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ESSENTIAL SETS FOR RANDOM OPERATORS CONSTRUCTED FROM AN ARRATIA FLOW

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ABSTRACT. In this paper we consider a strong random operator T_t which describes a shift of functions from $L_2(\mathbb{R})$ along an Arratia flow. We find a compact set in $L_2(\mathbb{R})$ that doesn't disappear under T_t, and estimate its Kolmogorov widths.

1. Introduction: Arratia Flow and Random Operators

In this paper we consider random operators in $L_2(\mathbb{R})$ which describe shifts of functions along an Arratia flow [1]. Let us recall the definition.

Definition 1.1 ([1])**.** A family of random processes $\{x(u,s), u \in \mathbb{R}, s \geq 0\}$ is called an *Arratia flow* if

1) for each $u \in \mathbb{R}$ $x(u,\cdot)$ is a Wiener process with respect to the joint filtration such that $x(u,0) = u$;

2) for any $u_1 \leq u_2$ and $t \geq 0$

$$x(u_1,t) \leq x(u_2,t) \text{ a.s.}$$

3) the joint characteristics are

$$d < x(u_1,\cdot), x(u_2,\cdot) > (t) = 1\!\!1_{\{x(u_1,t)=x(u_2,t)\}}dt.$$

In the informal language, Arratia flow is a family of Wiener processes started from each point of $\mathbb{R}$, which move independently up to the meeting, coalesce, and move together. It was proved in [4, 8] that for any $a, b \in \mathbb{R}$ and $t > 0$ the set $x([a;b],t)$ is finite a.s. Since Arratia flow has a right-continuous modification [3], $x(\cdot,t) : \mathbb{R} \to \mathbb{R}$ is a step function for any time $t > 0$. Hence, for any $a, b \in \mathbb{R}$ and $t > 0$ with probability one there exists a random point $y \in \mathbb{R}$ for which

$$\lambda\{u \in [a;b] : x(u,t) = y\} > 0, \tag{1.1}$$

where λ is Lebesgue measure on $\mathbb{R}$. Since $x(\cdot,t)$ is a right-continuous step function, for a fixed countable set A

$$\begin{aligned}\mathsf{P}\{x(\mathbb{R},t) \cap A \neq \emptyset\} &= \mathsf{P}\{x(\mathbb{Q},t) \cap A \neq \emptyset\} \\ &\leq \sum_{u\in\mathbb{Q}} \mathsf{P}\{x(u,t) \in A\} = 0.\end{aligned} \tag{1.2}$$

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Since for any $a < b$ the difference $\frac{x(b,\cdot)-x(a,\cdot)}{\sqrt{2}}$ is a Wiener processes until the collision happens, and $\frac{x(b,0)-x(a,0)}{\sqrt{2}} = \frac{b-a}{\sqrt{2}}$, one can find the distribution of the time of coalescence $\tau_{a,b} = \inf\{s \geq 0 | \ x(a,s) = x(b,s)\}$ of the processes $x(a,\cdot), x(b,\cdot)$, i.e. for any $t \geq 0$

$$\begin{aligned} \mathsf{P}\{\tau_{a,b} \leq t\} &= \mathsf{P}\{x(a,t) = x(b,t)\} \\ &= \sqrt{\frac{2}{\pi}} \int_{\frac{b-a}{\sqrt{2t}}}^{\infty} e^{-\frac{v^2}{2}} dv. \end{aligned} \tag{1.3}$$

Let us notice that for a fixed time $t > 0$ and an Arratia flow $X = \{x(u,s), u \in \mathbb{R}, s \in [0;t]\}$ there exists an Arratia flow $Y = \{y(u,r), u \in \mathbb{R}, r \in [0;t]\}$ such that trajectories of X and $\widetilde{Y} = \{y(u,t-r), u \in \mathbb{R}, r \in [0;t]\}$ don't cross [1, 7]. Y is called a conjugated (or dual) Arratia flow. It was proved in [10] the following change of variable formula for an Arratia flow.

Theorem 1.2 ([10])**.** *For any time $t > 0$ and nonnegative measurable function $h : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} h(u)du < \infty$*

$$\int_{\mathbb{R}} h(x(u,t))du = \int_{\mathbb{R}} h(u)dy(u,t) \quad a.s., \tag{1.4}$$

where the last integral is in sense of Lebesgue-Stieltjes.

In this paper we consider random operators T_t, $t > 0$, in $L_2(\mathbb{R})$ which are defined as follows

$$(T_t f)(u) = f(x(u,t)),$$

where $f \in L_2(\mathbb{R})$ and $u \in \mathbb{R}$. It was proved in [5] that T_t is a strong random operator [11] in $L_2(\mathbb{R})$, but, as it was shown in [10], is not a bounded one. Really, for the point y from (1.1) one can introduce a sequence of the intervals $A_i = [r_i; p_i]$ such that $y \in A_i$ for any $i \geq 1$ and $p_i - r_i \to 0, i \to \infty$. Thus for any $i \geq 1$

$$\|T_t 1\!\!1_{A_i}\|^2_{L_2(\mathbb{R})} \geq \lambda\{u \in [a;b] : \ x(u,t) = y\} > 0,$$

which can't be true if T_t was a bounded random operator. Hence, the image of a compact set under T_t may not be a random compact set. Moreover, as it was mentioned in [9], the image of a compact set under strong random operator may not exist. However, in [10] it was presented a family of compact sets in $L_2(\mathbb{R})$ whose images under T_t exist and are random compact sets. In this paper we consider a compact set of this type, and investigate the change of its Kolmogorov widths [12] under T_t.

2. T_t-essential Functions

If the support of the function $f \in L_2(\mathbb{R})$ is bounded, $supp f \subset [a;b]$, then $T_t f$ equals to 0 with positive probability. Really, by (1.4), one can check that

$$\mathsf{P}\left\{\int\limits_{-\infty}^{\infty} f^2(x(u,t))du = 0\right\} \geq \mathsf{P}\{\ x(\mathbb{R},t)\cap[a;b]=\emptyset\ \}$$

$$= \mathsf{P}\left\{\int\limits_{-\infty}^{\infty} 1\!\!\mathrm{I}_{[a;b]}(x(u,t))du = 0\right\}$$

$$= \mathsf{P}\left\{\int\limits_{-\infty}^{\infty} 1\!\!\mathrm{I}_{[a;b]}(u)dy(u,t) = 0\right\},$$

where $\{y(u,s),\ u\in\mathbb{R},\ s\in[0;t]\}$ is a conjugated Arratia flow. Since, by (1.3),

$$\mathsf{P}\left\{\int\limits_{-\infty}^{\infty} 1\!\!\mathrm{I}_{[a;b]}(u)dy(u,t) = 0\right\} = \mathsf{P}\{\ y(b,t)=y(a,t)\ \} > 0,$$

then $\mathsf{P}\left\{\ \|T_t f\|_{L_2(\mathbb{R})} = 0\ \right\} > 0$. This leads to the following definition.

Definition 2.1. For a fixed $t>0$ a function $f\in L_2(\mathbb{R})$ is said to be a T_t*-essential* if

$$\mathsf{P}\left\{\ \|T_t f\|_{L_2(\mathbb{R})} > 0\ \right\} = 1.$$

Example 2.2. Let $f\in L_2(\mathbb{R})$ be an analytic function which doesn't equal totally to zero. Denote the set of its zeroes $Z_f=\{u\in\mathbb{R}|\ f(u)=0\}$. Then, by (1.2), $\mathsf{P}\{x(\mathbb{R},t)\cap Z_f=\emptyset\}=1$, so f is a T_t-essential for any $t>0$.

Let us notice that if $t_1\neq t_2$ then T_{t_1}-essential function may not be a T_{t_2}-essential. To introduce a T_1-essential that is not T_2-essential function let us consider an increasing sequence $\{u_k\}_{k=0}^{\infty}$ such that $u_0=0, u_1=1$ and for any $n\in\mathbb{N}$

$$u_{2n+1}-u_{2n}=\frac{1}{2^n}, \qquad u_{2n}=u_{2n-1}+2n(\ln 2)^{\frac{1}{2}}.$$

Theorem 2.3. *The function* $f=\sum_{n=0}^{\infty} 1\!\!\mathrm{I}_{[u_{2n};u_{2n+1}]}$ *is a* T_1*-essential, and is not a* T_2*-essential.*

Proof. To prove that f is not a T_2 essential we show that $\mathsf{P}\{\ \|T_2 f\|_{L_2(\mathbb{R})}>0\ \}<1$. Since $[u_{2k};u_{2k+1}]\cap[u_{2j};u_{2j+1}]=\emptyset$ for any $k\neq j$ then, by (1.4),

$$\mathsf{P}\{\ \|T_2 f\|^2_{L_2(\mathbb{R})}>0\ \} = \mathsf{P}\left\{\int\limits_{-\infty}^{\infty}\left(\sum_{n=0}^{\infty} 1\!\!\mathrm{I}_{[u_{2n};u_{2n+1}]}(x(u,2))\right)^2 du > 0\right\}$$

$$= \mathsf{P}\left\{\sum_{n=0}^{\infty}\int\limits_{-\infty}^{\infty} 1\!\!\mathrm{I}_{[u_{2n};u_{2n+1}]}(x(u,2))du > 0\right\}$$

$$= \mathsf{P}\left\{ \sum_{n=0}^{\infty}(y(u_{2n+1},2) - y(u_{2n},2)) > 0 \right\}$$
$$= \mathsf{P}\left\{ \exists n \geq 0 : \ y(u_{2n+1},2) \neq y(u_{2n},2) \right\}$$
$$\leq \sum_{n=0}^{\infty} P\left\{ y(u_{2n+1},2) \neq y(u_{2n},2) \right\}.$$

Thus by (1.3),

$$\sum_{n=0}^{\infty} \mathsf{P}\left\{ y(u_{2n+1},2) \neq y(u_{2n},2) \right\} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{4\pi}} \int_{-\frac{1}{2^{n+1}}}^{\frac{1}{2^{n+1}}} e^{-\frac{v^2}{4}} dv \leq \frac{1}{\sqrt{\pi}} < 1.$$

Consequently, the function $f = \sum_{n=0}^{\infty} 1\!\!\mathrm{I}_{[u_{2n};u_{2n+1}]}$ is not a T_2-essential. To prove that $f = \sum_{n=0}^{\infty} 1\!\!\mathrm{I}_{[u_{2n};u_{2n+1}]}$ is a T_1-essential one can show the following estimation.

Lemma 2.4. *Let $\{w(u_n,\cdot)\}_{n=0}^{\infty}$ be a family of independent Wiener processes on $[0;1]$ such that $w(u_n,0) = u_n$. Then for any $n \in \mathbb{N}$*

$$\mathsf{P}\left\{ \max_{s\in[0;1]} \max_{j=\overline{0,2n-1}} w(u_j,s) \geq \min_{s\in[0;1]} w(u_{2n},s) \right\} < \frac{1}{2^{n^2}\sqrt{\pi \ln 2}}.$$

Proof. Let w_1, w_2 be an independent Wiener processes on $[0;1]$ started from point 0, i.e. $w_1(0) = w_2(0) = 0$. It can be noticed that

$$\mathsf{P}\left\{ \max_{s\in[0;1]} \max_{j=\overline{0,2n-1}} w(u_j,s) \geq \min_{s\in[0;1]} w(u_{2n},s) \right\}$$
$$= \mathsf{P}\left\{ \exists\, j = \overline{0,2n-1} : \ \max_{s\in[0;1]} w(u_j,s) - \min_{s\in[0;1]} w(u_{2n},s) \geq 0 \right\}$$
$$\leq \sum_{j=0}^{2n-1} \mathsf{P}\left\{ \max_{s\in[0;1]} w(u_j,s) - \min_{s\in[0;1]} w(u_{2n},s) \geq 0 \right\}$$
$$\leq \sum_{j=0}^{2n-1} \mathsf{P}\left\{ \max_{s\in[0;1]} w_1(s) - \min_{s\in[0;1]} w_2(s) \geq u_{2n} - u_j \right\}.$$

From the fact that $\{u_n\}_{n=0}^{\infty}$ is an increasing sequence we can estimate the last expression and complete the proof

$$\sum_{j=0}^{2n-1} \mathsf{P}\left\{ \max_{s\in[0;1]} w_1(s) - \min_{s\in[0;1]} w_2(s) \geq u_{2n} - u_j \right\}$$
$$\leq \frac{1}{\sqrt{\pi}} \sum_{j=0}^{2n-1} \frac{1}{u_{2n} - u_j} e^{-\frac{(u_{2n}-u_j)^2}{4}}$$
$$\leq \frac{2n-1}{\sqrt{\pi}(u_{2n} - u_{2n-1})} e^{-\frac{(u_{2n}-u_{2n-1})^2}{4}}$$
$$\leq \frac{1}{2^{n^2}\sqrt{\pi \ln 2}}.$$

□

Let us prove that the function $f = \sum_{n=0}^{\infty} 1\!\!\mathrm{I}_{[u_{2n};u_{2n+1}]}$ is a T_1-essential. Using the reasoning from the first part of the proof it can be checked that for the considered function f the following equality holds

$$\mathsf{P}\{\ \|T_1 f\|_{L_2(\mathbb{R})} > 0\ \} = \mathsf{P}\left\{\sum_{n=0}^{\infty}(y(u_{2n+1},1) - y(u_{2n},1)) > 0\right\}.$$

Let us prove that

$$\mathsf{P}\left\{\ \limsup_{n\to\infty}\,(y(u_{2n+1},1) - y(u_{2n},1)) \geq 1\ \right\} = 1. \tag{2.1}$$

Build a new processes $\{\widetilde{y}(u_n,\cdot)\}_{n=0}^{\infty}$ such that $\{\widetilde{y}(u_n,\cdot)\}_{n=0}^{\infty}$ and $\{y(u_n,\cdot)\}_{n=0}^{\infty}$ have the same distributions in $\mathcal{C}\,([0;1])^{\infty}$ in the following way [4]. Let $\{w(u_n,\cdot)\}_{n=0}^{\infty}$ be a given family of Wiener processes on $[0;1]$, $w(u_n,0) = u_n$. Let us denote collision time of $f, g \in \mathcal{C}\,([0;1])$ by $\tau[f,g] := \inf\{t \mid f(t) = g(t)\}$. Put $\widetilde{y}(u_0,\cdot) := w(u_0,\cdot)$. Then for any $n \in \mathbb{N}$, $s \in [0;1]$ one can define

$$\begin{aligned}\widetilde{y}(u_n,s) = w(u_n,s)1\!\!\mathrm{I}\{\ s < \tau[w(u_n,\cdot),\widetilde{y}(u_{n-1},\cdot)]\ \}\\ +\ \widetilde{y}(u_{n-1},s)1\!\!\mathrm{I}\{s \geq \tau[w(u_n,\cdot),\widetilde{y}(u_{n-1},\cdot)]\ \}.\end{aligned}$$

According to constructions of stochastic processes $\{\widetilde{y}(u_n,\cdot)\}_{n=0}^{\infty}$

$$\begin{aligned}&\mathsf{P}\,\{\ \exists\, N \in \mathbb{N}:\ \forall\, n \geq N \quad \widetilde{y}(u_{2n},t) = w(u_{2n},t),\\ &\ \widetilde{y}(u_{2n+1},t) = w(u_{2n+1},t)1\!\!\mathrm{I}\{\ t < \tau[w(u_{2n},\cdot),w(u_{2n+1},\cdot)]\}\\ &+w(u_{2n},t)1\!\!\mathrm{I}\{\ t \geq \tau[w(u_{2n},\cdot),w(u_{2n+1},\cdot)]\}\ \} = 1.\end{aligned} \tag{2.2}$$

Thus

$$\mathsf{P}\,\{\ \exists\, N \in \mathbb{N}:\ \forall\, n \geq N \quad \widetilde{y}(u_{2n+1},t) - \widetilde{y}(u_{2n},t) = w(u_{2n+1},t) - w(u_{2n},t)\} = 1.$$

For the considered sequence $\{u_n\}_{n=0}^{\infty}$ and any $n \in \mathbb{N}$ the following inequality holds

$$\mathsf{P}\{\ w(u_{2n+1},t) - w(u_{2n},t) \geq 1\ \} = \int_1^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{\left(v-\frac{1}{2^k}\right)^2}{4}}\,dv \quad \geq \frac{1}{\sqrt{4\pi}}\int_1^{\infty} e^{-\frac{v^2}{4}}\,dv.$$

Therefore, by the Borel-Cantelli lemma and (2.2),

$$\mathsf{P}\{\ \limsup_{n\to\infty}(\widetilde{y}(u_{2n+1},t) - \widetilde{y}(u_{2n},t)) \geq 1\ \} = 1.$$

□

Using the observation from Example 2.2 one can introduce a family of T_t-essential functions for all $t > 0$.

For any $\varepsilon > 0$ let us consider an integral operator K_ε in $L_2(\mathbb{R})$ with the kernel

$$k_\varepsilon(v_1,v_2) = \int_{\mathbb{R}} p_\varepsilon(u - v_1)p_\varepsilon(u - v_2)dy(u,t), \tag{2.3}$$

where $v_1, v_2 \in \mathbb{R}$, and $p_\varepsilon(u) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{u^2}{2\varepsilon}}$. By the change of variables formula for an Arratia flow [10],

$$(K_\varepsilon f, f) = \int_{\mathbb{R}} (f * p_\varepsilon)^2 (x(u,t)) du. \tag{2.4}$$

Lemma 2.5. *For any $\varepsilon > 0$ and nonzero function $f \in L_2(\mathbb{R})$*

$$\mathsf{P}\{\ (K_\varepsilon f, f) \neq 0\ \} = 1.$$

Proof. According to (2.1) it is sufficient to note that $f * p_\varepsilon$ is an analytic function. Consequently, for any $t > 0$ the following relations are true

$$\begin{aligned} \mathsf{P}\{\ (K_1 f, f) > 0\ \} &= \mathsf{P}\left\{\ \|T_t(f * p_1)\|_{L_2(\mathbb{R})} > 0\ \right\} \\ &= \mathsf{P}\{\ x(\mathbb{R}, t) \cap Z_{f*p_1} = \emptyset\ \} = 1. \end{aligned}$$

□

According to the last theorem and (2.4), for any $\varepsilon > 0$ and nonzero $f \in L_2(\mathbb{R})$ the function $f * p_\varepsilon$ is a T_t-essential for each $t > 0$.

3. Change of Compact Sets under a Strong Random Operator Generated by an Arratia Flow

As it was noticed in the introduction any function with bounded support isn't a T_t-essential. Consequently, if $K \subseteq L_2(\mathbb{R})$ is a compact set of functions with uniformly bounded supports such that $T_t(K)$ is well-defined, then the image $T_t(K)$ equals to $\{0\}$ with positive probability. It was shown in [10] that T_t may also change the geometry of K even in the case of a compact set K for which $T_t(K) \neq \{0\}$ a.s. For example, the image $T_t(K)$ of a convergent sequence and its limiting point may not have limiting points. In this section we build a compact set K for which $T_t(K) \neq \{0\}$ a.s. and investigate the change of its Kolmogorov-widths in $L_2(\mathbb{R})$ under random operator T_t.

Definition 3.1 ([12])**.** The *Kolmogorov n-width* of a set $C \subseteq H$ in a Hilbert space H is given by

$$d_n(C) = \inf_{\dim L \le n} \sup_{f \in C} \inf_{g \in L} \|f - g\|_H,$$

where L is a subspace of H.

We consider the following compact set in $L_2(\mathbb{R})$

$$K = \left\{\ f \in W_2^1(\mathbb{R}) |\ \int_{\mathbb{R}} f^2(u)(1+|u|)^3 du + \int_{\mathbb{R}} \left(f'(u)\right)^2 (1+|u|)^7 du \le 1\ \right\}. \tag{3.1}$$

Estimations on its Kolmogorov-widths in $L_2(\mathbb{R})$ are presented in the next lemma.

Lemma 3.2. *There exist positive constants C_1, C_2 such that for any $n \in \mathbb{N}$*

$$\frac{C_1}{n} \le d_n(K) \le \frac{C_2}{n^{\frac{3}{10}}}.$$

Proof. Let $n \in \mathbb{N}$ be fixed. To estimate $d_n(K)$ from above one can consider the partition $\{u_k\}_{k=0}^{n}$ of $[-n^{\frac{1}{5}}; n^{\frac{1}{5}}]$ into n segments $\{[u_k; u_{k+1}], k = \overline{0, n-1}\}$ with equal lengths. Let us show that for the n-dimensional subspace $L_n = LS\{\mathbb{1}_{[u_k;u_{k+1}]},\ k = \overline{0, n-1}\}$

$$\sup_{f\in K} \inf_{g\in L_n} \|f-g\|_{L_2(\mathbb{R})} \le \frac{C_2}{n^{\frac{3}{10}}}.$$

If $f \in K$ then $\int_{\mathbb{R}} f^2(u)(1+|u|)^3 du \le 1$. Thus for any $C > 0$

$$\int_{|u|>c} f^2(u)du \le \frac{1}{(1+C)^3} \int_{|u|>c} f^2(u)(1+|u|)^3 du \le \frac{1}{C^3}.$$

So, for the function $g_f = \sum_{k=0}^{n-1} f(u_k)\mathbb{1}_{[u_k;u_{k+1}]} \in L_n$ the following estimation is true

$$\|f-g_f\|^2_{L_2(\mathbb{R})} \le \frac{1}{n^{\frac{3}{5}}} + \int_{|u|\le n^{\frac{1}{5}}} (f(u)-g_f(u))^2 du.$$

By the Cauchy inequality, for $f \in K$ and $u \in [u_k; u_{k+1}]$

$$\left(\int_{u_k}^{u} f'(v)dv \right)^2 \le \int_{u_k}^{u} \frac{dv}{(1+|v|)^7} \le u - u_k.$$

Consequently,

$$\int_{|u|\le n^{\frac{1}{5}}} (f(u)-g_f(u))^2\, du = \sum_{k=0}^{n-1} \int_{u_k}^{u_{k+1}} \left(\int_{u_k}^{u} f'(v)dv \right)^2 du$$

$$\le \frac{1}{2} \sum_{k=0}^{n-1} (u_{k+1}-u_k)^2 = \frac{2}{n^{\frac{3}{5}}},$$

and the upper estimation for $d_n(K)$ holds with the constant $C_2 = 3^{\frac{1}{2}}$.

To get a lower estimation for $d_n(K)$ we use the theorem about n-width of $(n+1)$-dimensional ball [12]. Let $\{u_k\}_{k=0}^{2(n+1)}$ be a partition of $[0; 1]$ into $2(n+1)$ segments $\{[u_k; u_{k+1}],\ k = \overline{0, 2n+1}\}$ with equal lengths. Consider $(n+1)$-dimensional space $L_{n+1} = LS\{f_k,\ k = \overline{0, n}\}$, where the functions f_k, $k = \overline{0, n}$, are defined as follows

$$f_k = \begin{cases} 0, & u \notin [u_{2k}; u_{2k+1}], \\ 1, & u \in [u_{2k} + \frac{1}{6(n+1)}; u_{2k} + \frac{2}{6(n+1)}], \\ 6(n+1)(u-u_{2k}), & u \in [u_{2k}; u_{2k} + \frac{1}{6(n+1)}], \\ -6(n+1)(u-u_{2k+1}), & u \in [u_{2k} + \frac{2}{6(n+1)}; u_{2k+1}]. \end{cases} \tag{3.2}$$

We show that if $c = \frac{2^3(5+2^9\cdot 3^3)}{5}$ then the ball $B_{n+1} = \{f \in L_{n+1} \mid \|f\|_{L_2(\mathbb{R})} \le \frac{1}{\sqrt{cn}}\}$ is a subset of K. Since $\|f_k\|^2_{L_2(\mathbb{R})} = \frac{5}{18(n+1)}$, $k = \overline{0, n}$, then for any $f \in B_{n+1}$ such that $f = \sum_{k=0}^{n} c_k f_k$ the following relation holds $\sum_{k=0}^{n} c_k^2 \le \frac{36}{5cn}$. Thus according to (3.2),

$$\int_{\mathbb{R}} f^2(u)(1+|u|)^3du + \int_{\mathbb{R}} \left(f'(u)\right)^2 (1+|u|)^7du$$
$$\leq 2^3\|f\|^2_{L_2(\mathbb{R})} + 2^7 \cdot \sum_{k=0}^{n} c_k^2 \left(\int_{u_{2k}}^{u_{2k}+\frac{1}{6(n+1)}} (6(n+1))^2du + \int_{u_{2k}+\frac{2}{6(n+1)}}^{u_{2k+1}} (6(n+1))^2du \right)$$
$$\leq \frac{2^3}{cn^2} + 2^{10}\cdot 3n\frac{36}{5cn} \leq \frac{1}{c}\cdot\frac{2^3(5+2^9\cdot 3^3)}{5} = 1.$$

Consequently, $B_{n+1} \subset K$ and $d_n(K) \geq d_n(B_{n+1})$. Due to the theorem about n-width of $(n+1)$-dimensional ball, $d_n(B_{n+1}) = \frac{1}{\sqrt{cn}}$ [12]. So the lower estimation for $d_n(K)$ holds with $C_1 := \sqrt{c}$. □

To show that estimations from above for the Kolmogorov-widths of the considered compact set K don't change under T_t one may use the same idea as in Lemma 2.

Theorem 3.3. *There exists $\widetilde{\Omega}$ of probability one such that for any $\omega \in \widetilde{\Omega}$ and $n \in \mathbb{N}$*

$$d_n\left(T_t^\omega(K)\right) \leq \frac{C(\omega)}{n^{\frac{3}{10}}}, \tag{3.3}$$

where the constant $C(\omega) > 0$ doesn't depend on n.

Proof. For a fixed $n \in \mathbb{N}$ let us consider a partition $\{u_k\}_{k=0}^n$ of $[-n^{\frac{1}{5}}; n^{\frac{1}{5}}]$ into n segments with equal lengths. To prove (3.3) it's sufficient to show the following inequality for the linear space $L_n^\omega = LS\{\ T_t^\omega 1\!\!1_{[u_k;u_{k+1}]},\ k = \overline{0, n-1}\ \}$ with dimension at most n

$$\sup_{h_1\in T_t^\omega(K)} \inf_{h_2\in L_n^\omega} \|h_1 - h_2\|_{L_2(\mathbb{R})} \leq \frac{C(\omega)}{n^{\frac{3}{10}}}.$$

According to the change of variable formula for an Arratia flow, one can check the equality for any $f \in K$

$$\left\| T_t^\omega f - T_t^\omega\left(\sum_{k=0}^{n-1} f(u_k)1\!\!1_{[u_k;u_{k+1}]}\right)\right\|^2_{L_2(\mathbb{R})} = \int_{|u|>n^{\frac{1}{5}}} f^2(u)dy(u,t,\omega)$$
$$+\int_{|u|\leq n^{\frac{1}{5}}} \left(f(u) - \sum_{k=0}^{n-1} f(u_k)1\!\!1_{[u_k;u_{k+1}]}(u)\right)^2 dy(u,t,\omega).$$

To estimate from above the last integral let us notice that

$$\int_{|u|\le n^{\frac{1}{5}}}\left(f(u)-\sum_{k=0}^{n-1} f(u_k)1\!\!1_{[u_k;u_{k+1}]}(u)\right)^2 dy(u,t,\omega)$$
$$\le \sum_{k=0}^{n-1}\int_{u_k}^{u_{k+1}}\left(\int_{u_k}^{u}|f'(v)|\,dv\right)^2 dy(u,t,\omega).$$

Due to (3.1), for any $f\in K$ and $u\in[u_k;u_{k+1}]$

$$\left(\int_{u_k}^{u}|f'(v)|\,dv\right)^2 \le \int_{u_k}^{u}\frac{dv}{(1+|v|)^7}\le u_{k+1}-u_k.$$

Thus

$$\sum_{k=0}^{n-1}\int_{u_k}^{u_{k+1}}\left(\int_{u_k}^{u}|f'(v)|\,dv\right)^2 dy(u,t,\omega)\le \sum_{k=0}^{n-1}(u_{k+1}-u_k)\int_{u_k}^{u_{k+1}} dy(u,t,\omega)$$
$$=\frac{2}{n^{\frac{4}{5}}}(y(n^{\frac{1}{5}},t,\omega)-y(-n^{\frac{1}{5}},t,\omega)).$$

For an Arratia flow $\{y(u,s),u\in\mathbb{R},s\in[0;t]\}$ the following relation is true [2]

$$\lim_{|u|\to\infty}\frac{|y(u,t)|}{|u|}=1 \text{ a.s.}$$

Consequently, for any $\omega\in\widetilde{\Omega}=\{\omega'\in\Omega|\ \lim_{|u|\to\infty}\frac{|y(u,t,\omega')|}{|u|}=1\}$ the estimation holds

$$\int_{|u|\le n^{\frac{1}{5}}}\left(f(u)-\sum_{k=0}^{n-1} f(u_k)1\!\!1_{[u_k;u_{k+1}]}(u)\right)^2 dy(u,t,\omega)\le\frac{4c(\omega)}{n^{\frac{3}{5}}} \tag{3.4}$$

with the constant

$$c(\omega)=\sup_{|u|\ge 1}\frac{|y(u,t,\omega)|}{|u|}. \tag{3.5}$$

Let us prove that for any $\omega\in\widetilde{\Omega}$ there exists a constant $\widetilde{c}(\omega)$ such that

$$\int_{|u|>n^{\frac{1}{5}}} f^2(u)dy(u,t,\omega)\le\frac{\widetilde{c}(\omega)}{n^{\frac{3}{5}}}.$$

It can be noticed that $\int_{|u|>n^{\frac{1}{5}}} f^2(u)dy(u,t)\le\frac{1}{n^{\frac{3}{5}}}\int_{|u|>n^{\frac{1}{5}}} f^2(u)(1+|u|)^3dy(u,t)$. Denote by $\{\theta_j\}_{j=1}^{\infty}$ a sequence of jump points of the function $y(\cdot,t)$ on $\mathbb{R}_+$. Thus one may show

$$\int_{u>n^{\frac{1}{5}}} f^2(u)(1+u)^3 dy(u,t) = \sum_{\theta_i \geq n^{\frac{1}{5}}} f^2(\theta_i)(1+\theta_i)^3 \Delta y(\theta_i,t)$$
$$= \sum_{k=1}^{\infty} \sum_{\{i:\ \theta_i \in [\,k;k+1)\,\}} f^2(\theta_i)(1+\theta_i)^3 \Delta y(\theta_i,t)$$
$$\leq \sum_{k=1}^{\infty} (2+k)^3 \sum_{\{i:\ \theta_i \in [\,k;k+1)\,\}} f^2(\theta_i) \Delta y(\theta_i,t).$$

According to the Cauchy inequality and (3.1), for any $u \in \mathbb{R}_+$ the following relations hold

$$f^2(u) \leq \int_u^{\infty} \left(f'(v)\right)^2 (1+v)^7 dv \cdot \int_u^{\infty} \frac{dv}{(1+v)^7} \leq \frac{1}{6u^6}.$$

Consequently, due to (3.5), the inequalities are true

$$\sum_{k=1}^{\infty} (2+k)^3 \sum_{\{i:\ \theta_i \in [\,k;k+1)\,\}} f^2(\theta_i) \Delta y(\theta_i,t)$$
$$\leq \sum_{k=1}^{\infty} (2+k)^3 \frac{1}{6k^6} (y(k+1,t) - y(k,t))$$
$$\leq \frac{16c}{3} \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Hence, for any $\omega \in \widetilde{\Omega}$ there exists the constant $C_1(\omega) = \frac{16c(\omega)}{3}$ such that

$$\int_{u>n^{\frac{1}{5}}} f^2(u) dy(u,t,\omega) \leq \frac{C_1(\omega)}{n^{\frac{3}{5}}}.$$

Similarly, it can be proved that $\int\limits_{u<-n^{\frac{1}{5}}} f^2(u) dy(u,t,\omega) \leq \frac{C_1(\omega)}{n^{\frac{3}{5}}}$. According to this and (3.4), for any $\omega \in \widetilde{\Omega}$ an upper estimation for $d_n\left(T_t^{\omega}(K)\right)$ is true. □

The functions from Lemma 2 that were used to build the $(n+1)$-dimensional subspace are not T_t-essential for any $t > 0$. Thus the image of this subspace under the random operator T_t may be equal to $\{0\}$ with positive probability. So, one can ask about the existence of a finite-dimensional subspace such that for any $t > 0$ its image under T_t is a linear subspace with the same dimension.

4. A Subspace Preserving the Dimension under a Random Operator Generated by an Arratia Flow

In this section for any $t > 0$ and $n \in \mathbb{N}$ we present a family $\{g_k, k = \overline{0,n}\}$ of linearly independent T_t-essential functions such that their images under T_t are linearly independent. Such a family generates a subspace which preserves the dimension under a random operator generated by an Arratia flow. It can be used to get a lower estimation of $d_n(T_t(K))$.

Let us fix any $n \in \mathbb{N}$, and build a family of $(n+1)$ linearly independent functions in the following way. Let $\{u_k\}_{k=0}^{2(n+1)}$ be a partition of $[0; n^{-2}]$ into $2(n+1)$ segments with equal lengths. For any $k = \overline{0,n}$ define f_k by

$$f_k = \begin{cases} 0, & u \notin [u_{2k}; u_{2k+1}], \\ 1, & u \in [u_{2k} + \frac{n^{-2}}{6(n+1)}; u_{2k} + \frac{2n^{-2}}{6(n+1)}], \\ \frac{6(n+1)}{n^{-2}}(u - u_{2k}), & u \in [u_{2k}; u_{2k} + \frac{n^{-2}}{6(n+1)}], \\ -\frac{6(n+1)}{n^{-2}}(u - u_{2k+1}), & u \in [u_{2k} + \frac{2n^{-2}}{6(n+1)}; u_{2k+1}]. \end{cases} \tag{4.1}$$

Lemma 4.1. *There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \le \varepsilon_0$ the functions $\{f_k * p_\varepsilon, k = \overline{0,n}\}$ are linearly independent.*

Proof. Since the considered functions $\{f_k, k = \overline{0,n}\}$ are linearly independent, its Gram determinant doesn't equal to 0, i.e. $G(f_0, \dots, f_n) \neq 0$. For each $k = \overline{0,n}$

$$f_k * p_\varepsilon \to f_k,\ \varepsilon \to 0.$$

Hence, due to the continuity of the Gram determinant, one may notice that there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \le \varepsilon_0$

$$G(f_0 * p_\varepsilon, \dots, f_n * p_\varepsilon) \neq 0,$$

and the desired result is proved. □

Theorem 4.2. *There exists a set Ω_0 of probability one such that for any $\omega \in \Omega_0$ the functions $T_t^\omega(f_0 * p_\varepsilon), \dots, T_t^\omega(f_n * p_\varepsilon)$ are linearly independent.*

Proof. Denote by K_ε the integral operator in $L_2(\mathbb{R})$ with the kernel k_ε. To prove the statement of the theorem it's enough to show that on some Ω_0 of probability one the following inequality holds $(K_\varepsilon f, f) > 0$, for any nonzero $f \in LS\{f_0, \dots, f_n\}$. Due to (1.4)

$$(K_\varepsilon f, f) = \sum_\theta (f * p_\varepsilon)^2(\theta)\Delta y(\theta, t), \tag{4.2}$$

where θ is a point of jump of the function $y(\cdot, t)$.

It was proved in [6] that there exists Ω_0 of probability one such that for any $\omega \in \Omega_0$ a linear span of the functions $\{p_\varepsilon(\cdot - \theta(\omega))|_{[0;1]}\}_{\theta(\omega)}$ is dense in $L_2([0;1])$. Thus on the set Ω_0 for any $f \in LS\{f_0, \dots, f_n\} \subset L_2([0;1])$ one can find a random point θ_f such that $(f(\cdot), p_\varepsilon(\cdot - \theta_f)) \neq 0$. Since $y(\cdot, t) : \mathbb{R} \to \mathbb{R}$ is nondecreasing, $\Delta y(\theta, t) > 0$ for any jump-point θ. Consequently, on the set Ω_0

$$\begin{aligned} \sum_\theta (f * p_\varepsilon)^2(\theta)\Delta y(\theta, t) &= \sum_\theta (f(\cdot), p_\varepsilon(\cdot - \theta))^2 \Delta y(\theta, t) \\ &\ge (f(\cdot), p_\varepsilon(\cdot - \theta_f))^2 \Delta y(\theta_f, t) > 0, \end{aligned}$$

which proves the theorem. □

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