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REGULAR SEMIGROUP OF LINEAR TRANSFORMATIONS ON CLASS FUNCTIONS OF A FINITE GROUP

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Abstract : In this paper we intend to construct a bilinear form, with the help of the character values of a finite group G, on the vector space of all class functions defined on G. The bilinear form obtained thus is non-degenerate. Here we construct the regular semigroup associated with this bilinear form.

Keywords: Regular semigroup, character values, bilinear form, class functions.

INTRODUCTION:

In [1] we see that by starting with an arbitrary bilinear form on the Cartesian product of two finite dimensional vector spaces, a regular semigroup of pairs of linear maps is constructed. The character table of a finite group gives rise to a bilinear form on the Cartesian product of the space of all class functions on this group. The bilinear form obtained is non-degenerate. We show that there exists a regular semigroup associated with the bilinear form. For basic definitions, concepts and theorems that are required in this sequel refer [2,3,4,5].

NON-DEGENERATE BILINEAR FORM THE CHARACTER VALUES OF A FINITE GROUP

Let G be a finite group of order n. Let $C_1, C_2, \ldots\ldots\ldots, C_m$ be the conjugacy classes and $\chi_1, \chi_2, \ldots\ldots\ldots, \chi_m$ be the irreducible characters. Define $C = [c_{ij}], i, j = 1,2,\ldots\ldots, m$ where c_{ij} are the character values in the character table of the group G. The matrix C of order m is called character matrix of the group G. Let X be the vector space of all class functions defined on G.

Let $\theta, \theta' \in C$. Then $\theta = \sum_i \alpha_i \chi_i$ and $\theta' = \sum_i \beta_i \chi_i$, for unique values of $\{\alpha_1, \alpha_2, \ldots\ldots., \alpha_m\}$ and $\{\beta_1, \beta_2, \ldots\ldots., \beta_m\}$. Define $B : X \times X \to \mathbf{C}$ by

$$B(\theta, \theta') = (\alpha_1, \alpha_2, \ldots\ldots., \alpha_m)(c_{ij})(\beta_1, \beta_2, \ldots\ldots., \beta_m) = \sum_j \sum_i \alpha_i c_{ij} \beta_j \quad (1)$$

Theorem 2.1 : Let X be the vector space of all class functions defined on a finite group G of order n. Then $B : X \times X \to \mathbf{C}$ by, $B(\theta, \theta') = \sum_j \sum_i \alpha_i c_{ij} \beta_j$ is a bilinear form and the matrix of B is the character matrix in the basis given by the simple characters.

The bilinear form B defines two linear maps $B_1 : X \to X^*$ defined by $B_1(\theta) = B_1(\theta, -)$ and $B_2 : X \to X^*$ defined by $B_2(\theta) = B_2(\_, \theta)$.

Theorem 2.2 : If B is the bilinear form associated with the character matrix, then the map $B_1 : X \to X^*$ is an injection.

Theorem 2.3 : The bilinear form B associated with the character matrix is non-degenerate.

Proof : dim(X) = dim(X^*) = m. Since B_1 is an injection, $\eta(B_1) = \{0\}$. Hence $n(B_1) = 0$. This implies $r(B_1) = \dim(X) = m$. Since the matrix of B is C and it is a square matrix of order m, it follows that $r(B_2) = m$.

Hence $n(B_2) = 0 \Rightarrow \eta(B_2) = 0$

Thus $\eta(B_1) = 0 = \eta(B_2)$. Hence B is a non-degenerate bilinear form.

From the theorem 2.3 and the necessary and sufficient condition for a bilinear form to be degenerate [5], we get an interesting result which can be stated as follows.

Theorem 2.4 : The character matrix of a finite group is non-singular.

Theorem 2.5 : If B is the bilinear form associated with the character matrix then the linear map $B_2 : X \to X^*$ is an isomorphism of X onto X^* .

Theorem 2.6 : Every $f \in L(X)$ there exist $g \in L(X)$ such that (f, g) is an adjoint pair with respect to B.

Proof : since B is non-generate B_1 and B_2 are isomorphisms of X onto X^* . Therefore B_1^{-1} and B_2^{-1} exists. Let $f \in L(X)$. Define f^* on X^* by

$$tf^* = ft \text{ for all } t \in X^*$$

Then $f^* \in L(X^*)$. Now define

$$g = B_2 f^* B_2^{-1}$$

Clearly $g \in L(X)$.

$$g = B_2 f^* B_2^{-1} \Rightarrow gB_2 = B_2 f^*$$

Hence (f, g) is an adjont pair.

Theorem 2.7 : The set of all adjoint pairs arising from the non-degenerate bilinear form B associated with the character matrix C form a regular subsemigroup of $L(X) \times L(X)^{OP}$.

Proof : Let $S(B)=\{(f,g)\in L(X)\times L(X)^{OP}:(f,g)$ is an adjoint pair with respect to B $\}$. Since $(I_X,I_X)\in S(B)$, $S(B)$ is non empty where I_X is the identity map on X. Let $(f_1,g_1),(f_2,g_2)\in S(B)$ Then for all $x,y\in X$, we have

$$\begin{aligned} B(x(f_1f_2),y) &= B((xf_1)f_2,y) \\ &= B(xf_1,yg_2) \\ &= B(x,(yg_2)g_1) \;. \\ &= B(x,yg_2g_1) \end{aligned}$$

$$\Rightarrow (f_1f_2,g_2g_1)\in S(B)$$

Similarly we can prove the associativity. Since B is non-denerate, we have $g=B_2f^*B_2^{-1}$. Since $L(X)$ is regular there exist $f'\in L(X)$ such that $ff'f=f$. Since $f'\in L(X)$ there exist $g'\in L(X)^{OP}$ such that (f',g') is an adjoint pair. Hence $g'=B_2f'^*B_2^{-1}$ where $tf'^*\ f't$ for all $t\in X^*$. Since $ff'f=f$ it is enough if we show $gg'g=g$.

$$\begin{aligned} gg'g &= (B_2f^*B_2^{-1})(B_2f'^*B_2^{-1})(B_2f^*B_2^{-1}) \\ &= (B_2f^*)(B_2^{-1}B_2)f'^*(B_2^{-1}B_2)(f^*B_2^{-1}) \\ &= B_2(f^*f'^*f^*)B_2^{-1} \end{aligned}$$

Now for all $t\in L(X^*)$,

$$\begin{aligned} t(f^*f'^*f^*) &= (tf^*)(f'^*f^*) \\ &= ft(f'^*f^*) \\ &= f(tf'^*)f^* \\ &= ff'(tf^*) \\ &= (ff'f)t \\ &= ft \end{aligned}$$

Therefore $S(B)$ is a regular subsemigroup of $L(X)\times L(X)^{OP}$.

Theorem 2.8 : $(f,g)\in S(B)$ if and only if $M(f)C=CM(g)^T$.

Proof : Let $M(f)=(a_{ij})$: $i,j=1,2,.....,m$ and $M(g)=(a_{ij})$: $i,j=1,2,.....,m$ $M(B)=(c_{ij})=C, i,j=1,2,.....,m$. Then $\chi_i f=\sum_j a_{ij}\chi_j$ and $\chi_i g=\sum_j b_{ij}\chi_j$

$(f,g)\in S(B)\Rightarrow B(xf,y)=B(x,yg)$ for all $x\in X$ and $y\in Y$. In particular,

$$B(\chi_i f,\chi_j)=B(\chi_i,\chi_j f) \text{ for all } i \text{ and } j.$$

$$\Rightarrow \sum_k a_{ik}c_{kj} = \sum_k c_{ik}b_{kj} \text{ for all } i \text{ and } j.$$

Let $\theta, \theta' \in C$ .Then $\theta = \sum_i \alpha_i \chi_i$ and $\theta' = \sum_i \beta_i \chi_i$, for unique values of $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ and $\{\beta_1, \beta_2, ..., \beta_m\}$. Now

$$\begin{aligned}\theta f &= \sum_j \alpha_j \chi_j f \\ &= \sum_j \alpha_j (\sum_i a_{ij} \chi_i) \\ &= \sum_i (\sum_j a_{ji} \alpha_j) \chi_i\end{aligned}$$

and

$$\begin{aligned}\theta' g &= \sum_j \beta_j \chi_j g \\ &= \sum_i (\sum_j b_{ji} \beta_j) \chi_i\end{aligned}$$

Therefore $B(\theta f, \theta') = \sum_k \sum_i \sum_j a_{ji} \alpha_j c_{ik} \beta_k$ and $B(\theta, \theta' g) = \sum_k \sum_i \sum_j \alpha_i c_{ik} b_{jk} \beta_j$

It follows that $B(\theta f, \theta') = B(\theta, \theta' g)$. Implies $(f, g) \in S(B)$.

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