j'}}{F_{j'}}\right) + \upsilon\{l(m+n) + e + rsn\} + \sum_{i=1}^{t} q_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0,$$

$$|\arg(z)| < \frac{T\pi}{2}, |\arg(z_i)| < \frac{T_i\pi}{2},$$

$$i = 1, \ldots, t;\ j = 1, \ldots, M_i;\ j' = 1, \ldots, m'.$$

Second Integral

$$\int_a^b \frac{(cx+d)^{\eta}}{(x-a)^{1-\rho}} \frac{(gx+f)^{\omega}}{(b-x)^{1-\sigma}} H_{p',q'}^{m',n'}\left[\frac{z(cx+d)^{\lambda}(gx+f)^{\delta}}{(x-a)^{\zeta}(b-x)^{\xi}}\middle| \begin{matrix}(e_{p'}, E_{p'})\\(f_{q'}, F_{q'})\end{matrix}\right]$$

$$\times S_n^{\alpha,\beta,\tau}\left[\frac{h(cx+d)^{\gamma}(gx+f)^{\theta}}{(x-a)^{\mu}(b-x)^{\upsilon}}\right] H\left[\frac{z_1(cx+d)^{r_1}(gx+f)^{s_1}}{(x-a)^{p_1}(b-x)^{q_1}}, \ldots, \frac{z_t(cx+d)^{r_t}(gx+f)^{s_t}}{(x-a)^{p_t}(b-x)^{q_t}}\right] dx$$

$$= \sum_{g'=1}^{m'} \sum_{G=0}^{\infty} \sum_{u=0}^{m+n} \sum_{\upsilon=0}^{u} \sum_{e=0}^{m+n} \sum_{p=0}^{e} \frac{(-1)^G \phi(\eta_G) z^{\eta_G} B^{qn} (-1)^e (-e)_p (\alpha)_e (-u)_{\upsilon}}{G!\, F_{g'}\, u!\, \upsilon!\, e!\, p!}$$

$$\times \frac{(-\alpha - qn)_p}{(1-\alpha-e)_p}\left(-\frac{\beta}{\tau} - sn\right)_u l^{m+n} (-\tau)^u \left(\frac{p+k+r\upsilon}{l}\right)_{m+n} \left(\frac{A}{B}\right)^e \frac{h^R}{\Gamma(u - sn)}$$

$$\times (ac+d)^{-\eta-\lambda\eta}\ G^{+\gamma R}\,(bg+f)^{-\omega-\delta\eta}{}_{G}{}^{+\theta R}\,(b-a)^{\rho+\sigma-(\zeta+\xi)\eta}{}_{G}{}^{-(\mu+\upsilon)R-1}$$

$$\times H^{0,\,M+4:N_1,\,M_1;\,...;\,N_t,\,M_t;\,1,1;\,1,0;\,1,0}_{Q+4,\,P+3:Q_1,\,P_1;\,...;\,Q_t,\,P_t;\,1,1;\,0,1;\,0,1}\left[\frac{(b-a)^{p_1+q_1}}{z_1(ac+d)^{r_1}(bg+f)^{s_1}},\ldots,\frac{(b-a)^{p_t+q_t}}{z_t(ac+d)^{r_t}(bg+f)^{s_t}},\right.$$

$$\frac{-(b-a)^{(\mu+\upsilon)r}}{\tau h^r(ac+d)^{\gamma r}(bg+f)^{\theta r}},\frac{c(b-a)}{(ac+d)},\frac{-g(b-a)}{(bg+f)}\left|\begin{matrix}(1+\eta+\lambda\eta_G+\gamma R:r_1,\ldots,r_t,\gamma r,1,0),\\(1+\eta+\gamma\eta_G+\lambda R:r_1,\ldots,r_t,\gamma r,0,0),\end{matrix}\right.$$

$$(1+\omega+\delta\eta_G+\theta R:s_1,\ldots,s_t,\theta r,0,1),\ (1-\rho+\zeta\eta_G+\mu R:p_1,\ldots,p_t,\mu r,1,0),$$
$$(1+\omega+\delta\eta_G+\theta R:s_1,\ldots,s_t,\theta r,0,0),\ (1-\rho-\sigma+(\zeta+\xi)\eta_G+(\mu+\upsilon)R:$$

$$(1-\sigma+\xi\eta_G+\upsilon r:q_1,\ldots,q_t,\upsilon r,0,1),\left(1-b_j:\beta_j^{(1)},\ldots,\beta_j^{(t)},0,0,0\right)_{1,Q}:\left(1-d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1}$$

$$p_1+q_1,\ldots,p_t+q_t,(\mu+\upsilon)r,1,1),\left(1-a_j:\alpha_j^{(1)},\ldots,\alpha_j^{(t)},0,0,0\right)_{1,P}:\left(1-c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1}$$

$$\left.\begin{matrix};\ldots;\left(1-d_j^{(t)},\delta_j^{(t)}\right)_{1,Q_t};(1,1);-;-\\;\ldots;\left(1-c_j^{(t)},\gamma_j^{(t)}\right)_{1,P_t};(u-sn,1);(0,1);(0,1)\end{matrix}\right], \tag{2.2}$$

where $R = l(m+n) + ru + e$.

The above result holds under the following conditions:

$$\text{Re}\,(\rho, \sigma, r) > 0,$$

$\min(\mu, \upsilon, \gamma, \theta, p_i, q_i, r_i, s_i) \geq 0$ ($i = 1, \ldots, t$), (not all zero simultaneously)

$$\max\left\{\left|\frac{(b-a)c}{(ac+d)}\right|,\left|\frac{(b-a)g}{(bg+f)}\right|\right\}<1;\ b\neq a,$$

$$\text{Re}\left[\rho-\zeta\left(\frac{e_{j'}-1}{F_{j'}}\right)-\mu\{l(m+n)+e+rsn\}-\sum_{i=1}^{t}p_i\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]>0,$$

$$\text{Re}\left[\sigma-\xi\left(\frac{e_{j'}-1}{F_{j'}}\right)-\upsilon\{l(m+n)+e+rsn\}-\sum_{i=1}^{t}q_i\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]>0,$$

$$|\arg(z)| < \frac{T\pi}{2}, |\arg(z_i)| < \frac{T_i\pi}{2},$$

$$i = 1, ..., t; j = 1, ..., N_i; j' = 1, ..., n'.$$

Proof of (2..1): To establish the integral (2.1), we first express the Fox's *H*-function in series form, the generalized polynomial set and the multivariable H-function occurring in the left hand side of (2.1) with the help of (1.10), (1.4) and (1.6) respectively, then expressing in contour integral representation of the term $(1 - \tau x^r)^{sn-u}$ as appearing in the series expression (1.4) of $S_n^{\alpha,\beta,\tau}[x]$,

$$\left(1-\tau x^r\right)^{sn-u} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(-\xi_{t+1})\Gamma(u-sn+\xi_{t+1})}{\Gamma(u-sn)}\left\{-\tau x^r\right\}^{\xi_{t+1}} d\xi_{t+1}, \tag{2.3}$$

Now collect the power of $(cx + d)$ and $(gx + f)$ and applying the following formula for $x \in [a, b]$,

$$(cx+d)^{\eta'} = (ac+d)^{\eta'} \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(-\xi_{t+2})\Gamma(-\eta'+\xi_{t+2})}{\Gamma(-\eta')}\left\{\frac{(x-a)c}{ac+d}\right\}^{\xi_{t+2}} d\xi_{t+2}, \tag{2.4}$$

$$|(x - a)c| < |ac + d|$$

$$(gx+f)^{\omega'} = (bg+f)^{\omega'} \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(-\xi_{t+3})\Gamma(-\omega'+\xi_{t+3})}{\Gamma(-\omega')}\left\{\frac{-(b-x)g}{bg+f}\right\}^{\xi_{t+3}} d\xi_{t+3}, \tag{2.5}$$

$$|(b - x)g| < |bg + f|$$

with η' and ω' replaced by $-\eta - \lambda\eta_G - \gamma R - \sum_{i=1}^{t} r_i\xi_i - \gamma r\xi_{t+1}$ and $-\omega - \delta\eta_G - \theta R - \sum_{i=1}^{t} s_i\xi_i - \theta r\xi_{t+1}$ respectively, then changing the order of summation and integrations (which is justified under the conditions stated) and evaluating the inner integral with the help of well-known Eulerian integral, finally we arrive at the desired result (2.1) after a little simplification.

Proof of (2..2): To prove this result, we follow the same lines as in the proof of (2.1) except the following changes. Here, we apply the formulas (2.4) and (2.5) with η'

and ω' replaced by $\eta + \lambda\eta_G + \gamma R - \sum_{i=1}^{t} r_i\xi_i - \gamma r\xi_{t+1}$ and $+ \omega + \delta\eta_G^{+} \theta R - \sum_{i=1}^{t} s_i\xi_i - \theta r\xi_{t+1}$ respectively and then interpret the resulting Mellin Barnes contour integral as an H-function of $(t + 3)$ variables and arrive at the right hand side of (2.2).

3. SPECIAL CASES

(i) If we take $N = P$, $M_i = 1$, $N_i = P_i$, $Q_i = Q_i + 1$ ($\forall\ i = 1, \ldots, t$), the multivariable H-function is reduced to the generalized Lauricella function of several complex variables [3, p. 454] and further on taking $A = 1$, $B = 0$, $q = s = m = k = 0$, $l = -1$ and letting $\tau \to 0$, the polynomial set $S_n^{\alpha,\beta,\tau}[x]$ reduced to the Gould and Hopper polynomial $H_n^{(r)}(x; \alpha, \beta)$ [1] in our integral formula (2.1) and we arrive at following result after a little simplification,

$$\int_a^b \frac{(x-a)^{\rho-1}}{(cx+d)^{\eta}} \frac{(b-x)^{\sigma-1}}{(gx+f)^{\omega}} H_{p',q'}^{m',n'}\left[\frac{z(x-a)^{\zeta}(b-x)^{\xi}}{(cx+d)^{\lambda}(gx+f)^{\delta}}\middle|\begin{matrix}(e_{p'}, E_{p'})\\(f_{q'}, F_{q'})\end{matrix}\right]$$

$$\times H_n^{(r)}\left[\frac{h(x-a)^{\mu}(b-x)^{\upsilon}}{(cx+d)^{\gamma}(gx+f)^{\theta}}; \alpha, \beta\right] F_{Q:Q_1;\ldots;Q_t}^{P:P_1;\ldots;P_t}\left[-\frac{z_1(x-a)^{p_1}(b-x)^{q_1}}{(cx+d)^{r_1}(gx+f)^{s_1}}, \ldots,\right.$$

$$\left.-\frac{z_t(x-a)^{p_t}(b-x)^{q_t}}{(cx+d)^{r_t}(gx+f)^{s_t}}\middle|\begin{matrix}\left(1-a_j : \alpha_j^{(1)}, \ldots, \alpha_j^{(t)}\right)_{1,P} : \left(1-c_j^{(1)}, \gamma_j^{(1)}\right)_{1,P_1}; \ldots; \left(1-c_j^{(t)}, \gamma_j^{(t)}\right)_{1,P_t}\\ \left(1-b_j : \beta_j^{(1)}, \ldots, \beta_j^{(t)}\right)_{1,Q} : \left(1-d_j^{(1)}, \delta_j^{(1)}\right)_{1,Q_1}; \ldots; \left(1-d_j^{(t)}, \delta_j^{(t)}\right)_{1,Q_t}\end{matrix}\right] dx$$

$$= \sum_{g'=1}^{m'}\sum_{G=0}^{\infty}\sum_{u=0}^{n}\sum_{\upsilon=0}^{u} \frac{(-1)^G \phi(\eta_G) z^{\eta_G} (-1)^n (-u)_{\upsilon} (-\alpha - r\upsilon)_n \beta^u h^{R'}}{G!\,F_{g'}\,u!\,\upsilon!}$$

$$\times (ac + d)^{-\eta-\lambda}\,\eta_G^{-\gamma R'} (bg + f)^{-\omega-\delta}\,\eta_G^{-\theta R'} (b - a)^{\rho+\sigma+(\zeta+\xi)}\,\eta_G^{+(\mu+\upsilon)R'-1}$$

$$\times \frac{\Gamma(\rho + \zeta\eta_G + \mu R')\,\Gamma(\sigma + \xi\eta_G + \upsilon R')}{\Gamma(\rho + \sigma + (\zeta + \xi)\eta_G + (\mu + \upsilon)R')}$$

$$F_{Q+3:Q_1;\ldots;Q_t;1;1}^{P+4:P_1;\ldots;P_t;0;0}\left[-\frac{z_1(b-a)^{p_1+q_1}}{(ac+d)^{r_1}(bg+f)^{s_1}}, \ldots, -\frac{z_t(b-a)^{p_t+q_t}}{(ac+d)^{r_t}(bg+f)^{s_t}}, -\frac{c(b-a)}{(ac+d)},\right.$$

$$\left.\frac{g(b-a)}{(bg+f)}\right| \begin{matrix} (\eta+\lambda\eta_G+\gamma R' : r_1, ..., r_t, 1, 0), (\omega+\delta\eta_G+\theta R' : s_1, ..., s_t, 0, 1), \\ (\eta+\lambda\eta_G+\gamma R' : r_1, ..., r_t, 0, 0), (\omega+\delta\eta_G+\theta R' : s_1, ..., s_t, 0, 0), \end{matrix}$$

$$\begin{matrix} (\rho+\zeta\eta_G+\mu R' : p_1, ..., p_t, 1, 0), (\sigma+\xi\eta_G+\upsilon R' : q_1, ..., q_t, 0, 1), \left(1-a_j : \alpha_j^{(1)}, ..., \alpha_j^{(t)}, 0, 0\right)_{1, P} \\ (\rho+\sigma+(\zeta+\xi)\eta_G+(\mu+\upsilon)R' : p_1+q_1, ..., p_t+q_t, 1, 1), \left(1-b_j : \beta_j^{(1)}, ..., \beta_j^{(t)}, 0, 0\right)_{1, Q} \end{matrix}$$

$$\left.\begin{matrix} : \left(1-c_j^{(1)}, \gamma_j^{(1)}\right)_{1, P_1} ; ... ; \left(1-c_j^{(t)}, \gamma_j^{(t)}\right)_{1, P_t} ; - ; - \\ : \left(1-d_j^{(1)}, \delta_j^{(1)}\right)_{1, Q_1} ; ... ; \left(1-d_j^{(t)}, \delta_j^{(t)}\right)_{1, Q_t} ; (0, 1) ; (0, 1) \end{matrix}\right], \tag{3.1}$$

where $R' = ru - n$.

The conditions of existence of this result can easily be derived from those mentioned with (2.1).

(ii) If we set $m = n = q = k = B = 0$, $l = r = -1$ and $A = 1$ in our integral formula (2.1), the generalized polynomial set reduces to unity and we arrive at following result after a little simplification,

$$\int_a^b \frac{(x-a)^{\rho-1}}{(cx+d)^{\eta}} \frac{(b-x)^{\sigma-1}}{(gx+f)^{\omega}} H_{p', q'}^{m', n'} \left[\frac{z(x-a)^{\zeta}(b-x)^{\xi}}{(cx+d)^{\lambda}(gx+f)^{\delta}} \middle| \begin{matrix} (e_{p'}, E_{p'}) \\ (f_{q'}, F_{q'}) \end{matrix} \right]$$

$$\times H \left[\frac{z_1 (x-a)^{p_1} (b-x)^{q_1}}{(cx+d)^{r_1} (gx+f)^{s_1}}, ..., \frac{z_t (x-a)^{p_t} (b-x)^{q_t}}{(cx+d)^{r_t} (gx+f)^{s_t}} \right] dx$$

$$= \sum_{g'=1}^{m'} \sum_{G=0}^{\infty} \frac{(-1)^G \phi(\eta_G) z^{\eta_G}}{G! F_{g'}} (ac+d)^{-\eta-\lambda} \, \eta_G (bg+f)^{-\omega-\delta} \, \eta_G (b-a)^{\rho+\sigma+(\zeta+\xi)} \, \eta_G{}^{-1}$$

$$\times H_{P+4, Q+3: P_1, Q_1; ...; P_t, Q_t; 0, 1; 0, 1}^{0, N+4: M_1, N_1; ...; M_t, N_t; 1, 0; 1, 0} \left[\frac{z_1 (b-a)^{p_1+q_1}}{(ac+d)^{r_1} (bg+f)^{s_1}}, ..., \frac{z_t (b-a)^{p_t+q_t}}{(ac+d)^{r_t} (bg+f)^{s_t}}, \frac{c(b-a)}{(ac+d)}, \right.$$

$$\left.\frac{-g(b-a)}{(bg+f)}\right| \begin{matrix} (1-\eta-\lambda\eta_G : r_1, ..., r_t, 1, 0), (1-\omega-\delta\eta_G : s_1, ..., s_t, 0, 1), (1-\rho-\zeta\eta_G : p_1, ..., p_t, 1, 0), \\ (1-\eta-\lambda\eta_G : r_1, ..., r_t, 0, 0), (1-\omega-\delta\eta_G : s_1, ..., s_t, 0, 0), (1-\rho-\sigma(\zeta+\xi)\eta_G : \end{matrix}$$

$$\left(1-\sigma-\xi\eta_G : q_1,\ldots,q_t,0,1\right),\left(a_j:\alpha_j^{(1)},\ldots,\alpha_j^{(t)},0,0\right)_{1,P}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1};\ldots;\left(c_j^{(t)},\gamma_j^{(t)}\right)_{1,P_t};$$

$$p_1+q_1,\ldots,p_t+q_t,1,1),\left(b_j:\beta_j^{(1)},\ldots,\beta_j^{(t)},0,0\right)_{1,Q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1};\ldots;\left(d_j^{(t)},\delta_j^{(t)}\right)_{1,Q_t};$$

$$\left.\begin{matrix} -; & - \\ (0,1);(0,1) \end{matrix}\right], \tag{3.2}$$

The conditions of existence of this result can easily be derived from those mentioned with (2.1).

(iii) On taking $\omega = 0$, $m = n = q = k = B = 0$, $l = r = -1$, $A = 1$, $s_i = 0$ $(i = 1, \ldots, t)$, reducing the Fox's H-function to unity (by giving suitable values to parameters) in integral formula (2.1) and reducing the multivariable H-function to product of 't' Fox's H-function (by taking $P = Q = N = 0$), then on taking $t = 1$, we arrive at the integral evaluated recently by Saxena and Saigo [9, p. 37, Eq. (2.1)].

(iv) On taking $m = n = q = k = B = 0$, $l = r = -1$, $A = 1$, $p_i = q_i = 0$ $(i = 1, \ldots, t)$ and reducing the Fox's H-function to unity (by giving suitable values to parameters) in integral formulas (2.1) and (2.2), we arrive at two known results given by Srivastava and Hussain [5, Eqs.(2.5), (2.6), p.79] after a little simplification.

In the results thus obtained if we further take $\omega = 0$, $s_i = 0$ $(i = 1, \ldots, t)$ and reduce the multivariable H-function to product of 't' Fox's H-function (by taking $P = Q = N = 0$), then on taking $t = 1$, we arrive at the integral evaluated recently by Saxena and Nishimoto [8, Eq. (4.1), p. 69].

4. APPLICATIONS

For b = y, $\upsilon = \xi = q_i = 0 (i = 1, \ldots, t)$ each of Eulerian integrals (2.1), (2.2), (3.1), (3.2) can easily be expressed as fractional integral formula involving the operator ${}_cD_y^{-\upsilon}$ defined by (1.1). To illustrate, we express below the integrals (2.1) and (2.2) as fractional integral formulae, which are valid under the conditions stated with them,

$${}_aD_y^{-\sigma}\left\{\frac{(y-a)^{\rho-1}}{(cy+d)^{\eta}(gy+f)^{\omega}}H_{p',q'}^{m',n'}\left[\frac{z(y-a)^{\zeta}}{(cy+d)^{\lambda}(gy+f)^{\delta}}\middle|\begin{matrix}(e_{p'},E_{p'})\\(f_{q'},F_{q'})\end{matrix}\right]\right.$$

$$\left.\times S_n^{\alpha,\beta,\tau}\left[\frac{h(y-a)^{\mu}}{(cy+d)^{\gamma}(gy+f)^{\theta}}\right]H\left[\frac{z_1(y-a)^{p_1}}{(cy+d)^{r_1}(gy+f)^{s_1}},\ldots,\frac{z_t(y-a)^{p_t}}{(cy+d)^{r_t}(gy+f)^{s_t}}\right]\right\}$$

$$= \sum_{g'=1}^{m'} \sum_{G=0}^{\infty} \sum_{u=0}^{m+n} \sum_{\upsilon=0}^{u} \sum_{e=0}^{m+n} \sum_{p=0}^{e} \frac{(-1)^G \phi(\eta_G) z^{\eta_G} B^{qn} (-1)^e (-e)_p (\alpha)_e (-u)_\upsilon}{G!\, F_{g'}\, u!\, \upsilon!\, e!\, p!}$$

$$\times \frac{(-\alpha - qn)_p}{(1-\alpha-e)_p} \left(-\frac{\beta}{\tau} - sn\right)_u l^{m+n} (-\tau)^u \left(\frac{p+k+r\upsilon}{l}\right)_{m+n} \left(\frac{A}{B}\right)^e \frac{h^R}{\Gamma(u-sn)\Gamma(\sigma)}$$

$$\times (ac+d)^{-\eta-\lambda\eta_G-\gamma R} (bg+f)^{-\omega-\delta\eta_G-\theta R} (b-a)^{\rho+\sigma+\zeta\eta_G+\mu R-1}$$

$$\times H^{0,\, N+3: M_1,\, N_1; \ldots; M_t,\, N_t; 1,1; 1,0; 1,1}_{P+3,\, Q+3: P_1,\, Q_1; \ldots; P_t,\, Q_t; 1,1; 0,1; 1,1} \left[\frac{z_1 (y-a)^{p_1}}{(ac+d)^{r_1} (gy+f)^{s_1}}, \ldots, \frac{z_t (y-a)^{p_t}}{(ac+d)^{r_t} (gy+f)^{s_t}}, \right.$$

$$\left. \frac{-\tau h^r (y-a)^{\mu r}}{(ac+d)^{\gamma r} (gy+f)^{\theta r}}, \frac{c(y-a)}{(ac+d)}, \frac{-g(y-a)}{(gy+f)} \middle| \begin{matrix} C : K; (1-u+sn, 1); -; & (1-\sigma, 1) \\ D : L; \quad (0,1); \quad (0,1); & (0,1) \end{matrix} \right],$$

where

$$C = (1-\eta-\lambda\eta_G-\gamma R : r_1, \ldots, r_t, \gamma r, 1, 0), (1-\omega-\delta\eta_G-\theta R : s_1, \ldots, s_t, \theta r, 0, 1),$$

$$(1-\rho-\zeta\eta_G-\mu R : p_1, \ldots, p_t, \mu r, 1, 0), \left(a_j : \alpha_j^{(1)}, \ldots, \alpha_j^{(t)}, 0, 0, 0\right)_{1,P},$$

$$D = (1-\eta-\lambda\eta_G-\gamma R : r_1, \ldots, r_t, \gamma r, 0, 0), (1-\omega-\delta\eta_G-\theta R : s_1, \ldots, s_t, \theta r, 0, 0),$$

$$(1-\rho-\sigma-\zeta\eta_G-\mu R : p_1, \ldots, p_t, \mu r, 1, 1), \left(b_j : \beta_j^{(1)}, \ldots, \beta_j^{(t)}, 0, 0, 0\right)_{1,Q},$$

$$K = \left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,P_1}; \ldots; \left(c_j^{(t)}, \gamma_j^{(t)}\right)_{1,P_t},$$

$$L = \left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,Q_1}; \ldots; \left(d_j^{(t)}, \delta_j^{(t)}\right)_{1,Q_t},$$

$$R = l(m+n) + ru + e.$$

$${}_aD_y^{-\sigma} \left\{ \frac{(cy+d)^\eta (gy+f)^\omega}{(y-a)^{1-\rho}} H^{m',n'}_{p',q'} \left[\frac{z(cy+d)^\lambda (gy+f)^\delta}{(y-a)^\zeta} \middle| \begin{matrix} (e_{p'}, E_{p'}) \\ (f_{q'}, F_{q'}) \end{matrix} \right] \right.$$

$$\times S_n^{\alpha,\beta,\tau}\left[\frac{h(cy+d)^{\gamma}(gy+f)^{\theta}}{(y-a)^{\mu}}\right]H\left[\frac{z_1(cy+d)^{r_1}(gy+f)^{s_1}}{(y-a)^{p_1}},\ldots,\frac{z_t(cy+d)^{r_t}(gy+f)^{s_t}}{(y-a)^{p_t}}\right]\Bigg\}$$

$$=\sum_{g'=1}^{m'}\sum_{G=0}^{\infty}\sum_{u=0}^{m+n}\sum_{\upsilon=0}^{u}\sum_{e=0}^{m+n}\sum_{p=0}^{e}\frac{(-1)^G\phi(\eta_G)z^{\eta_G}B^{qn}(-1)^e(-e)_p(\alpha)_e(-u)_\upsilon}{G!F_{g'}\,u!\upsilon!e!\,p!}$$

$$\times\frac{(-\alpha-qn)_p}{(1-\alpha-e)_p}\left(-\frac{\beta}{\tau}-sn\right)_u l^{m+n}(-\tau)^u\left(\frac{p+k+r\upsilon}{l}\right)_{m+n}\left(\frac{A}{B}\right)^e\frac{h^R}{\Gamma(u-sn)\Gamma(\sigma)}$$

$$\times(ac+d)^{\eta+\lambda\eta_G+\gamma R}(bg+f)^{\omega+\delta\eta_G+\theta R}(b-a)^{\rho+\sigma-\zeta\eta_G-\mu R-1}$$

$$\times H^{0,\,M+3:N_1,\,M_1;\ldots;\,N_t,\,M_t;\,1,1;\,1,0;\,1,1}_{Q+3,\,P+3:Q_1,\,P_1;\ldots;\,Q_t,\,P_t;\,1,1;\,0,1;\,1,1}\left[\frac{(y-a)^{p_1}}{z_1(ac+d)^{r_1}(gy+f)^{s_1}},\ldots,\frac{(y-a)^{p_t}}{z_t(ac+d)^{r_t}(gy+f)^{s_t}},\right.$$

$$\left.\frac{-(y-a)^{\mu r}}{\tau h^r(ac+d)^{\gamma r}(gy+f)^{\theta r}},\frac{c(y-a)}{(ac+d)},\frac{-g(y-a)}{(gy+f)}\middle|\begin{matrix}C':K'; & (1,1); & -; & (1-\sigma,1)\\ D':L'; & (u-sn,1); & (0,1); & (0,1)\end{matrix}\right],$$

where

$$\begin{aligned}
C' &= (1+\eta+\lambda\eta_G+\gamma R:r_1,\ldots,r_t,\gamma r,1,0),(1+\omega+\delta\eta_G+\theta R:s_1,\ldots,s_t,\theta r,0,1),\\
&\quad(1-\rho+\zeta\eta_G+\mu R:p_1,\ldots,p_t,\mu r,1,0),\left(1-b_j:\beta_j^{(1)},\ldots,\beta_j^{(t)},0,0,0\right)_{1,Q},\\
D' &= (1+\eta+\lambda\eta_G+\gamma R:r_1,\ldots,r_t,\gamma r,0,0),(1+\omega+\delta\eta_G+\theta R:s_1,\ldots,s_t,\theta r,0,0),\\
&\quad(1-\rho-\sigma+\zeta\eta_G+\mu R:p_1,\ldots,p_t,\mu r,1,1),\left(1-a_j:\alpha_j^{(1)},\ldots,\alpha_j^{(t)},0,0,0\right)_{1,P},\\
K' &= \left(1-d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1};\ldots;\left(1-d_j^{(t)},\delta_j^{(t)}\right)_{1,Q_t},\\
L' &= \left(1-c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1};\ldots;\left(1-c_j^{(t)},\gamma_j^{(t)}\right)_{1,P_t},\\
R &= l(m+n)+ru+e.
\end{aligned}$$

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