J M M
Journal of Mechanics and MEMS

ISSN: 0974-8407
© Serials Publications
9(2) 2017: pp. 217-241

Special Issue dedicated to Prof. K Y Ye edited by Z M Ye, P Liu and B Sun

SOME PROBLEMS FOR DIFFERENT YOUNG'S MODULUS WHEN TENSION AND COMPRESSION

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ABSTRACT

Many materials have different Young's modulus when tension and compression. Great error may occur if the classical theory of elasticity is used to analyze the deformation and intensity of the structure or the component made of the materials. Based on an improvement to the 2-D FEM with different modulus, in this paper, the influence of variety of the ratio ϖ between tensile and compressive modulus is further discussed by the problem of a bending beam under various conditions, such as geometric model, loadings. Numerical results show that with the increase of ϖ, the deflection and the stress continuously change under different operation conditions. When the simply-supported beam with different modulus is exerted symmetric loadings, the point of the max deflection is located in the middle section. However, with the bending deformation of the beam with different modulus increasing, not only bending deformation but also excursion occurs on the neutral axis. The middle of the beam with different modulus is no more the neutral axis. Great relative differences of computational results using different modulus model and uniform modulus model indicate that great errors of rigidity and intension exist when the deformation and the stress are analyzed by uniform modulus model that $E = E^+ = E^-$. Also, under the condition of the same ϖ, the relative differences respectively keep mainly the same values under various conditions. It shows that ϖ is the key influence factor leading to computational error by uniform modulus model.

Keywords: Different Young's modulus when tension via compression stresses; Finite element method; Dimensionless; Simply-supported beam; Neutral axis

1. INTRODUCTION

As is well known tensile modulus is assumed the same as compressive modulus in the classical theory of elasticity. Most metal materials, like steels, follow the assumption. However, many materials have different modulus when they are loaded with tension and with compression. Some engineering materials, especially new materials, are widely developed and applied, such as powder metallurgical materials, composite materials, etc. Most of them have distinctive mechanical characteristics, one of which is different Young's modulus when tension via compression stresses. As early as in 1864, Saint-Venant [1] observed that some materials have different elastic behavior when they are loaded with tension and with compression. The concept of a material having different modulus in tension and in compression was originated to Timoshenko [2] who considered the bending stresses in such a material undergoing simple bending in 1941. This problem was not paid much attention to until Ambartsumyan, Khachatryan, etc advanced the bi-modulus concept again in 1960s when they studied the axisymmetric problem of a circular cylindrical shell [3]. In solving some 2-D and 3-D problems [4~6], they created the theory of elasticity with different Young's modulus when tension via compression stresses [7].

With the advance of science and technology, some researchers turn into a new trend to research and develop new materials and to explore potency of material speciality in itself. However, great error may occur if the classical theory of elasticity is used to analyze the deformation and intensity of the structure or the component made of the above-mentioned materials.

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The finite element method (FEM) has been applied to many engineering fields. As the mechanical properties of the materials have not been clarified [8~12], the finite element method for different Young's modulus when tension via compression stresses [13~29] is not widely developed and applied in practical engineering. With the development of contemporary engineering materials, the materials with different modulus will be widely used in engineering. So we have to develop an effective numerical method to correctly analyze the mechanical properties of these materials.

2. THEORETICAL FUNDAMENT OF ELASTICITY WITH DIFFERENT YOUNG'S MODULUS WHEN TENSION VIA COMPRESSION STRESSES

2.1. The Fundamental Assumption of the Materials

The following assumptions are made [7]

(1) The media is elastic and continuous.

(2) The media is line-elastic case, with small strain and displacement.

(3) The general law of continuous medium is satisfied.

(4) There is no initial stress.

Between this theory and the classical theory of elasticity, the difference only exists in constitutive equations. A large number of experimental data have shown that the constitutive relationship between the stress and the strain is linear but with different slops [7] (Fig. 1). We assume tensile modulus as E^+ and the corresponding Poisson's ratio as ν^+, and compressive modulus as E^- and the Poisson's ratio ν^-.

2.2. The Elasticity Equations for Different Modulus [30~32]

2.2.1. The Stress Vector and Strain Vector

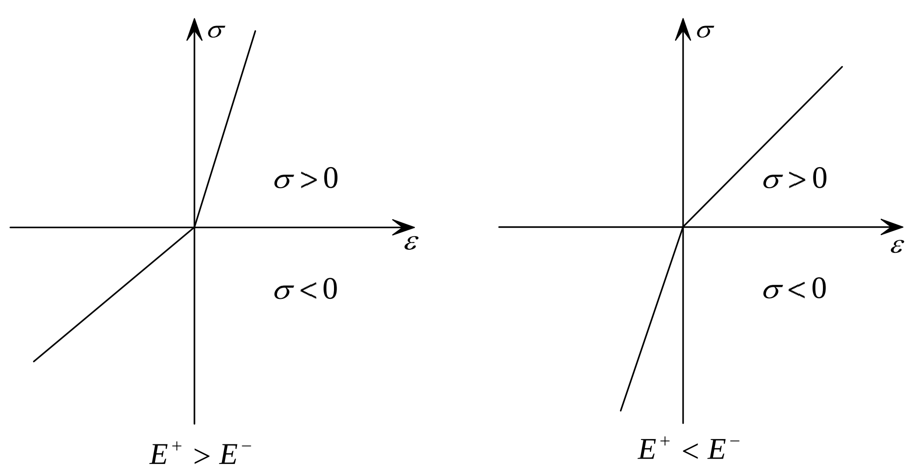

Figure 1: The Constitutive Relationship of Different Modulus when Tension via Compression Stresses

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}, \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{1}$$

Here, {σ}, {ε} are relatively stress vector and stain vector.

2.2.2. The Equilibrium Equations and the Geometrical Equations

I. The equilibrium equations

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+\bar{f}_x=0\\ \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+\bar{f}_y=0\\ \frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+\bar{f}_z=0\end{aligned}\right\} \tag{2-1}$$

$$\begin{bmatrix}\frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix}+\begin{Bmatrix}\bar{f}_x\\ \bar{f}_y\\ \bar{f}_z\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix} \tag{2-2}$$

or

$$[A]\{\sigma\}+\{\bar{f}\}=\{0\} \tag{2-3}$$

II. The geometrical equations

$$\varepsilon_x=\frac{\partial u}{\partial x},\varepsilon_y=\frac{\partial v}{\partial y},\varepsilon_z=\frac{\partial w}{\partial y},$$

$$\gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\gamma_{xz}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z},\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \tag{3-1}$$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix}=\begin{bmatrix}\frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\end{bmatrix}\begin{Bmatrix}u\\ v\\ w\end{Bmatrix} \tag{3-2}$$

or

$$\{\varepsilon\} = [L]\{u\} \tag{3-3}$$

where

$$[L] = [A]^T \tag{3-4}$$

2.2.3. The Constitutive Equations

The bi-linear stress-strain relationship when tension via compression stresses will be discussed in details. To investigate the mechanical behaviors in such elastic media, it is necessary to establish the general constitutive equations. Inner variable will be introduced and the elastic constitutive equations will be derived. Also, this paper deals with the method of reconstructing the constitutive relationship by means of the symbol of principal stress or principal strain.

I. The partition of stress zone

According to the sign of principal stress, four kinds of combination are obtained

(1) When $\sigma_\alpha > 0$, $\sigma_\beta > 0$, $\sigma_\gamma > 0$, all principal stresses are tension stresses.

(2) When $\sigma_\alpha < 0$, $\sigma_\beta < 0$, $\sigma_\gamma < 0$, all principal stresses are compression stresses.

(3) When $\sigma_\alpha > 0$, $\sigma_\beta < 0$, $\sigma_\gamma < 0$ one principal stress is tension stress.

(4) When $\sigma_\alpha > 0$, $\sigma_\beta > 0$, $\sigma_\gamma < 0$, one principal stress is compression stress.

The material model has the following zones

(1) Zone I: the signs of all principal stresses are uniform (Tension Stress Zone or Compression Stress Zone).

(2) Zone II: the signs of one principal stress and of the others are different (Complex Stress Zone).

II. Criterion of the Principal Stress [7]

The constitutive equations are represented using principal stress and principal strain relation

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix} \tag{4-1}$$

or

$$\{\varepsilon_I\} = [a_I]\,\{\sigma_I\} \tag{4-2}$$

Based on the sign of the principal stress, the corresponding multiplied compliance in each row is adopted: If $\sigma_\alpha > 0$, $a_{i1} = a_{i1}^{+}$. Otherwise, $a_{i1} = a_{i1}^{-}$. a_{i2}, a_{i3} are obtained by similar regulation. The constitutive relation is concretely expressed by

(1) Zone I (Tension Stress Zone or Compression Stress Zone)

When $\sigma_\alpha > 0$, $\sigma_\beta > 0$, $\sigma_\gamma > 0$

$$[a_I] = \begin{bmatrix} \dfrac{1}{E^+} & -\dfrac{\nu^+}{E^+} & -\dfrac{\nu^+}{E^+} \\ -\dfrac{\nu^+}{E^+} & \dfrac{1}{E^+} & -\dfrac{\nu^+}{E^+} \\ -\dfrac{\nu^+}{E^+} & -\dfrac{\nu^+}{E^+} & \dfrac{1}{E^+} \end{bmatrix} \tag{5-1}$$

When $\sigma_\alpha < 0$, $\sigma_\beta < 0$, $\sigma_\gamma < 0$

$$[a_I] = \begin{bmatrix} \dfrac{1}{E^-} & -\dfrac{\nu^-}{E^-} & -\dfrac{\nu^-}{E^-} \\ -\dfrac{\nu^-}{E^-} & \dfrac{1}{E^-} & -\dfrac{\nu^-}{E^-} \\ -\dfrac{\nu^-}{E^-} & -\dfrac{\nu^-}{E^-} & \dfrac{1}{E^-} \end{bmatrix} \tag{5-2}$$

In Zone I, the compliance matrix is directly adopted $[a_I]^+$ or $[a_I]^-$, which is consistent with the classical theory of elasticity.

(2) Zone II (Complex Stress Zone)

When $\sigma_\alpha > 0$, $\sigma_\beta < 0$, $\sigma_\gamma < 0$

$$[a_I] = \begin{bmatrix} \dfrac{1}{E^+} & -\dfrac{\nu^-}{E^-} & -\dfrac{\nu^-}{E^-} \\ -\dfrac{\nu^+}{E^+} & \dfrac{1}{E^-} & -\dfrac{\nu^-}{E^-} \\ -\dfrac{\nu^+}{E^+} & -\dfrac{\nu^-}{E^-} & \dfrac{1}{E^-} \end{bmatrix} \tag{5-3}$$

When $\sigma_\alpha > 0$, $\sigma_\beta < 0$, $\sigma_\gamma < 0$

$$[a_I] = \begin{bmatrix} \dfrac{1}{E^+} & -\dfrac{\nu^+}{E^+} & -\dfrac{\nu^-}{E^-} \\ -\dfrac{\nu^+}{E^+} & \dfrac{1}{E^+} & -\dfrac{\nu^-}{E^-} \\ -\dfrac{\nu^+}{E^+} & -\dfrac{\nu^+}{E^+} & \dfrac{1}{E^-} \end{bmatrix} \tag{5-4}$$

According to the previous analysis, the following is obtained

$$[D_I] = [a_I]^{-1} \tag{6}$$

III. Criterion of the Principal Strain [24, 33, 34]

Depending on the sign of principal strain, the constitutive relationship is also expressed by principal stress and principal strain relation

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix} \tag{7-1}$$

or

$$\{\sigma_I\} = [D_I]\,\{\varepsilon_I\} \tag{7-2}$$

According to the sign of principal strain, the corresponding multiplied compliance in each row is adopted: If $\varepsilon_\alpha > 0$, $d_{i1} = d_{i1}^+$. Otherwise, $d_{i1} = d_{i1}^-$. d_{i2}, d_{i3} are given by similar regulation.

IV. The constitutive equations

The transforming matrix is used

$$[L]=\begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & m_1n_1 & l_1n_1 & l_1m_1 \\ l_2^2 & m_2^2 & n_2^2 & m_2n_2 & l_2n_2 & l_2m_2 \\ l_3^2 & m_3^2 & n_3^2 & m_3n_3 & l_3n_3 & l_3m_3 \end{bmatrix} \tag{8}$$

The following formulation is obtained

$$[D] = [L]^T [D_l] [L] \tag{9}$$

where $[D]$ is the elastic matrix with different Young's modulus when tension via compression.

Then, the constitutive equations can be derived

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{10-1}$$

or

$$\{\sigma\} = [D]\{\varepsilon\} \tag{10-2}$$

2.2.4. The Boundary Conditions

I. The force boundary conditions

Assume that there are the inner force components T_x, T_y, T_z and the surface forces components $\overline{T}_x, \overline{T}_y, \overline{T}_z$ on the boundary S_σ of an elastic body. According to the equilibrium conditions

$$T_x = \overline{T}_x, T_y = \overline{T}_y, T_z = \overline{T}_z \tag{11-1}$$

On the assumption that **N** is the outer normal on the boundary whose direction cosines are n_x, n_y, n_z, the inner forces are represented by

$$\begin{cases} T_x = n_x\sigma_x + n_y\tau_{xy} + n_z\tau_{xz} \\ T_y = n_x\tau_{xy} + n_y\sigma_y + n_z\tau_{yz} \\ T_z = n_x\tau_{xz} + n_y\tau_{yz} + n_z\sigma_z \end{cases} \tag{11-2}$$

The above formulations are rewritten in matrix expression as

$$\{T\} = \{\overline{T}\} \ (\text{on } S_\sigma) \tag{11-3}$$

where

$$\{T\} = [n]\{\sigma\} \tag{11-4}$$

$$[n]=\begin{bmatrix} n_x & 0 & 0 & n_y & 0 & n_z \\ 0 & n_y & 0 & n_x & n_z & 0 \\ 0 & 0 & n_z & 0 & n_y & n_x \end{bmatrix} \tag{11-5}$$

II. The geometric boundary condition

$\overline{u}, \overline{v}, \overline{w}$ are assumed as known displacement components on the boundary S_u of the elastic body. Then

$$u = \overline{u},\ v = \overline{v},\ w = \overline{w} \tag{12-1}$$

The matrix expression is

$$\{u\} = \{\overline{u}\}\ \ (\text{on } S_u) \tag{12-2}$$

3. THE FINITE ELEMENT METHOD WITH DIFFERENT YOUNG'S MODULUS WHEN TENSION VIA COMPRESSION STRESSES [24,32~35]

Based on the finite element methods (FEM), many researchers have considered lots of relative factors (such as E^+, ν^+, σ_α, $\varepsilon\alpha$, etc) to modify the stiffness matrix and establish the corresponding finite element formulation with different modulus**.**

3.1. The Element Displacement Model and the Corresponding Strain Matrix and Stress Matrix

3.1.1. The Element Displacement Model

The approximate function in the polynomial expression is adopted in the element displacement mode. While the displacements in the element are similarly expressed by the coordinates function, the nodal displacements are obtained using nodal coordinates as the following.

$$\{u\} = [N]\ \{d^e\} \tag{13}$$

Here, $[N]$ is interpolating function matrix or shape function matrix.

3.1.2. The Strain Matrix and the Stress Matrix

After the element displacement is adopted, the strain and stress are accordingly derived based on the geometrical equations and the constitutive equations.

I. The strain matrix

Substituting Equation (13) into Equation (3-3), the expression (3-3) is rewritten as

$$\{\varepsilon\} = [L]\ \{u\} = [L]\ [N^e]\ \{d^e\} = [B^e]\ \{d^e\} \tag{14}$$

where, $[B]$ is the strain matrix.

II. The stress matrix

Replacing Equation (14) into Equation (10-2), the following is obtained by

$$\{\sigma\} = [D^e]\ \{\varepsilon\} = [D^e]\ [B^e]\ \{d^e\} = [S^e]\ \{d^e\} \tag{15}$$

where, $[S]$ is the stress matrix.

3.2. The Finite Element Formulation and the Iterative Formulation for Different Young's Modulus

The element volume strain energy is taken as

$$\overline{U} = \frac{1}{2}\{\varepsilon\}^T\{\sigma\} = \frac{1}{2}\{\varepsilon\}^T[D]\{\varepsilon\} \tag{16}$$

The system energy is

$$\Pi_p = \frac{1}{2}\int_V \{\varepsilon\}^T[D]\{\varepsilon\}\,dV - \int_V \{u\}^T\{f\}\,dV - \int_{S_\sigma} \{u\}^T\{T\}\,dS \tag{17-1}$$

Here, $\{f\}$ is the volume force vector. $\{T\}$ is the mass force vector on the boundary.

The potential energy of discrete model is given by

$$\Pi_p = \sum_e(\{d^e\}(\int_{V_e}\frac{1}{2}[B^e]^T[D^e][B^e]dV)\{d^e\}) - \sum_e(\{d^e\}^T(\int_{V_e}[N]^T\{f\}\,dV)) \\ -\sum_e(\{d^e\}^T(\int_{S_\sigma^e}[N]^T\{T\}\,dS)) \tag{17-2}$$

The element displacement vector, $\{d^e\}$ is expressed by the nodal displacement vector as follows

$$\{d^e\} = [G]\,\{d\} \tag{18-1}$$

where,

$$\{d^e\} = \{u_1\ v_1\ w_1 \ldots u_i\ v_i\ w_i \ldots u_n\ v_n\ w_n\}^T \tag{18-2}$$

Supposing

$$[K^e] = \int_{V_e}[B^e]^T[D^e][B^e]\,dV \tag{19}$$

$$\{P^e\} = \{P_f^e\} + \{P_S^e\} \tag{20-1}$$

where,

$$\{P_f^e\} = \int_{V_e}[N]^T\{f\}\,dV \text{ , and } \{P_S^e\} = \int_{S_\sigma^e}[N]^T\{T\}\,dS \tag{20-2}$$

$[K^e]$ is element stiffness matrix. $\{P^e\}$ is element equivalent nodal load vector.

Then

$$\Pi_p = \{d\}^T\frac{1}{2}\sum_e([G]^T[K^e][G])\{d\} - \{d\}^T\sum_e([G]^T\{P^e\}) \tag{17-3}$$

Supposing

$$[K] = \sum_e([G]^T[K^e][G]) \tag{21}$$

$$\{P\} = \sum_e([G]^T\{P^e\}) \tag{22}$$

Here, $[K]$ is system stiffness matrix. $\{P\}$ is system load vector.

Then

$$\Pi_p = \frac{1}{2}\{d\}^T[K]\{d\} - \{d\}^T\{P\} \tag{17-4}$$

According to minimum potential principle $\frac{\partial \Pi_p}{\partial\{d\}} = 0$. The variational expression of the energy Π_p is given by

$$[K]\,\{d\} = \{P\} \tag{23}$$

The elastic problem, with different modulus when tension via compression stresses, is materially non-linear. Based on piecewise linearization, the iterative technology is used to solve that problem. During every iterative step, different with the classical FEM computation, the principal stress state is judged in every element again so as to obtain the corresponding elastic matrix.

I. The FEM iterative procedure for different modulus

The computational procedure of the finite element analysis is expressed as the following.

(1) Assuming the same elastic modulus with tension and compression, the stress and strain fields are calculated.

(2) The principal stress and strain values and their directions can be gotten for each element.

(3) The corresponding elastic matrix and rigidity matrix can be re-obtained for each element.

(4) According to the new constitutive relation, the new stress and strain are calculated.

(5) The values in this step and those in the previous step are compared. When the differences are among a certain arrange we assume the final results are reached. Otherwise, the calculation returns to (2).

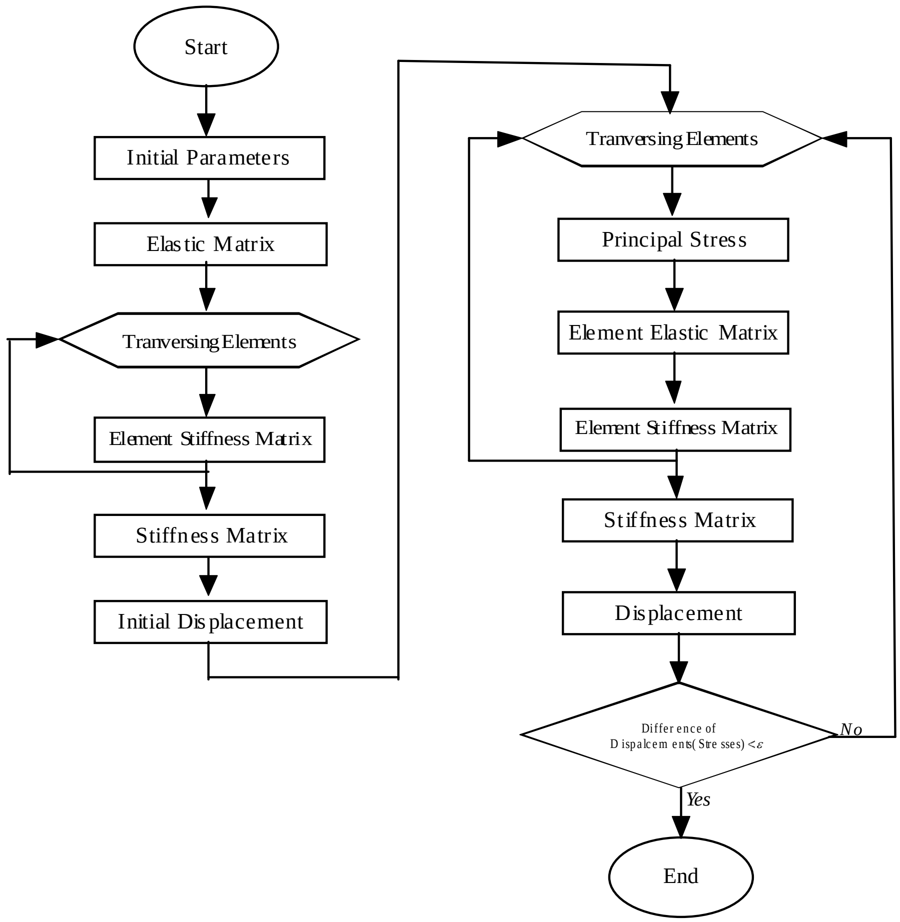

Figure 2: Flow Chart of the Problem with Different Modulus when Tension via Compression Stresses

II. Selection of initial material coefficients [36~38]

(1) As a rule, supposing $E^+ = E^- = E$, $v^+ = v^- = v$, initial displacement $\{d\}^0$ is reached using the same modulus model.

(2) If two stress zones could be estimated before computing, elasticity coefficients are properly chosen: E^+, v^+ are used in Tension Stress Zone. E^-, v^- are used in Compression Stress Zone.

III. The iterative formulation

The variable stiffness direct iterative method is usually taken. The iterative procedure is

$$[K]_{i-1}\{d\}_i = \{P\} \tag{24}$$

Yang, Wu, and so on [39, 40] applied the initial stress method to solve the different modulus problems. The iterative procedure was

$$[K]\{d\}_i = \{P\} - \{P^*\}_{i-1} \tag{25-1}$$

where

$$[K] = \sum_e \int_{V_e} [B]^T [D_0][B] dV \tag{25-2}$$

$$\{P^*\}_i = \sum_e \int_{V_e} [B]^T([D]-[D_0])[B]dV\{d\}_i \tag{25-3}$$

The same modulus elastic matrix $[D_0]$ is given in advance.

An effective method [38~42] was presented to accelerate convergence. A general formula of the shear modulus was put forward and an accelerating convergence factor η was applied to numerical computation. Supposing that $\theta = \nu^+/E^+ - \nu^-/E^- \neq 0$, the iterative formulation was described as

$$[K]_{i-1}\{u\}_i = \{P\} - \Delta[K]_{i-1}\{u\}_{i-1} \tag{26}$$

The error of supposing $\theta = 0$ was analyzed, and the upper limit of the error was given.

3.3. Ye's Improved Method

Ye, Yu, Yao, *et al.* [24, 33, 34] made an assumption that when the modulus was modified the Poisson's ratio was not changed, and presented a new finite element formulation for engineering application. The given principal stress and principal strain relationship was written as

$$[D_I] = \begin{bmatrix} \frac{E_1}{1-\nu^2} & \frac{E_1\nu}{1-\nu^2} & \frac{E_1\nu}{1-\nu^2} \\ \frac{E_2\nu}{1-\nu^2} & \frac{E_2}{1-\nu^2} & \frac{E_2\nu}{1-\nu^2} \\ \frac{E_3\nu}{1-\nu^2} & \frac{E_3\nu}{1-\nu^2} & \frac{E_3}{1-\nu^2} \end{bmatrix} \left(\text{or } [D_I] = \begin{bmatrix} \frac{E_1}{1-\nu^2} & \frac{E_2\nu}{1-\nu^2} & \frac{E_3\nu}{1-\nu^2} \\ \frac{E_1\nu}{1-\nu^2} & \frac{E_2}{1-\nu^2} & \frac{E_3\nu}{1-\nu^2} \\ \frac{E_1\nu}{1-\nu^2} & \frac{E_2\nu}{1-\nu^2} & \frac{E_3}{1-\nu^2} \end{bmatrix}\right) \tag{27}$$

When $\theta = \nu^+/E^+ - \nu^-/E^- \neq 0$, the governing equations could be represented as the following

$$\frac{[K]+[K]^T}{2}\{d\} = \{P\} \tag{28}$$

3.4. The 2-D Finite Element Formulation and Iterative Formulation for Different Modulus

The element volume strain energy is expressed as

$$\bar{U} = \frac{1}{2}\{\varepsilon\}^T\{\sigma\} = \frac{1}{2}\{\varepsilon\}^T[D]\{\varepsilon\} \tag{29}$$

The system energy is written by

$$\Pi_p = \frac{1}{2}\int_\Omega \{\varepsilon\}^T[D]\{\varepsilon\}\, t\, d\Omega - \int_\Omega \{u\}^T\{f\}\, t\, d\Omega - \int_{S_\sigma} \{u\}^T\{T\}\, t\, ds \tag{30-1}$$

Here, $\{f\}$ is the volume force. t is the thickness of the 2-D body. $\{T\}$ is the mass force on the boundary. The potential energy of discrete model is given by

$$\begin{aligned}\Pi_p = &\sum_e (\{d^e\}(\frac{1}{2}\int_{\Omega_e}[B^e]^T[D^e][B^e]tdxdy)\{d^e\}) \\ &-\sum_e (\{d^e\}^T(\int_{\Omega_e}[N]^T\{f\}t\, dxdy) - \sum_e (\{d^e\}^T(\int_{S_\sigma^e}[N]^T\{T\}t\, ds))\end{aligned} \tag{30-2}$$

The element displacement vector $\{d^e\}$ is expressed using the nodal displacement vector as follows

$$\{d^e\} = [G]\{d\} \tag{31}$$

where $\{d^e\} = \{u_1 \quad v_1 \quad \ldots \quad u_i \quad v_i \quad \ldots \quad u_n \quad u_n\}^T$. i is the i-th nodal number.

Supposing

$$[K^e] = \int_{\Omega_e}[B^e]^T[D^e][B^e]t\, d\Omega \tag{32}$$

$$\{P_f^e\} = \int_{\Omega_e} [N]^T \{f\} t \, d\Omega, \{P_S^e\} = \int_{S_\sigma^e} [N]^T \{T\} t \, dS \tag{33-1}$$

$$\{P^e\} = \{P_f^e\} + \{P_S^e\} \tag{33-2}$$

Here, $[K^e]$ is element stiffness matrix. $\{P^e\}$ is element equivalent nodal load vector. Then

$$\Pi_p = \{d\}^T \frac{1}{2} \sum_e ([G]^T [K^e][G])\{d\} - \{d\}^T \sum_e ([G]^T \{P^e\}) \tag{30-3}$$

Supposing

$$[K] = \sum_e ([G]^T [K^e][G]) \tag{34}$$

$$\{P\} = \sum_e [G]^T \{P^e\} \tag{35}$$

where $[K]$ is system stiffness matrix. $\{P\}$ is system load vector.

Then

$$\Pi_p = \frac{1}{2} \{d\}^T [K]\{d\} - \{d\}^T \{P\} \tag{30-4}$$

According to minimum potential principle, $\frac{\partial \Pi_p}{\partial \{d\}} = 0$. The variational expression of the energy Π_p is given by

$$[K] \{d\} = \{P\} \tag{36}$$

3.5. The 3-node Triangular Finite Element Formulation and Iterative Formulation for Different Young's Modulus

The nodal displacement vector is represented using nodal coordinates as the following

$$\begin{aligned}
\{u\} = \begin{Bmatrix} u \\ v \end{Bmatrix} &= \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_i \\ v_j \\ u_m \\ v_m \end{Bmatrix} \\
&= \begin{bmatrix} \mathbf{I}N_i & \mathbf{I}N_j & \mathbf{I}N_m \end{bmatrix} \begin{Bmatrix} \mathbf{d}_i \\ \mathbf{d}_j \\ \mathbf{d}_k \end{Bmatrix} \\
&= \begin{bmatrix} \mathbf{N}_i & \mathbf{N}_j & \mathbf{N}_m \end{bmatrix} = [N]\{d^e\}
\end{aligned} \tag{37}$$

The strain matrix is written by

$$\begin{aligned}
\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} &= [L]\{u\} = [L][N]\{d^e\} = [L]\begin{bmatrix} \mathbf{N}_i & \mathbf{N}_j & \mathbf{N}_m \end{bmatrix}\{d^e\} \\
&= \begin{bmatrix} \mathbf{B}_i & \mathbf{B}_j & \mathbf{B}_m \end{bmatrix}\{d^e\} = [B]\{d^e\}
\end{aligned} \tag{38}$$

where $[B]$ is the strain matrix. The partitioned sub-matrix is expressed as

$$\mathbf{B}_i = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & 0 \\ 0 & \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_i}{\partial x} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} b_i & 0 \\ 0 & c_i \\ c_i & b_i \end{bmatrix} \; (i, j, m) \tag{39}$$

Here, A is the area of the triangular of the element.

The 3-node element strain matrix is expressed as follows

$$[B] = \begin{bmatrix} \mathbf{B}_i & \mathbf{B}_j & \mathbf{B}_m \end{bmatrix} = \begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix} \tag{40}$$

The stress matrix is obtained by

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D^e]\{\varepsilon\} = [D^e][B]\{d^e\} = [S^e]\{d^e\} \tag{41}$$

where $[S]$ is the stress matrix.

$$[S^e] = [D^e]\,[B] = [D^e]\,[\mathbf{B}_i\,\mathbf{B}_j\,\mathbf{B}_m] = [\mathbf{S}_i\,\mathbf{S}_j\,\mathbf{S}_m] \tag{42}$$

The 3-node element stiffness matrix can be written

$$[K^e] = [B]^T\,[D^e]\,[B]\,tA \tag{43}$$

The element equilibrium equations are established as the following

$$[K^e]\,\{d^e\} = \{P^e\} \tag{44}$$

Assume there is a rectangular with length L. The element equilibrium equations are rewritten by $[B]^T[\bar{D}^e][B]tA\{d^e\} = \{P^e\}$. When the element equilibrium equations are divided by the term EL^2 in two sides, the formulations can be derived as the following.

$$\frac{L}{2A}\begin{bmatrix} b_i & 0 & c_i \\ 0 & c_i & b_i \\ b_j & 0 & c_j \\ 0 & c_j & b_j \\ b_m & 0 & c_m \\ 0 & c_m & b_m \end{bmatrix} \cdot \frac{1}{E}\begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \cdot \frac{L}{2A}\begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix} \cdot \frac{t}{L} \cdot \frac{A}{L^2} \begin{Bmatrix} \dfrac{u_i}{L} \\ \dfrac{v_i}{L} \\ \dfrac{u_j}{L} \\ \dfrac{v_j}{L} \\ \dfrac{u_m}{L} \\ \dfrac{v_m}{L} \end{Bmatrix} = \begin{Bmatrix} \dfrac{P_{ix}}{EL^2} \\ \dfrac{P_{iy}}{EL^2} \\ \dfrac{P_{jx}}{EL^2} \\ \dfrac{P_{jy}}{EL^2} \\ \dfrac{P_{mx}}{EL^2} \\ \dfrac{P_{my}}{EL^2} \end{Bmatrix} \tag{45-1}$$

The simplified form is

$$[B']^T[\bar{D}^{e'}][B']t'A'\{d^{e'}\}=\{P^{e'}\} \tag{45-2}$$

Here, $$[B']=\frac{L}{2A}\begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix},\ t'=\frac{t}{L},\ A'=\frac{A}{L^2},$$

$$\{d^{e'}\}=\begin{Bmatrix} \frac{u_i}{L} \\ \frac{v_i}{L} \\ \frac{u_j}{L} \\ \frac{v_j}{L} \\ \frac{u_m}{L} \\ \frac{v_m}{L} \end{Bmatrix},\ \{P^{e'}\}=\begin{Bmatrix} \frac{P_{ix}}{EL^2} \\ \frac{P_{iy}}{EL^2} \\ \frac{P_{jx}}{EL^2} \\ \frac{P_{jy}}{EL^2} \\ \frac{P_{mx}}{EL^2} \\ \frac{P_{my}}{EL^2} \end{Bmatrix},\ [\bar{D}^{e'}]=\frac{1}{E}\begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \tag{45-3}$$

The stress and the strain are given as follow

$$\{\varepsilon'\}=[B']\{d^{e'}\} \tag{46}$$

$$\{\sigma'\}=[\bar{D}^{e'}]\{\varepsilon'\}=[\bar{D}^{e}][B']\{d^{e'}\}=[S^{e'}]\{d^{e'}\} \tag{47}$$

where E can be usually chosen as E^+ or E^-.

4. IMPROVEMENT TO THE 2-D FINITE ELEMENT FORMULATION

Based on Ye's model [25, 42, 43], an improvement is adopted in the 2-D finite element formulation with different modulus when tension via compression stresses in this paper. The equivalent principal stress and principal strain relation is written as

$$[\bar{D}_I]=\frac{[D_I]+[D_I]^T}{2} \tag{48}$$

And by complete concept, the modified material matrix is given as

$$[\bar{\bar{D}}_I]=\begin{bmatrix} [\bar{D}_I] & 0 \\ 0 & \bar{D}_{33} \end{bmatrix} \tag{49}$$

where $\bar{D}_{33}=d_{33}$

The corresponding element elastic matrix is

$$[\bar{D}]=[\bar{L}]^T[\bar{\bar{D}}_I][\bar{L}] \tag{50}$$

where

$$[\bar{L}]=\begin{bmatrix} l_1^2 & m_1^2 & l_1m_1 \\ l_2^2 & m_2^2 & l_2m_2 \\ 2l_1l_2 & 2m_1m_2 & l_1m_2+l_2m_1 \end{bmatrix} \tag{51}$$

The potential energy of discrete model is expressed as

$$\begin{aligned}\Pi_p &= \sum_e (\{d^e\}^T \left(\int_{\Omega_e} \frac{1}{2}[B^e]^T[\bar{D}^e][B^e]t\ dxdy)\{d^e\} \right) \\ &-\sum_e (\{d^e\}^T (\int_{\Omega_e} [N]^T\{f\}t\ dxdy) - \sum_e (\{d^e\}^T \left(\int_{S_\sigma^e} [N]^T\{T\}t\ ds \right))\end{aligned} \tag{52}$$

And the element stiffness matrix is

$$[\bar{K}^e]=\int_{\Omega_e}[B^e]^T[\bar{D}^e][B^e]t\ dxdy \tag{53}$$

Then

$$\Pi_p = \frac{1}{2}\{d\}^T[\bar{K}]\{d\}-\{d\}^T\{P\} \tag{54}$$

According to the minimum potential principle, $\frac{\partial \Pi_p}{\partial\{d\}}=0$. The variational expression of the energy Π_p is given as

$$[\bar{K}]\{d\}=\{P\} \tag{55}$$

Here, $[\bar{K}]=\frac{[K]+[K]^T}{2}$, which is the derived systemic stiffness matrix. The matrix expression is the same as Ye's.

5. THE INFLUENCE OF VARIETY OF THE RATIO BETWEEN TENSILE AND COMPRESSIVE MODULUS FOR THE BENDING BEAM WITH DIFFERENT MODULUS

The influence of variety of the ratio between tensile and compressive modulus is further discussed by the problem of a bending beam under various conditions, such as geometric model, loadings. Defining $\omega = E^+ / E^-$, changing from 1/4 to 4/1, the ratio ω is applied to the dimensionless computational forms. The following examples are based on the simply-supported beam case. The geometry model and boundary conditions are: $L = 1$, $H = 0.0833$, $P = 1.035 \times 10^{-7}$.

Example 1

In the same condition of two concentrated loadings, the height of the simply-supported beam changes: (1) H = 0.0714, (2) $H = 0.0833$, (3) H = 0.1.

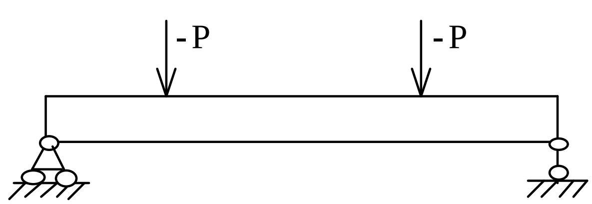

Figure 3: The two Concentrated Loadings

Example 2

Considering the equivalent reaction of support, the loadings are respectively adopted: (3) evenly distributed loadings, (2) two concentrated loadings, (1) single concentrated loading.

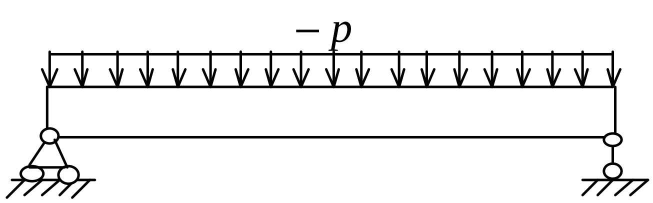

Figure 4.1: The Evenly Distributed Loadings

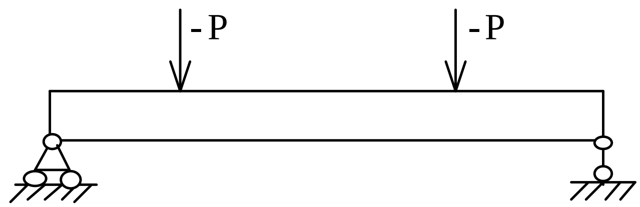

Figure 4.2: The two Concentrated Loadings

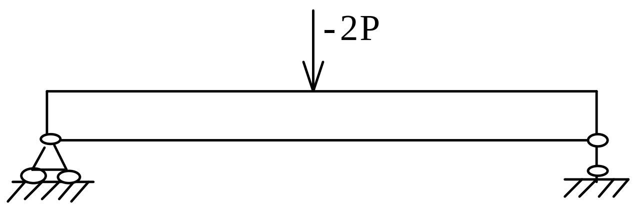

Figure 4.3: The Single Concentrated Loading

5.1. The Locations of the Max Deflection and the Neutral Axis

5.1.1. The Location of the Max Deflection

When the simply-supported beam is exerted symmetric loadings, the point of the max deflection is located in the middle section in classical bending beam theory. This may be similarly applicable to the beam with different modulus.

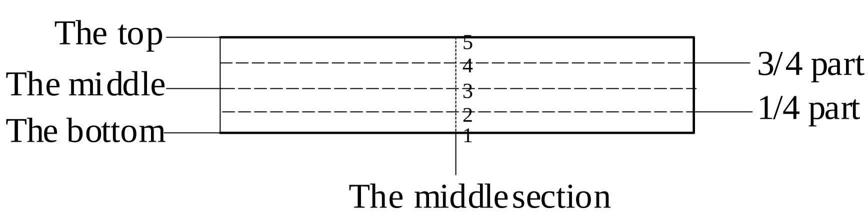

Figure 5: The Lable of Different Locations of the Beam

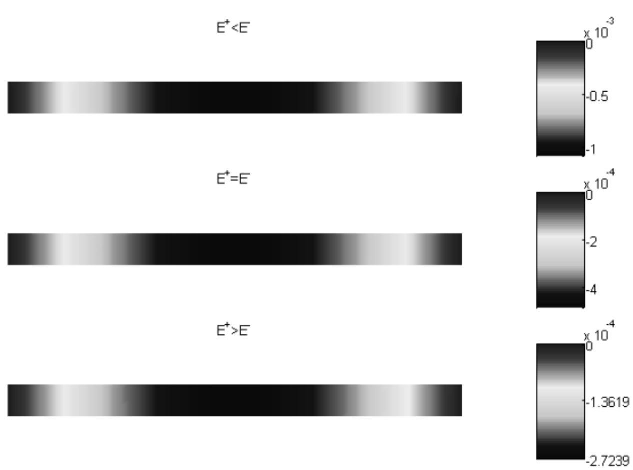

Figure 6.1: Reflection (H = 0.0714)

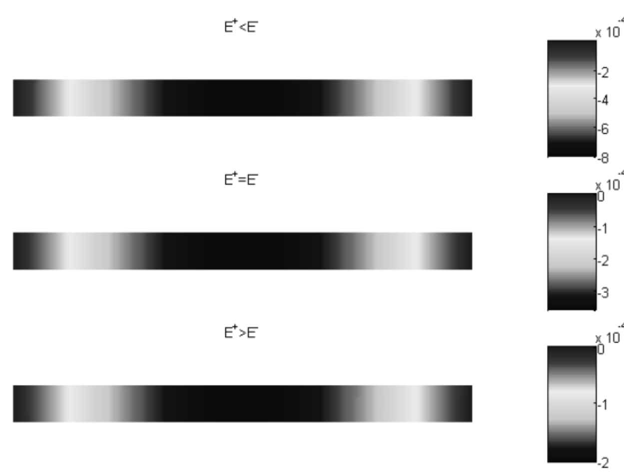

Figure 6.2: Reflection (H = 0.0833)

Figure 6.3: Reflection (H = 0.1)

Figure 6: Variety of Reflections under the Conditions of Different Heights

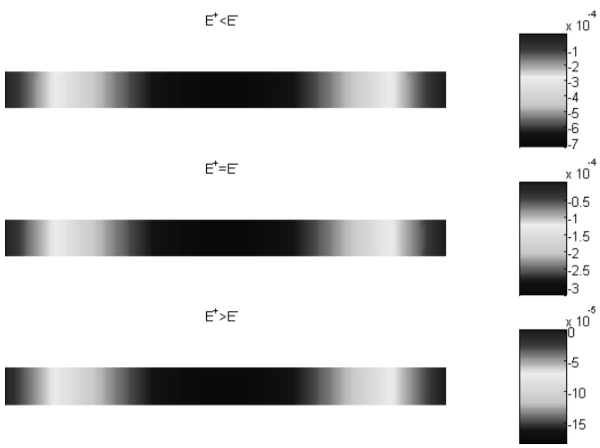

Figure 7.1: Reflection (Evenly Distributed Loadings)

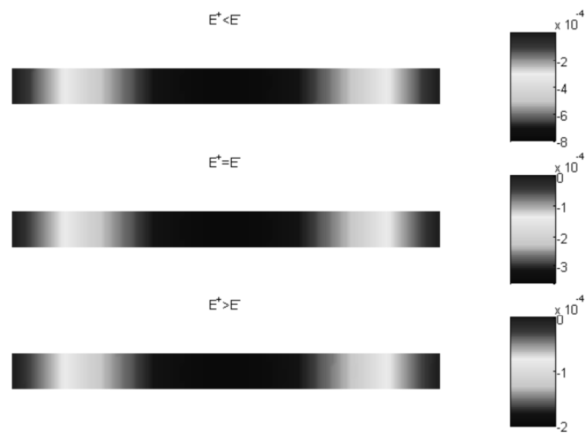

Figure 7.2: Reflection (Two Concentrated Loadings)

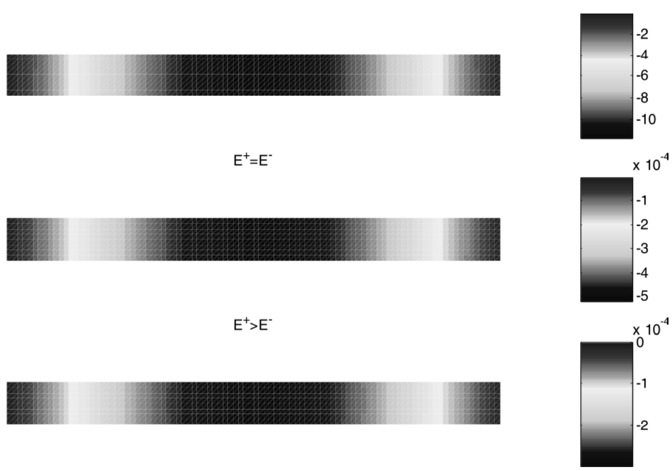

Figure 7.3: Reflection (Single Concentrated Loading)

Figure 7: Variety of Reflections under the Conditions of Different

In each figure of Fig. 6 and Fig. 7, there are three pictures respectively denoting the computational results under the conditions $E^+ < E^-$, $E^+ = E^-$, and $E^+ > E^-$. The results show that the location of the max deflection is located in the middle section under different conditions. Also, the reflections in middle section are mainly the same shown as the following Table 1~ Table 5.

Table 1
The Deflections in the Middle Section (Two Concentrated Loadings, H = 0.0714)

	Point 1	*Point 2*	*Point 3*	*Point 4*	*Point 5*
ϖ = 1/4	-1.0846E-03	-1.0853E-03	-1.0857E-03	-1.0857E-03	-1.0849E-03
ϖ = 1/3.5	-9.9303E-04	-9.9369E-04	-9.9401E-04	-9.9392E-04	-9.9323E-04
ϖ = 1/3	-8.9945E-04	-9.0005E-04	-9.0034E-04	-9.0022E-04	-8.9960E-04
ϖ = 1/2.5	-7.9946E-04	-7.9999E-04	-8.0020E-04	-8.0011E-04	-7.9960E-04
ϖ = 1/2	-7.0183E-04	-7.0231E-04	-7.0250E-04	-7.0239E-04	-7.0193E-04
ϖ = 1/1.5	-5.9716E-04	-5.9757E-04	-5.9773E-04	-5.9762E-04	-5.9722E-04
ϖ = 1	-4.8306E-04	-4.8339E-04	-4.8350E-04	-4.8339E-04	-4.8306E-04
ϖ = 1.5	-3.9858E-04	-3.9885E-04	-3.9892E-04	-3.9881E-04	-3.9854E-04
ϖ = 2	-3.5157E-04	-3.5180E-04	-3.5185E-04	-3.5175E-04	-3.5151E-04
ϖ = 2.5	-3.2050E-04	-3.2070E-04	-3.2074E-04	-3.2065E-04	-3.2044E-04
ϖ = 3	-2.9780E-04	-2.9799E-04	-2.9802E-04	-2.9794E-04	-2.9775E-04
ϖ = 3.5	-2.8012E-04	-2.8029E-04	-2.8032E-04	-2.8025E-04	-2.8007E-04
ϖ = 4	-2.7218E-04	-2.7236E-04	-2.7237E-04	-2.7228E-04	-2.7210E-04

Table 2
The Deflections in the Middle Section (Two Concentrated Loadings, H = 0.0833)

	Point 1	*Point 2*	*Point 3*	*Point 4*	*Point 5*
ϖ = 1/4	-8.0068E-04	-8.0139E-04	-8.0176E-04	-8.0173E-04	-8.0099E-04
ϖ = 1/3.5	-7.3310E-04	-7.3376E-04	-7.3408E-04	-7.3399E-04	-7.3331E-04
ϖ = 1/3	-6.6407E-04	-6.6467E-04	-6.6496E-04	-6.6484E-04	-6.6422E-04
ϖ = 1/2.5	-5.9297E-04	-5.9352E-04	-5.9376E-04	-5.9363E-04	-5.9307E-04
ϖ = 1/2	-5.1947E-04	-5.1995E-04	-5.2015E-04	-5.2002E-04	-5.1945E-04
ϖ = 1/1.5	-4.4100E-04	-4.4141E-04	-4.4157E-04	-4.4146E-04	-4.4106E-04
ϖ = 1	-3.5684E-04	-3.5717E-04	-3.5729E-04	-3.5717E-04	-3.5684E-04
ϖ = 1.5	-2.9456E-04	-2.9482E-04	-2.9490E-04	-2.9479E-04	-2.9452E-04
ϖ = 2	-2.5991E-04	-2.6013E-04	-2.6019E-04	-2.6009E-04	-2.5985E-04
ϖ = 2.5	-2.3817E-04	-2.3840E-04	-2.3845E-04	-2.3835E-04	-2.3813E-04
ϖ = 3	-2.2242E-04	-2.2262E-04	-2.2266E-04	-2.2257E-04	-2.2237E-04
ϖ = 3.5	-2.1059E-04	-2.1079E-04	-2.1081E-04	-2.1072E-04	-2.1053E-04
ϖ = 4	-2.0137E-04	-2.0156E-04	-2.0156E-04	-2.0147E-04	-2.0129E-04

Table 3
The Deflections in the Middle Section (Two Concentrated Loadings, H = 0.1)

	Point 1	*Point 2*	*Point 3*	*Point 4*	*Point 5*
ϖ = 1/4	-5.6048E-04	-5.6120E-04	-5.6156E-04	-5.6153E-04	-5.6079E-04
ϖ = 1/3.5	-5.1326E-04	-5.1392E-04	-5.1425E-04	-5.1415E-04	-5.1347E-04
ϖ = 1/3	-4.6499E-04	-4.6559E-04	-4.6587E-04	-4.6576E-04	-4.6513E-04
ϖ = 1/2.5	-4.1527E-04	-4.1581E-04	-4.1606E-04	-4.1593E-04	-4.1537E-04
ϖ = 1/2	-3.6302E-04	-3.6350E-04	-3.6369E-04	-3.6358E-04	-3.6313E-04
ϖ = 1/1.5	-3.0902E-04	-3.0943E-04	-3.0959E-04	-3.0948E-04	-3.0908E-04
ϖ = 1	-2.5021E-04	-2.5054E-04	-2.5065E-04	-2.5054E-04	-2.5021E-04
ϖ = 1.5	-2.0670E-04	-2.0696E-04	-2.0704E-04	-2.0693E-04	-2.0666E-04
ϖ = 2	-1.8249E-04	-1.8272E-04	-1.8278E-04	-1.8268E-04	-1.8244E-04
ϖ = 2.5	-1.6650E-04	-1.6671E-04	-1.6675E-04	-1.6666E-04	-1.6645E-04
ϖ = 3	-1.5483E-04	-1.5501E-04	-1.5504E-04	-1.5497E-04	-1.5477E-04
ϖ = 3.5	-1.4806E-04	-1.4826E-04	-1.4829E-04	-1.4819E-04	-1.4801E-04
ϖ = 4	-1.4162E-04	-1.4181E-04	-1.4182E-04	-1.4172E-04	-1.4155E-04

Table 4
The Deflections in the Middle Section (Evenly Distributed Loadings, H = 0.0833)

	Point 1	*Point 2*	*Point 3*	*Point 4*	*Point 5*
ϖ = 1/4	-7.2854E-04	-7.2926E-04	-7.2963E-04	-7.2963E-04	-7.2895E-04
ϖ = 1/3.5	-6.6711E-04	-6.6777E-04	-6.6810E-04	-6.6803E-04	-6.6741E-04
ϖ = 1/3	-6.0432E-04	-6.0492E-04	-6.0522E-04	-6.0513E-04	-6.0456E-04
ϖ = 1/2.5	-5.3951E-04	-5.4006E-04	-5.4031E-04	-5.4023E-04	-5.3978E-04
ϖ = 1/2	-4.7241E-04	-4.7289E-04	-4.7311E-04	-4.7302E-04	-4.7264E-04
ϖ = 1/1.5	-4.0149E-04	-4.0190E-04	-4.0207E-04	-4.0198E-04	-4.0161E-04
ϖ = 1	-3.2496E-04	-3.2529E-04	-3.2541E-04	-3.2532E-04	-3.2501E-04
ϖ = 1.5	-2.6828E-04	-2.6855E-04	-2.6863E-04	-2.6854E-04	-2.6828E-04
ϖ = 2	-2.3673E-04	-2.3696E-04	-2.3702E-04	-2.3693E-04	-2.3671E-04
ϖ = 2.5	-2.1702E-04	-2.1725E-04	-2.1731E-04	-2.1722E-04	-2.1702E-04
ϖ = 3	-2.0270E-04	-2.0291E-04	-2.0296E-04	-2.0287E-04	-2.0268E-04
ϖ = 3.5	-1.9195E-04	-1.9215E-04	-1.9217E-04	-1.9209E-04	-1.9192E-04
ϖ = 4	-1.8355E-04	-1.8374E-04	-1.8375E-04	-1.8366E-04	-1.8350E-04

Table 5
The Deflections in the Middle Section (Single Concentrated Loading, H = 0.0833)

	Point 1	*Point 2*	*Point 3*	*Point 4*	*Point 5*
ϖ = 1/4	-1.1687E-03	-1.1701E-03	-1.1710E-03	-1.1714E-03	-1.1718E-03
ϖ = 1/3.5	-1.0700E-03	-1.0713E-03	-1.0721E-03	-1.0724E-03	-1.0729E-03
ϖ = 1/3	-9.6908E-04	-9.7027E-04	-9.7102E-04	-9.7134E-04	-9.7182E-04
ϖ = 1/2.5	-8.6533E-04	-8.6641E-04	-8.6705E-04	-8.6734E-04	-8.6785E-04
ϖ = 1/2	-7.5741E-04	-7.5837E-04	-7.5890E-04	-7.5920E-04	-7.5976E-04
ϖ = 1/1.5	-6.4289E-04	-6.4373E-04	-6.4418E-04	-6.4447E-04	-6.4507E-04
ϖ = 1	-5.2003E-04	-5.2071E-04	-5.2111E-04	-5.2138E-04	-5.2204E-04
ϖ = 1.5	-4.2907E-04	-4.2962E-04	-4.2997E-04	-4.3024E-04	-4.3095E-04
ϖ = 2	-3.7848E-04	-3.7895E-04	-3.7929E-04	-3.7956E-04	-3.8030E-04
ϖ = 2.5	-3.4687E-04	-3.4735E-04	-3.4766E-04	-3.4791E-04	-3.4867E-04
ϖ = 3	-3.2386E-04	-3.2431E-04	-3.2460E-04	-3.2485E-04	-3.2562E-04
ϖ = 3.5	-3.0660E-04	-3.0703E-04	-3.0729E-04	-3.0754E-04	-3.0832E-04
ϖ = 4	-2.9309E-04	-2.9350E-04	-2.9374E-04	-2.9399E-04	-2.9479E-04

5.1.2. The Location of the Neutral Axis

The neutral axis is the borderline of the tension stress zone and the compression zone. In classical bending beam theory, the middle of the beam is considered as the neutral axis. When the beam has bending deformation, only bending deformation exists on the neutral axis. However, the above rule cannot be applicable to the beam with different modulus.

In each figure of Fig. 8 and Fig. 9, there are also three pictures respectively denoting the computational results under the conditions $E^+ < E^-$, $E^+ = E^-$, and $E^+ > E^-$. The zone near the top is Compression Stress Zone, and the one near the bottom is Tension Stress Zone in every picture. The field of $\sigma_x = 0$ is located in the middle of the two zones. The neutral axis can be obtained by the computational results. When $E^+ = E^-$, the neutral axis is located in the middle. When $E^+ < E^-$, the neutral axis deviates to the top. When $E^+ > E^-$, the neutral axis deviates to the bottom.

The results for the beam with different modulus show that with the bending deformation of the beam increasing, not only bending deformation but also excursion occurs on the neutral axis (Fig. 10~Fig. 14). Under the condition that ϖ = 1/4 or ϖ = 4/1, the relative difference can reach 26% or so by different modulus model and uniform modulus. Particularly, the simple bending part exists in Example 2. In the part, the relative difference stands on 25%

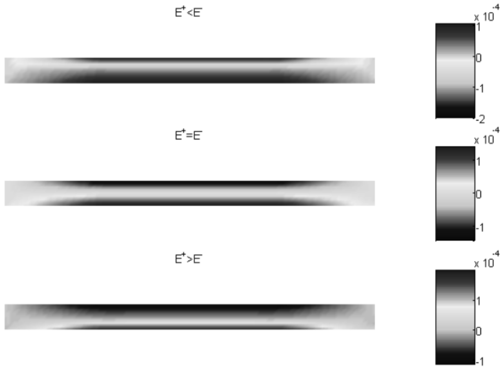

Figure 8.1: σ_x (H = 0.0714)

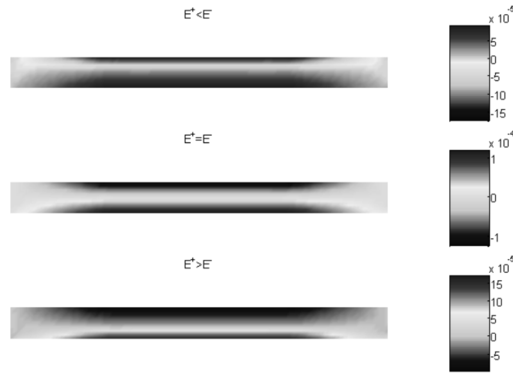

Figure 8.2: σ_x (H = 0.0833)

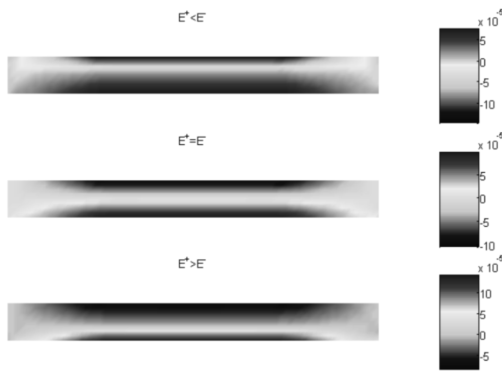

Figure 8.3: σ_x (H = 0.1)

Figure 8: The Influence of Variety of ϖ to Bending Beam under the Conditions of Different Heights

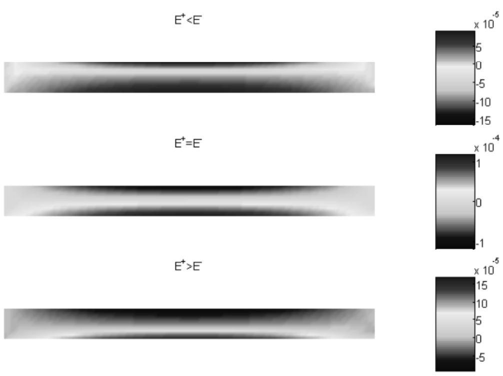

Figure 9.1: σ_x (Evenly Distributed Loadings)

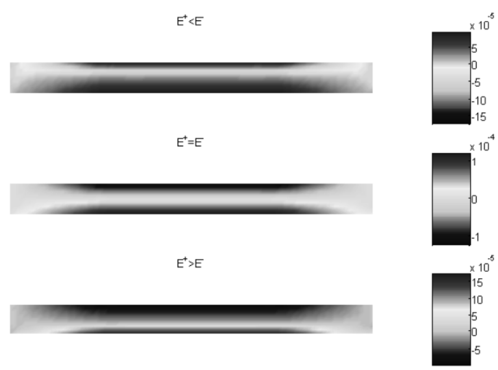

Figure 9.2: σ_x (Two Concentrated Loadings)

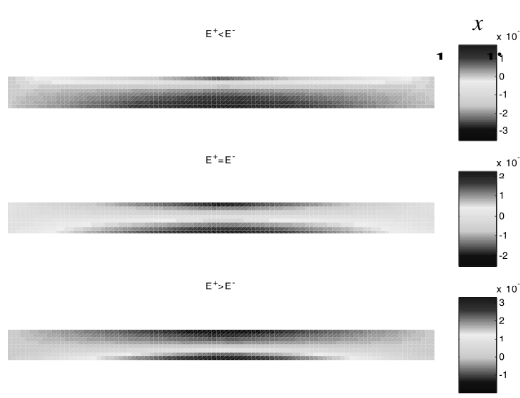

Figure 9.3: σ_x (Single Concentrated Loading)

Figure 9: The Influence of Variety of ϖ to Bending Beam under the Conditions of Different Loadings

using different modulus model and uniform modulus. Also under the condition of the same ϖ, the relative difference of the reflection keeps mainly the same value under various conditions.

So the middle of the beam with different modulus is no more the neutral axis. The deflection of the middle cannot reflect the deformation of the neutral axis. ϖ is the key influence factor leading to computational error by uniform modulus model.

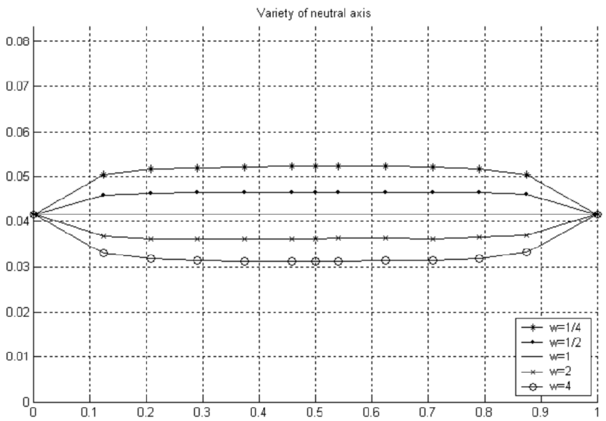

Figure 10.1: Variety of the Neutral Axis (evenly Distributed Loadings, H = 0.0833)

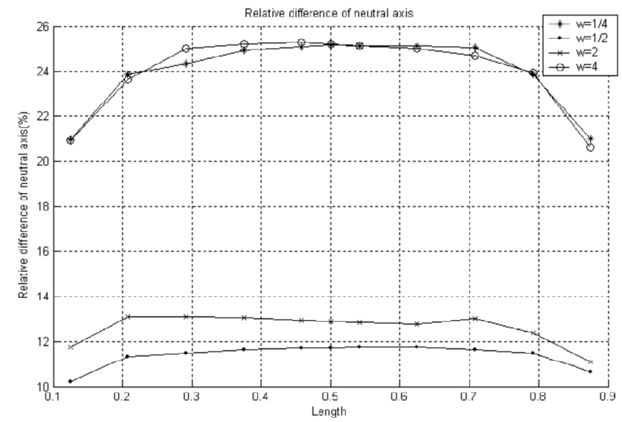

Figure 10.2: The Relative Difference of the Neutral Axis by Different Modulus Model and Uniform Modulus Model (Evenly Distributed Loadings, H = 0.0833)

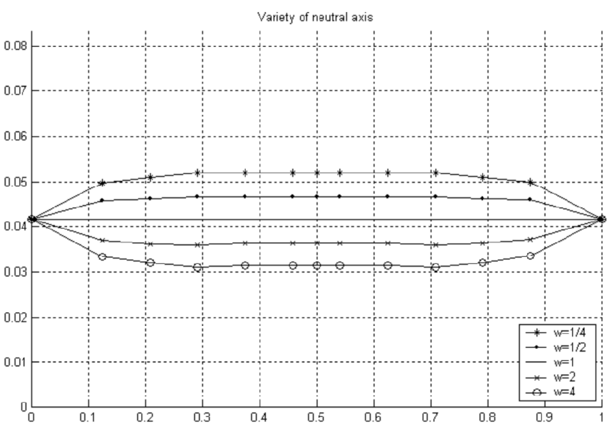

Figure 11.1: Variety of the Neutral Axis (two Concentrated Loadings, H = 0.0833)

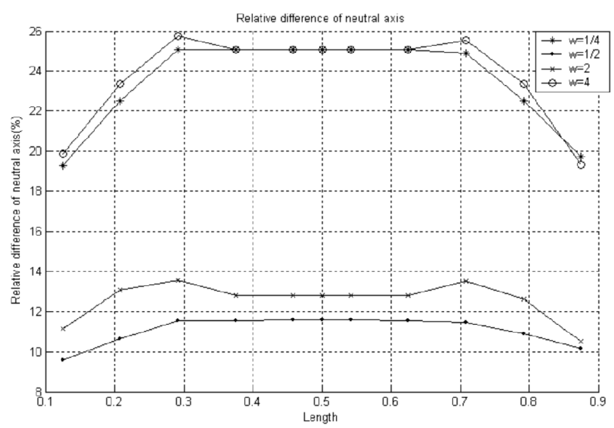

Figure 11.2: The Relative Difference of the Neutral Axis by Different Modulus Model and Uniform Modulus Model (Two Concentrated Loadings, H = 0.0833)

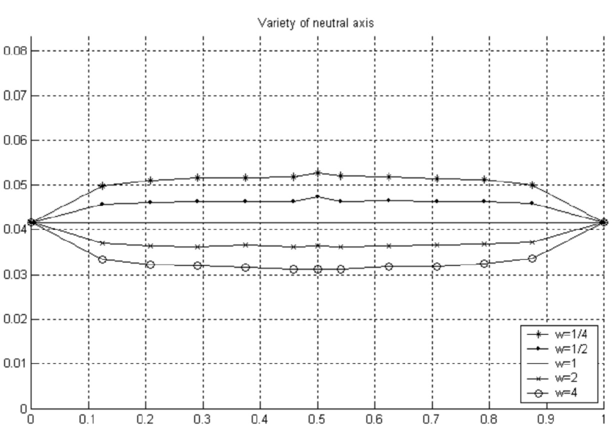

Figure 12.1: Variety of the Neutral Axis (Single Concentrated Loading, H = 0.0833)

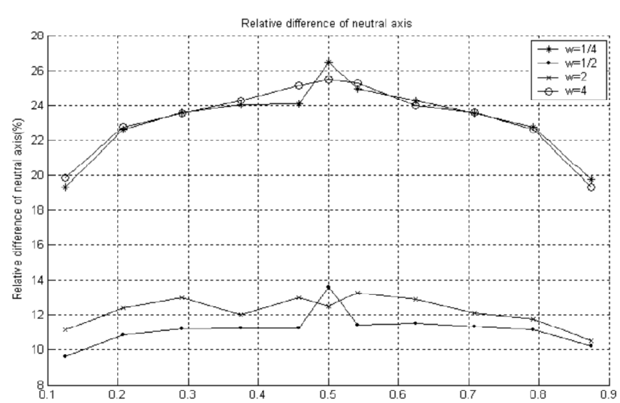

Figure 12.2: The Relative Difference of the Neutral Axis by Different Modulus Model and Uniform Modulus Model (Single Concentrated Loading, H = 0.0833)

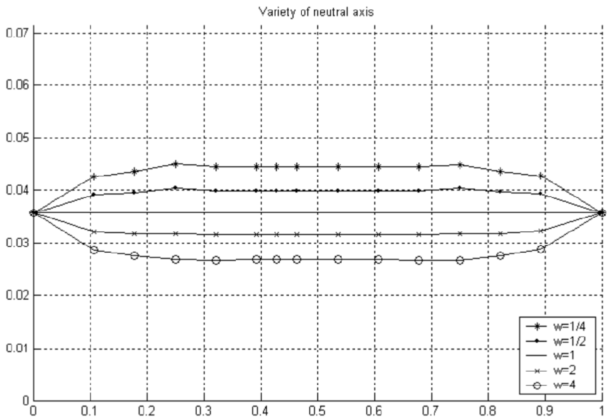

Figure 13.1: Variety of the Neutral Axis (Two Concentrated Loadings, H = 0.0714)

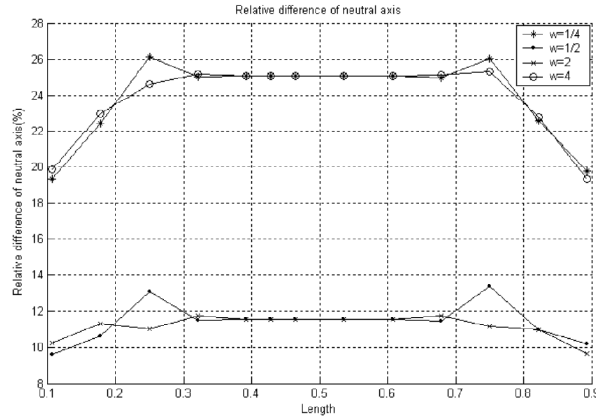

Figure 13.2: The Relative Difference of the Neutral Axis by Different Modulus Model and Uniform Modulus Model (Two Concentrated Loadings, H = 0.0714)

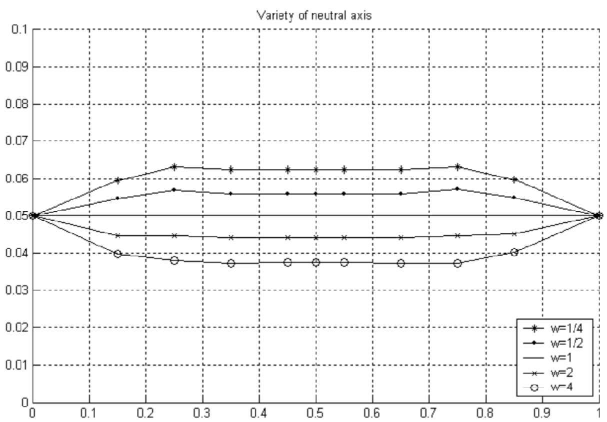

Figure 14.1: Variety of the Neutral Axis (Two Concentrated Loadings, H = 0.1)

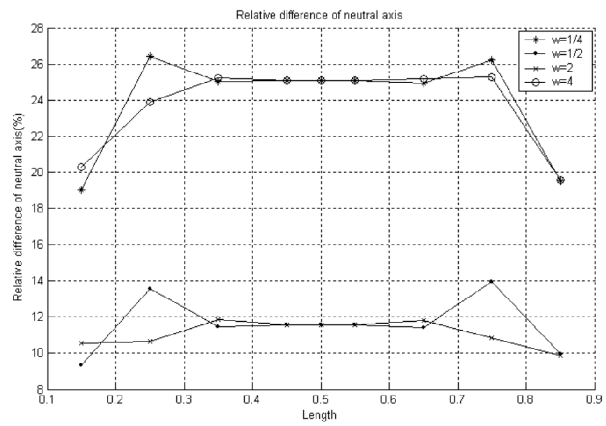

Figure 14.2: The Relative Difference of the Neutral Axis by Different Modulus Model and Uniform Modulus Model (Two Concentrated Loadings, H = 0.1)

5.2. The Max Deflection and the Max Stress

5.2.1. The Max Deflection

With the increase of ϖ, not only under the conditions of different heights but also of different loadings, the deflection successively decreases. With the increase of the height of the beam, the deflection continuously reduces. With the variety of the loadings, the deflection increases by degrees (Fig. 15).

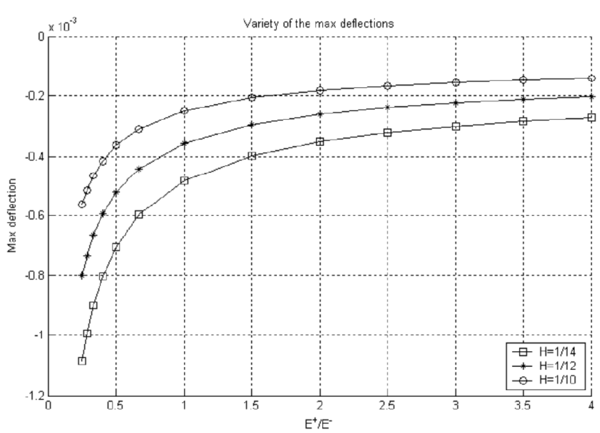

Figure 15.1: Variety of the Max Deflections Under the Conditions of Different Heights

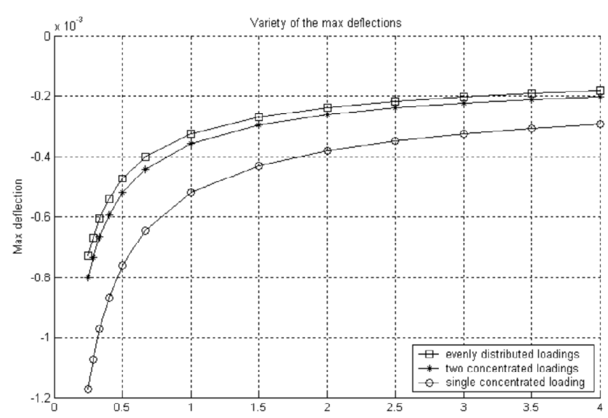

Figure 15.2: Variety of the Max Deflections Under the Conditions of Different Loadings

According to the computational results (Fig. 16), with the increase of the difference of E^+ and E^-, the relative difference of the max deflection using different modulus model and uniform modulus model increases under different conditions. Under the condition that ϖ = 1/4, the relative difference of the max deflection can reach about 125% by two models. Under the condition that ϖ = 4/1, the relative difference can reach or so 43%.

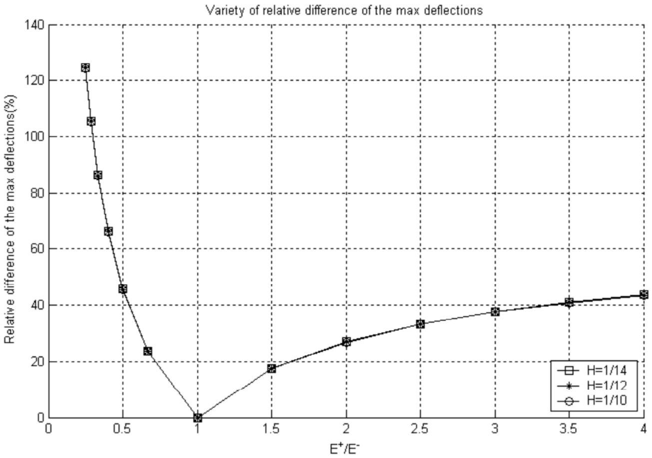

Figure 16.1: Relative Difference of the Max Deflections Under the Conditions of Different Heights

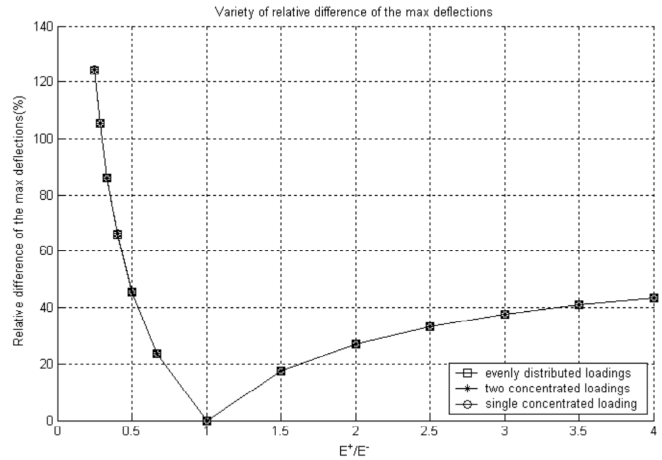

Figure 16.2: Relative Difference of the Max Deflections Under the Conditions of Different Loadings

5.2.2. The Max Stress

With the increase of ϖ, not only under the conditions of different heights but also of different loadings, the max tension stress successively increases and the max compression stress successively deceases. With the increase of the height of the beam, the max tension stress and compression stress continuously reduce. With the variety of the loadings, the max tension stress and compression stress increase to a certain extent (Fig. 17~Fig. 18).

With the difference of E^+ and E^- increasing, based on the numerical results, the relative difference of the max stress by different modulus model and uniform modulus model increases under different conditions. Under the condition that ϖ = 1/4, the relative differences of the max tension stress and the max compression stress can respectively reach about 23.5% and 40% by two models. Under the condition that ϖ = 4/1, the relative differences can reach 43% and 22% or so using different modulus model and uniform modulus model (Fig. 19~ Fig. 20).

Under the condition of the same ϖ, the relative differences of the max deflection and the max stress respectively keep mainly the same values under various conditions. So, ϖ is the key influence factor leading to computational error using uniform modulus model.

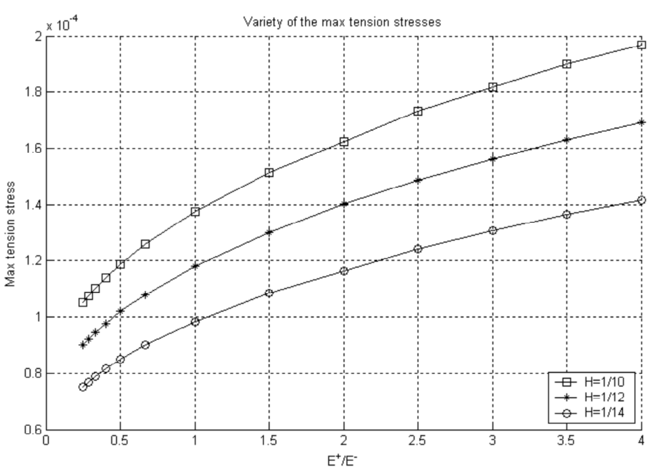

Figure 17.1: Variety of The Max Tension Stress Under the Conditions of Different Heights

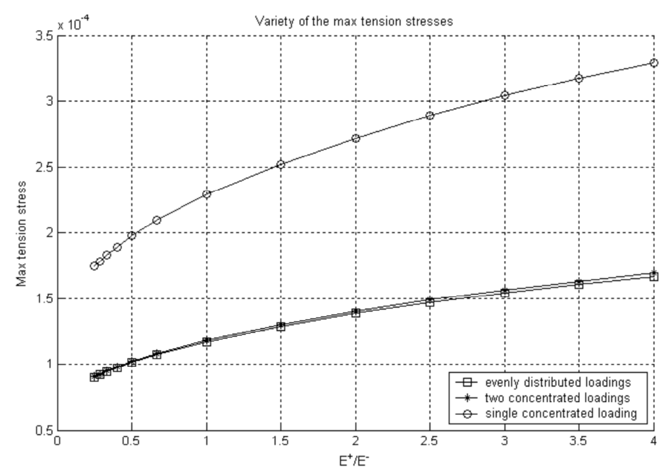

Figure 17.2: Variety of the Max Tension Stress Under the Conditions of Different Loadings

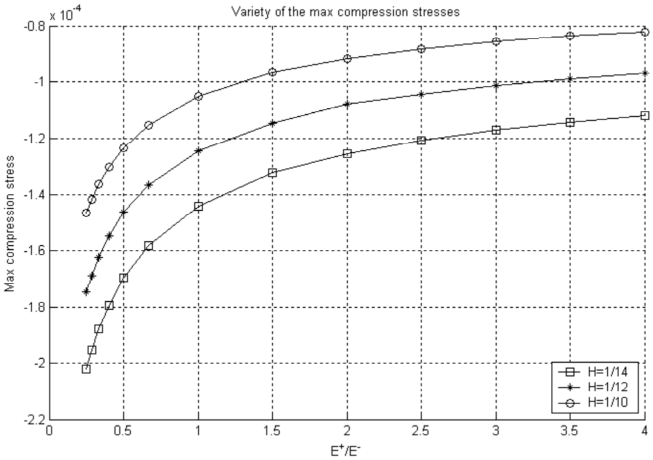

Figure 18.1: Variety of The Max Compression Stress Under the Conditions of Different Heights

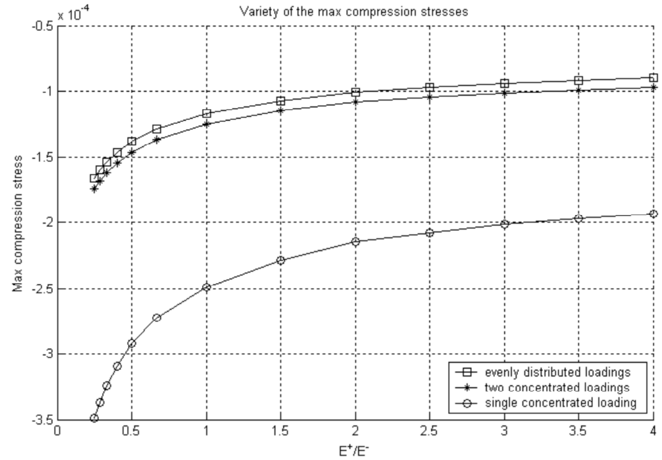

Figure 18.2: Variety of The Max Compression Stress Under the Conditions of Different Loadings

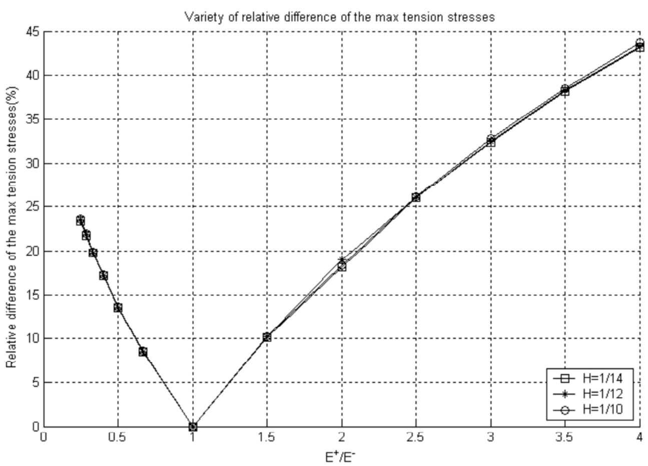

Figure 19.1: Relative Difference of the Max Tension Stress Under the Conditions of Different Heights

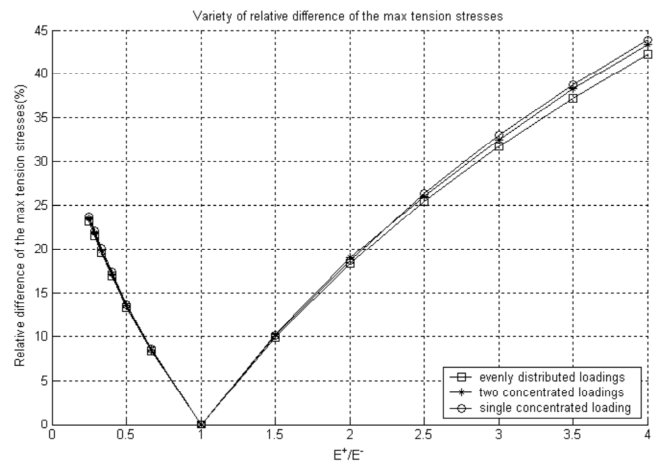

Figure 19.2: Relative Difference of the Max Tension Stress Under the Conditions of Different Loadings

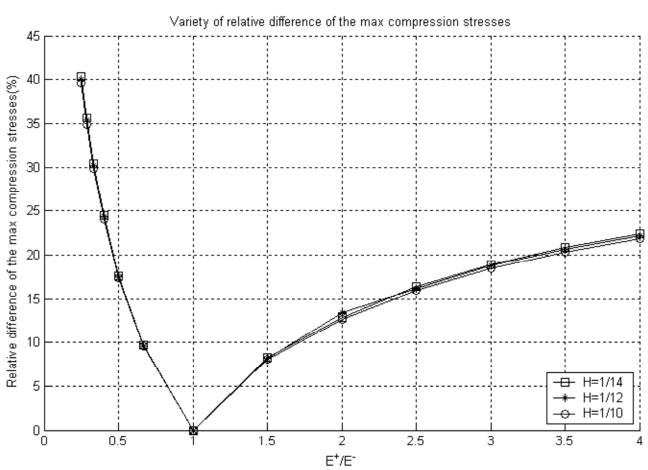

Figure 20.1: Relative Difference of the Max Compression Stress Under the Conditions of Different Heights

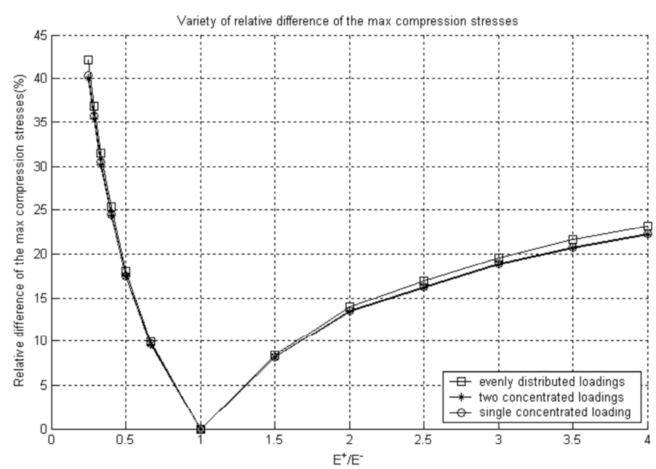

Figure 20.2: Relative Difference of the Max Compression Stress Under the Conditions of Different Loadings

5. CONCLUSIONS

With the increase of ϖ, the deflection and stress continuously change under different operation conditions. When the simply-supported beam with different modulus is exerted symmetric loadings, the point of the max deflection is located in the middle section. However, with the bending deformation of the beam increasing, not only bending deformation but also excursion occurs on the neutral axis. The middle of the beam with different modulus is no more the neutral axis. Great relative differences of computational results using different modulus model and uniform modulus model indicate that great errors of rigidity and intension exist when the deformation and the stress are analyzed by uniform modulus model that $E = E^{+} = E^{-}$. Also, under the condition of the same ϖ, the relative differences respectively keep mainly the same values under various conditions. It shows that ϖ is the key influence factor leading to computational error by uniform modulus model.

Finally, it should be mentioned that the influence of different Poisson's values [44] and the beam-column problems [45] is not considered in the present examples. However, the inclusion of this is straightforward.

Acknowledgement

The authors would like to acknowledge the support received from Shanghai Educational Development Fund and Shanghai Post-Doctoral Research Program Foundation.

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