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GLOBAL ATTRACTIVITY OF A RATIONAL RECURSIVE SEQUENCE

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Abstract

In this paper we investigate the boundedness, the periodic character and global attractivity of the recursive sequence

$$x_{n+1} = \frac{a + bx_{n-k}}{A - x_n},\ n = 0,1,\ldots,$$

where $a \geqslant 0$, $b, A > 0$ are real numbers, $k \in \{1,2, \ldots\}$ and the initial conditions $x_{-k}, \ldots, x_0$ are arbitrary real numbers. We show that the positive equilibrium of the equation is a global attractor with a basin that depends on certain conditions posed on the coefficients.

1. INTRODUCTION

Our goal in this paper is to investigate the boundedness, the periodic character and global attractivity of all positive solutions of the recursive sequence

$$x_{n+1} = \frac{a + bx_{n-k}}{A - x_n}, n = 0,1,\ldots, \tag{1.1}$$

where $a \geqslant 0$, $b, A > 0$ are real numbers, $k \in \{1, 2, \ldots\}$ and the initial conditions $x_{-k}, \ldots, x_0$ are arbitrary real numbers.

In [12], Li and Sun investigated the global asymptotic stability of the rational recursive sequence

$$x_{n+1} = \frac{a - bx_{n-k}}{A - x_n}, n = 0,1,\ldots,$$

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where a, b, A are nonnegative real numbers and $k \in \{1, 2, \ldots\}$. In [6], He et al. investigated the global asymptotic stability of the rational recursive sequences

$$x_{n+1} = \frac{a - bx_{n-1}}{A + x_n}, n = 0, 1, \ldots,$$

where $a \geqslant 0, A, b > 0$ are real numbers, and the initial conditions x_{-1}, x_0 are arbitrary positive real numbers. For the global behavior of solutions of some related equations, see [1, 2, 3, 14]. Other related results refer to [4, 5, 7, 8, 9, 10, 11, 13, 15, 16].

Here, we recall some results which will be useful in the sequel.

Let I be some interval of real numbers and let F be continuous function defined on I^{k+1}. Then, for initial conditions $x_{-k}, \ldots, x_0 \in I$, it is easy to see that the difference equation

$$x_{n+1} = F(x_n, \ldots, x_{n-k}), n = 0, 1, \ldots \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x}$ is called an equilibrium of Eq. (1.2), when $\overline{x} = F(\overline{x}, \ldots, \overline{x})$, that is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(1.2), or equivalently, is fixed point of F.

The linearized equation with Eq.(1.2) about an equilibrium $\overline{x}$ is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial x_i}(\overline{x}, \ldots, \overline{x}) y_{n-i}, n = 0, 1, \ldots,$$

and its characteristic equation is

$$\lambda^{n+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial x_i}(\overline{x}, \ldots, \overline{x}) \lambda^{n-i}, n = 0, 1, \ldots .$$

Definition 1.1. An interval $J \subset I$ is called an invariant interval of Eq.(1.2), if $x_{-k}, \ldots, x_0 \in J \Rightarrow x_n \in J$ for all $n \geq 0$. That is, every solution of Eq.(1.2) with initial conditions in J remains in J.

Theorem 1.1 Assume that F is a C^1 function and let $\overline{x}$ be the equilibrium of Eq.(1.2). Then the following statements are ture.

(a) If all the roots of the polynomial equation

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial F}{\partial x_i}(\overline{x}, \cdots, \overline{x}) \lambda^{k-i} = 0 \tag{1.3}$$

lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\bar{x}$ of Eq.(1.2) is locally asymptotically stable.

(b) If at least one of the roots of Eq.(1.3) has absolute value greater than one, then the equilibrium $\bar{x}$ of Eq.(1.2) is unstable.

Also, we need the following well-known explicit condition for local asymptotic stability.

Theorem1.2. Assume that $p_1, p_2, \ldots, p_k \in R$. Then $\sum_{i=1}^{k} | p_i | < 1$ is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0,\ n = 0, 1, \ldots.$$

Theorem 1.3. Consider the difference equation

$$x_{n+1} = F(x_n, \ldots, x_{n-k}),\ n = 0, 1, \ldots,$$

where $k \in \{1, 2, \ldots\}$, $F \in C[(0, \infty)^{k+1}, (0, \infty)]$ is increasing in each of its arguments and the initial condition $x_{-k}, \ldots, x_0$ are positive. Assume that Eq.(1.2) has a unique positive equilibrium $\bar{x}$ and that the function h difined by $h(x) = F(x, \ldots, x)$, $x \in (0, \infty)$ satisfies $(h(x) - x)(x - \bar{x}) < 0$ for $x \neq \bar{x}$. Then $\bar{x}$ is a global attractor of all positive solution of Eq. (1.2).

2. ASYMPTOTIC STABILITY

Consider the difference equation (1.1) with

$$a > 0 \text{ and } A > b > 0. \tag{2.1}$$

The equilibria of Eq.(1.1) are the nonnegative solution of the quadratic equation

$$\bar{x}^2 - (A - b)\bar{x} + a = 0$$

and the linearized equation of Eq.(1.1) about $\bar{x}$ is

$$y_{n+1} - \frac{\bar{x}}{A - \bar{x}} y_n - \frac{b}{A - \bar{x}} y_{n-k} = 0, n = 0,1, \cdots.$$

Its characteristic equation

$$\lambda^{k+1} - \frac{\bar{x}}{A - \bar{x}} \lambda^k - \frac{b}{A - \bar{x}} = 0. \tag{2.2}$$

According to our assumption, we have

if (2.1) holds and $a = (A - b)^2/4$, then Eq.(1.1) has a unique positive equilibrium $\bar{x}_0 = (A - b)/2$;

if (2.1) holds and $a < (A - b)^2/4$, then Eq.(1.1) has two positive equilibria

$$\bar{x}_1 = \frac{A - b + \sqrt{(A-b)^2 - 4a}}{2} \quad \text{and} \quad \bar{x}_2 = \frac{A - b - \sqrt{(A-b)^2 - 4a}}{2}$$

Thus, we obtain the following results.

Theorem 2.1. Assume (2.1) holds. Then

(a) the positive equilibrium $\bar{x}_2$ of Eq.(1.1) is locally asymptotically stable (In the sequel, we will denote $\bar{x}_2$ as $\bar{x}$).

(b) the positive equilibria $\bar{x}_0$ and $\bar{x}_1$ of Eq.(1.1) are unstable.

Proof. (a) By (2.2), the characteristic equation (2.2) about $\bar{x}_2$ is

$$\lambda^{k+1} - \frac{A - b - \sqrt{(A-b)^2 - 4a}}{A + b + \sqrt{(A-b)^2 - 4a}}\lambda^k - \frac{2b}{A + b + \sqrt{(A-b)^2 - 4a}} = 0.$$

By Theorem 1.2, we have

$$\begin{aligned}&\left| -\frac{A - b - \sqrt{(A-b)^2 - 4a}}{A + b + \sqrt{(A-b)^2 - 4a}} \right| + \left| \frac{2b}{A + b + \sqrt{(A-b)^2 - 4a}} \right| \\ &= \left| \frac{A + b - \sqrt{(A-b)^2 - 4a}}{A + b + \sqrt{(A-b)^2 - 4a}} \right| < 1.\end{aligned}$$

Hence, the positive equilibrium $\bar{x}_2$ is locally asymptotically stable.

(b) Similarly, the positive equilibria $\bar{x}_0$ and $\bar{x}_1$ of Eq. (1.1) are unstable for $k \in \{1, 2, \ldots\}$.

3. GLOBAL ATTRACTIVITY

In this section, we study the global attractivity of all positive solution of Eq.(1.1). We show that the positive equilibrium $\bar{x}$ of Eq.(1.1) is a global attractor with a basin that depends on certain conditions posed on the coefficients.

Lemma 3.1. Assume (2.1) holds. Then the following statements are true.

(a) $0<\bar{x}<\bar{x}_1<A.$

(b) Let $F(x_n,\ldots,x_{n-k})=\dfrac{a+b\,x_{n-k}}{A-x_n}$, so F is a strictly increasing function in $(-\infty,\ \bar{x}_1)$.

Proof. The proof of (b) is obviously and will be omitted. We only prove (a). In view of

$$A-\bar{x}_1=A-\frac{A-b+\sqrt{(A-b)^2-4a}}{2}=\frac{A+b-\sqrt{(A-b)^2-4a}}{2}>0.$$

and the definitions of $\bar{x}$ and $\bar{x}_1$, the proof is obvious and complete.

Lemma 3.2. Suppose that the function h defined by

$$h(x)=F(x,\cdots,x)=\frac{a+bx}{A-x},\ x\in(0,\bar{x}_1).$$

Then $(h(x)-x)(x-\bar{x})<0$ for $x\neq\bar{x}$.

Proof. By the definition of h, we have

$$\begin{aligned}
&(h(x)-x)(x-\bar{x})\\
=\ &\left(\frac{a+bx}{A-x}-x\right)(x-\bar{x})=\frac{1}{A-x}(x^2-(A-b)x+a)\ (x-\bar{x})\\
=\ &\frac{1}{A-x}\left(x-\frac{(A-b)+\sqrt{(A-b)^2-4a}}{2}\right)\left(x-\frac{(A-b)-\sqrt{(A-b)^2-4a}}{2}\right)(x-\bar{x})\\
=\ &\frac{1}{A-x}(x-\bar{x}_1)(x-\bar{x})^2<0\ \ for\ \ x\neq\bar{x}.
\end{aligned}$$

The proof is complete.

Theorem 3.1. Assume that $0<a<(A-b)^2/4$ and $A>b>0$ hold. Let $\{x_n\}$ be a solution of Eq.(1.1). If $(x_{-k},\ldots,x_0)\in(-\infty,\bar{x}_1]^k\times[-a/b,\bar{x}_1],$ then $0\leq x_n\leqslant\bar{x}_1$ for $n\geq1$.

Proof. By part (b) of Lemma 3.1, we have

$$0 = F(-a/b, x_{-1}, \ldots, x_{-k}) \le x_1 = F(x_0, x_{-1}, \ldots, x_{-k}) \le F(\bar{x}_1, \bar{x}_1, \ldots, \bar{x}_1) = \bar{x}_1,$$

and

$$0 < F(0, x_{-1}, \ldots, x_{-k}) \le x_2 = F(x_1, x_0, \ldots, x_{-k+1}) \le F(\bar{x}_1, \bar{x}_1, \ldots, \bar{x}_1) = \bar{x}_1.$$

Hence, the result follows by induction. The proof is complete.

Theorem 3.2. Assume that $0 < a < (A - b)^2/4$ and $A > b > 0$ hold. Then the positive equilibrium $\bar{x}$ of Eq.(1.1) is a global attractor with a basin $S = (0, \bar{x}_1)^{k+1}$.

Proof. Let $\{x_n\}$ is a solution of Eq.(1.1) with initial conditions $(x_{-k}, \ldots, x_0) \in S$. Then, by part (b) of Lemma 3.1 and Theorem 3.1, the function $F(x_n, \ldots, x_{n-k})$ is a strictly increasing function in each of arguments.

By Lemma 3.2, we have $(h(x) - x)(x - \bar{x}) < 0$. Furthermore, Theorem 1.3 implies that $\bar{x}$ is a global attaractor of all positive solution of Eq.(1.1). Thus

$$\lim_{n\to\infty} x_n = \bar{x} = \frac{A - b - \sqrt{(A-b)^2 - 4a}}{2}$$

The proof is complete.

4. THE PRIME PERIOD TWO SOLUTION

In the section, we discuss whether Eq. (1.1) has the prime period two solution.

Theorem 4.1. Assume that $a > 0$ and $A > b > 0$ hold. Then Eq. (1.1) has no positive solution with prime period two.

Proof. Assume for the sake of contradiction that there exist distinctive positive real numbers ϕ and φ such that

$$\ldots, \phi, \varphi, \phi, \varphi, \ldots$$

is a period two solution of Eq.(1.1). There are two cases to be considered.

Case (a) k is odd. In this case $x_{n+1} = x_{n-k}$, ϕ and φ satisfy the system

$$\phi = \frac{a + b\phi}{A - \varphi}, \quad \varphi = \frac{a + b\varphi}{A - \phi}.$$

Hence $(\phi - \varphi)(A - b) = 0$. According to the condition $A > b > 0$, we obtain $\phi = \varphi$, which contradicts the hypothesis $\phi \ne \varphi$.

Case (b) k is even. In this case $x_n = x_{n-k}$, ϕ and φ statisfy the system

$$(A - \varphi)\,\phi = a + b\varphi \text{ and } (A - \phi)\varphi = a + b\phi$$

Subtracting the two equations above, we obtain

$$(\phi - \varphi)(A + b) = 0,$$

and so $A + b = 0$ or $\phi - \varphi = 0$, which contradicts the condition $A > b > 0$ and the hypothesis of $\phi \neq \varphi$. The proof is complete.

Corollary 4.1. If $A \neq b$, then Eq.(1.1) has no solution with prime period two for all $a \in (0, \infty)$.

5. THE CASE $A = 0$

In this section, we study the asymptotic stability of the difference equation

$$x_{n+1} = \frac{bx_{n-k}}{A - x_n},\ n = 0,1\cdots, \tag{5.1}$$

where b, $A \in (0, +\infty)$, $k \in \{1, 2, \ldots\}$ and the initial conditon $x_{-k}, \ldots, x_0$ are arbitrary real numbers.

By putting $x_n = by_n$, Eq.(5.1) yields

$$y_{n+1} = \frac{y_{n-k}}{C - y_n}, n=0,1..., \tag{5.2}$$

where $C = A/b > 0$. Eq.(5.2) has two equilibria $\overline{y}_1 = 0,\ \overline{y}_2 = C - 1.$ The linearized equation of Eq. (5.2) about the equilibria $\overline{y}_i, i=1,2,$ is

$$z_{n+1} - \frac{\overline{y}_i}{C - \overline{y}_i} z_n - \frac{1}{C - \overline{y}_i} z_{n-k} = 0, i = 1, 2, n = 0,1\cdots.$$

For $\overline{y}_2 = C - 1,$ Theorem 1.2 implies that $\overline{y}_2$ is unstable. For $\overline{y}_1 = 0$, we have

$$z_{n+1} - \frac{1}{C} z_{n-k} = 0,\ n = 0,1\cdots. \tag{5.3}$$

The characteristic equation of Eq.(5.3) is $\lambda^{k+1} - 1/c = 0$. Hence, by Theorem 1.1, we have

(i) if $A > b$, then $\overline{y}_1$ is locally asymptotically stable;

(ii) if $A < b$, then $\bar{y}_1$ is a repeller;

(iii) if $A = b$, then linearized stability analysis fails.

In the sequel, we will discuss the global attractivity of the zero equilibrium of Eq. (5.2). So, we assume that $A > b$, namely, $C > 1$.

Lemma 5.1. Assume that the initial conditions $(y_{-k}, \dots, y_0) \in [-C+1, C-1]^{k+1}$. Then $y_n \in [-C+1, C-1]$ for $n \geq 1$.

Proof. It is easy to see that

$$-C+1 \leqslant \frac{-C+1}{C-(-C+1)} \leqslant y_1 = \frac{y-1}{C-y_0} \leqslant \frac{C-1}{C-(C-1)} = C-1,$$

and

$$-C+1 \leqslant \frac{-C+1}{C-(-C+1)} \leqslant y_2 = \frac{y_0}{C-y_1} \leqslant \frac{C-1}{C-(C-1)} = C-1.$$

The result follows by induction, and this completes the proof.

Obviously, Lemma 5.1 implies that the following result is true.

Theorem 5.1. The equilibrium $\bar{y}_1 = 0$ of Eq. (5.2) is a global attractor with a basin $S = [-C+1, C-1]^{k+1}$.

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