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SOLUTIONS OF FRACTIONAL KINETIC EQUATIONS INVOLVING GENERALIZED MULTIINDEX BESSEL FUNCTION

Mehar Chand, Rakesh Kumar and Daljeet Kaur

Abstract: A new and further generalized form of the fractional kinetic equation involving generalized multiindex Bessel function $J_{(v_j)_m,q}^{(\lambda_j)_m,\gamma}(z)$ is developed. The manifold generality of the generalized multiindex Bessel Function is discussed in terms of the solution of fractional kinetic equation in the present paper. The graphical interpretation and numerical results are presented of the solutions of fractional kinetic equations. The results obtained here are capable of yielding a very large number of known and (presumably) new results.

Mathematics Subject Classification: Fractional Calculus, Fractional Differential Equation, Multiindex-Bessel function, Laplace Transform.

Keywords: 26A33, 33C45, 33C60, 33C70.

1. INTRODUCTION AND PRELIMINARIES

The importance of fractional differential equations in the field of applied science has gained more attention not only in mathematics but also in physics, dynamical systems, control systems and engineering, to create the mathematical model of many physical phenomena. Especially, the kinetic equations describe the continuity of motion of substance. The extension and generalization of fractional kinetic equations involving many fractional operators were found in [1, 2, 9, 4, 8, 12, 6].

If an arbitrary reaction is described by a time dependent quantity $N = N(t)$, then the fractional differential equation between rate of change of the reaction, the destruction rate and the production rate of N was established by Haubold and Mathai in [9], is given as follows:

$$\frac{dN}{dt} = -d(N_t) + p(N_t), \tag{1}$$

where $N = N(t)$ the rate of reaction, $d = d(N)$ the rate of destruction, $p = p(N)$ the rate of production and N_t denotes the function defined by $N_t(t^*) = N(t - t^*)$, $t^* > 0$.

A special case of (1), when spatial fluctuations and inhomogeneities in the quantity $N(t)$ are neglected, is given by the following differential equation as:

$$\frac{dN}{dt} = -c_i N_i(t), \tag{2}$$

such that $N_i\ (t = 0) = N_0$ is the number density of the species i at time $t = 0$ and $c_i > 0$.

If we remove the index i and integrate the standard kinetic equation (2), we have

$$N(t) - N_0 = -c\,{}_0D_t^{-1}N(t), \tag{3}$$

where c is a constant and ${}_0D_t^{-1}$ is the special case of the Riemann-Liouville fractional integral operator ${}_0\,D_t^{-\nu}$ defined as

$${}_0D_t^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}f(s)ds, \qquad (t > 0, R(\nu) > 0). \tag{4}$$

The fractional generalization of the standard kinetic equation (3) is given by Haubold and Mathai [9] as follows:

$$N(t) - N_0 = -c^{\nu}{}_0D_t^{-\nu}N(t), \tag{5}$$

and obtained the solution of (5) as follows:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu k+1)}(ct)^{\nu k}. \tag{6}$$

Further, Saxena and Kalla [16] considered the the following fractional kinetic equation:

$$N(t) - N_0 f(t) = -c^{\nu}{}_0D_t^{-\nu}N(t), \qquad (\Re(\nu) > 0, c > 0), \tag{7}$$

where $N(t)$ denotes the number density of a given species at time t, $N_0 = N(0)$ is the number density of that species at time $t = 0$, c is a constant and $f \in \mathcal{L}(0, \infty)$.

By applying the Laplace transform to (7) (see [12]),

$$L\{N(t);p\} = N_0 \frac{F(p)}{1+c^{\nu}p^{-\nu}} = N_0\left(\sum_{n=0}^{\infty} (-c^{\nu})^n p^{-\nu n}\right)F(p), \tag{8}$$

$$\left(n \in N_0, \left|\frac{c}{p}\right| < 1\right).$$

Let $f(t)$ be a real or complex valued function of variable t and p is a real or complex parameter, then Laplace transform of $f(t)$ is defined as (see [17])

$$F(p) = L\{f(t);p\} = \int_0^{\infty} e^{-pt}f(t)dt, \qquad (\Re(p) > 0). \tag{9}$$

In view of the effectiveness and a great importance of the kinetic equation in certain astrophysical problems the authors develop a further generalized form of the fractional kinetic equation involving generalized multiindex Bessel function $J_{(\nu_j)_m,q}^{(\lambda_j)_m,\gamma}(z)$.

For our present study we start by recalling the previous work.The Bessel-Maitland function $J_\nu^\lambda(z)$ is given as (see Marichev [13]):

$$J_\nu^\lambda(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\nu+\lambda n+1)n!}, \qquad \lambda > 0; z \in \mathbb{C} \tag{10}$$

The generalized form of Bessel function $J_{\nu,\mu}^\lambda(z)$ is given by Jain and Agarwal [10] as:

$$J_{\nu,\mu}^\lambda(z) = \sum_{n=0}^{\infty} \frac{(-1)^r\left(\frac{z}{2}\right)^{\nu+2\mu+2n}}{\Gamma(\nu+\mu+\lambda n+1)\Gamma(\mu+n+1)}, \tag{11}$$

$$\lambda > 0, \nu, \mu \in \mathbb{C}; z \in \mathbb{C}\backslash(-\infty, 0].$$

Further, Pathak [15] gave the following more generalized form of the generalized Bessel-Maitland function $J_{\nu,\mu}^{\lambda,\gamma}(.)$ as:

$$J_{\nu,q}^{\lambda,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\nu+\lambda n+1)n!}, \tag{12}$$

$\lambda, \nu, \gamma \in \mathbb{C}, \Re(\lambda) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0$ and $q \in (0,1) \cup \mathbb{N}$.

If ν is replaced by $\nu - 1$ and z by $-z$ then generalized Bessel-Maitland function given in equation (12) reduces to well known Mittag-Leffler function as follows:

$$J_{\nu-1,q}^{\lambda,\gamma}(-z) = E_{\lambda,\nu}^{\gamma,q}(z), \tag{13}$$

$$\lambda, \nu, \gamma \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0, \Re(\gamma) > 0; q \in (0,1) \cup \mathbb{N},$$

where $E_{\lambda,\nu}^{\gamma,q}(z)$ denotes generalized Mittag-Lefller function, was introduced by Shukla and Prajapati [19].

If $q = 1, \gamma = 1$, ν is replaced by $\nu-1$ and z by $-z$ then generalized Bessel-Maitland function given in equation (12) reduces to Mittag-Leffler function, studied by Wiman [21] as follows:

$$J_{\nu-1,1}^{\lambda,1}(-z) = E_{\lambda,\nu}(z), \qquad \lambda, \nu \in \mathbb{C}, \Re(\lambda) > 0, \Re(\nu) > 0. \tag{14}$$

The generalized multiindex Bessel function $J_{(\nu_j)_m,q}^{(\lambda_j)_m,\gamma}(z)$ studied by [5] is defined as follows:

$$J_{(\nu_j)_m,q}^{(\lambda_j)_m,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\prod_{j=1}^{m} \Gamma(\lambda_j n+\nu_j+1)} \frac{(-z)^n}{n!}, \tag{15}$$

where $m \in \mathbb{N}$, λ_j, ν_j, γ, q, $z \in \mathbb{C}$ ($j = 1,..., m$) such that

$$\sum_{j=1}^{m} \Re(\lambda_j) > max\{0; \Re(q) - 1\}; q > 0, \Re(\nu_j) > -1, \Re(\gamma) > 0 \text{ and } q \in (0,1) \cup \mathbb{N}.$$

On setting $m = 1$, $q = 0$, $\lambda_1 = 1$, $\nu_1 = \nu$ and replace z by $\frac{z^2}{4}$ in (15), we have

$$J_{\nu,0}^{1,\gamma}\left[\frac{z^2}{4}\right] = \left(\frac{2}{z}\right)^{\nu} J_\nu[z], \tag{16}$$

where $J_\nu\,[z]$ is a well known Bessel function of first kind defined by (see [11])

$$J_\nu[z] = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{\Gamma(n+\nu+1)}, \qquad \nu \in \mathbb{C}; z \in \mathbb{C}\backslash(-\infty, 0]. \tag{17}$$

For more details about the Bessel function one may refer to earlier work by Erdélyi *et al.* [7] and Watson [20].

2. FRACTIONAL KINETIC EQUATIONS

In this section, we investigated the solutions of the generalized fractional kinetic equations by considering generalized Multiindex Bessel function.

Remark 1: *The solutions of the fractional kinetic equations in this section are obtained in terms of the generalized Mittag-Leffler function $E_{\alpha,\beta}(x)$ (Mittag-Leffler[14]), which is defined as:*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \qquad \Re(\alpha) > 0, \Re(\beta) > 0. \tag{18}$$

Theorem 1 *If $a>0$, $d>0$, $\nu > 0$; $a \neq d$; $m \in \mathbb{N}$, λ_j, ν_j, γ, $q \in \mathbb{C}$ ($j = 1,...,m$) such that $\sum_{j=1}^{m} \Re(\lambda_j) > max\{0; \Re(q) - 1\}$; $q > 0$, $\Re(\nu_j) > -1$, $\Re(\gamma)>0$ and $q \in (0,1) \cup \mathbb{N}$, then the solution of the fractional kinetic equation:*

$$N(t) - N_0 J_{(\nu_j)m,q}^{(\lambda_j)m,\gamma}(d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t), \tag{19}$$

is given by the following formula

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\Gamma(\nu n+1)}{\prod_{j=1}^{m} \Gamma(\lambda_j n+\nu_j+1)} \frac{(-d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-a^\nu t^\nu). \tag{20}$$

Proof. Laplace transform of Riemann-Liouville fractional integral operator is given by (Erdelyi *et al.* [3], Srivastava and Saxena [18]):

$$L\{\,{}_0D_t^{-\nu} f(t); p\} = p^{-\nu} F(p), \tag{21}$$

where $F(p)$ is defined in (9).

Now, taking Laplace transform on (19) gives,

$$L\{N(t); p\} = N_0 L\left\{J_{(\nu_j)m,q}^{(\lambda_j)m,\gamma}(d^\nu t^\nu); P\right\} - a^\nu L\{\,{}_0D_t^{-\nu} N(t); P\} \tag{22}$$

$$N(p) = N_0 \left(\int_0^\infty e^{-pt} \sum_{n=0}^\infty \frac{(\gamma)_{qn}}{\prod_{j=1}^m \Gamma(\lambda_j n + \nu_j + 1)} \frac{(-d^\nu t^\nu)^n}{n!} dt\right) - a^\nu p^{-\nu} N(p), \tag{23}$$

interchanging the order of integration and summation in (23), we have

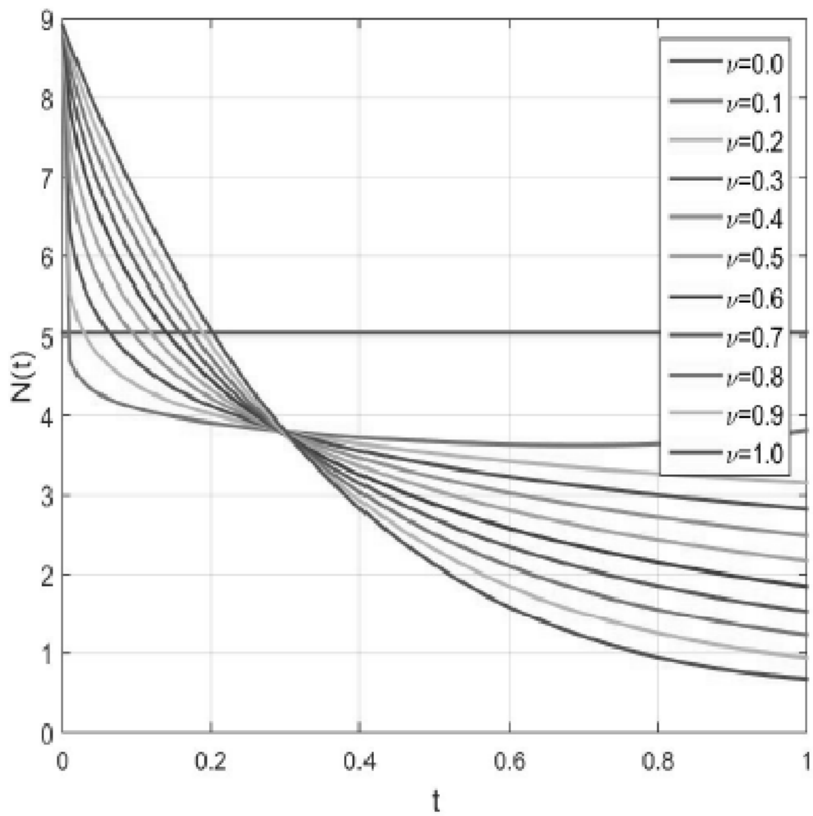

Figure 1: Solution of (19) for $N_0 = 5$, $\gamma = 7$, $\lambda_1 = .5$, $\lambda_2 = .8$, $\nu_1 = 1.7$, $\nu_2 = 2.8$, $q = .5$, $d = 3$, $a = 2$

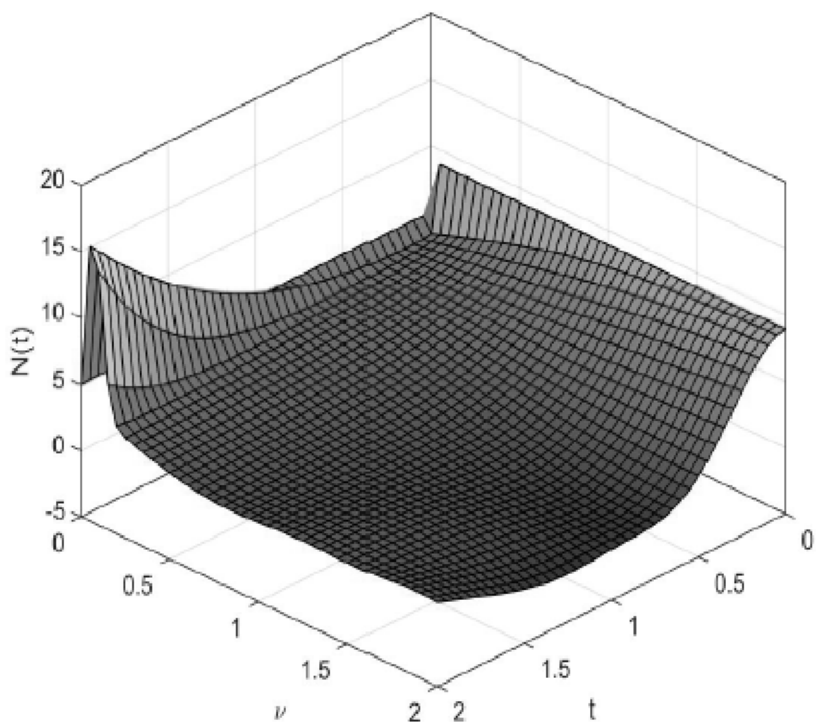

Figure 2: Solution of (19) for $N_0 = 5, \gamma = 7, \lambda_1 = .5, \lambda_2 = .8, v_1 = 1.7, v_2 = 2.8, q = .5, d = 3, a = 2$

$$N(p) + a^{\nu} p^{-\nu} N(p) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\prod_{j=1}^{m} \Gamma(\lambda_j n + \nu_j + 1)} \frac{(-d^{\nu})^n}{n!} \int_0^{\infty} e^{-pt} t^{\nu n} dt \tag{24}$$

$$= N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\prod_{j=1}^{m} \Gamma(\lambda_j n + \nu_j + 1)} \frac{(-d^{\nu})^n}{n!} \frac{\Gamma(\nu n + 1)}{p^{\nu n + 1}},$$

this leads to

$$N(p) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} \Gamma(\nu n + 1)}{\prod_{j=1}^{m} \Gamma(\lambda_j n + \nu_j + 1)} \frac{(-d^{\nu})^n}{n!} \left\{ p^{-(\nu n + 1)} \sum_{l=0}^{\infty} \left[-\left(\frac{p}{a}\right)^{-\nu} \right]^l \right\}. \tag{25}$$

Taking Laplace inverse of (25), and by using

$$L^{-1}\{p^{-\nu}; t\} = \frac{t^{\nu - 1}}{\Gamma(\nu)}, \qquad (\Re(\nu) > 0). \tag{26}$$

We can obtained from the equation (25), as follows:

$$L^{-1}\{N(p)\} = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} \Gamma(\nu n + 1)}{\prod_{j=1}^{m} \Gamma(\lambda_j n + \nu_j + 1)} \frac{(-d^{\nu})^n}{n!} \tag{27}$$

$$\times L^{-1}\left\{\sum_{l=0}^{\infty} (-1)^l a^{\nu l} p^{-[\nu(n+l)+1]}\right\}$$

i.e.

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\Gamma(\nu n+1)}{\prod_{j=1}^{m}\Gamma(\lambda_j n+\nu_j+1)} \frac{(-d^\nu t^\nu)^n}{n!} \left\{\sum_{l=0}^{\infty} (-1)^l \frac{(a^\nu t^\nu)^l}{\Gamma(\nu(n+l)+1)}\right\}, \tag{28}$$

now interpret the above equation (28) with the view of Mittag-Leffler function given in equation (18) we have the required result (20).

Theorem 2: *If $d > 0$, $\nu > 0$; $m \in \mathbb{N}$, λ_j, ν_j, γ, $q \in \mathbb{C}$ $(j = 1,...,m)$ such that $\sum_{j=1}^{m} \Re(\lambda_j) > \max\{0, \Re(q) - 1\}$; $q > 0$, $\Re(\nu_j) > -1$, $\Re(\gamma) > 0$ and $q \in (0, 1) \cup \mathbb{N}$, then the solution of the fractional kinetic equation:*

$$N(t) - N_0 J_{(\nu_j)m,q}^{(\lambda_j)m,\gamma}(d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{29}$$

is given by following formula

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\Gamma(\nu n+1)}{\prod_{j=1}^{m}\Gamma(\lambda_j n+\nu_j+1)} \frac{(-d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-d^\nu t^\nu)^l. \tag{30}$$

Theorem 3 *If $d > 0$, $\nu>0$; $m \in \mathbb{N}$, λ_j, ν_j, γ, $q \in \mathbb{C}$ $(j=1,...,m)$ such that $\sum_{j=1}^{m} \Re(\lambda_j) > \max\{0; \Re(q)-1\}$; $q>0$, $\Re(\nu_j) > -1$, $\Re(\gamma)>0$ and $q \in (0,1) \cup \mathbb{N}$, then the solution of the fractional kinetic equation:*

$$N(t) - N_0 J_{(\nu_j)m,q}^{(\lambda_j)m,\gamma}(t) = -d_0^\nu D_t^{-\nu} N(t) \tag{31}$$

is given by following formula

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\Gamma(\nu n+1)}{\prod_{j=1}^{m}\Gamma(\lambda_j n+\nu_j+1)} \frac{(-t^\nu)^n}{n!} E_{\nu,\nu n+1}(-d^\nu t^\nu)^l. \tag{32}$$

Proof. The proof of the Theorem 2 and 3 are similar as that of Theorem 1, therefore we omit the details.

3. SPECIAL CASES

If we choose $m = 1$, then all the Theorems 1, 2 and 3 are reduces to the following form involving the generalized Bessel-Maitland function $J_{\nu,\mu}^{\lambda,\gamma}(.)$ (see Pathak [15]):

Corollary 1: *If $a > 0$, $d > 0$, $v > 0$; $a \neq d$; λ, v, γ, $q \in \mathbb{C}$ such that $\Re(\lambda) > \max\{0; \Re(q)-1\}$; $q > 0$, $\Re(v) > -1$, $\Re(\gamma) > 0$ and $q \in (0,1) \cup \mathbb{N}$, then the solution of the fractional kinetic equation:*

$$N(t) - N_0 J_{(\nu),q}^{(\lambda),\gamma}(d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t), \tag{33}$$

is given by the following formula

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\Gamma(\nu n+1)}{\Gamma(\lambda n+\nu+1)} \frac{(-d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-a^\nu t^\nu). \tag{34}$$

Corollary 2: *If $d > 0$, $v > 0$; λ, v, γ, $q \in \mathbb{C}$ such that $\Re(\lambda) > \max\{0; \Re(q)-1\}$; $q>0$, $\Re(n) > -1$, $\Re(\gamma) > 0$ and $q \in (0, 1) \cup \mathbb{N}$, then the solution of the fractional kinetic equation:*

$$N(t) - N_0 J_{(\nu),q}^{(\lambda),\gamma}(d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{35}$$

is given by following formula

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\Gamma(\nu n+1)}{\Gamma(\lambda n+\nu+1)} \frac{(-d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-d^\nu t^\nu)^l. \tag{36}$$

Corollary 3: *If $d>0$, $v>0$; λ, v, γ, $q \in \mathbb{C}$ such that $\Re(\lambda) > \max\{0; \Re(q)-1\}$; $q>0$, $\Re(v)>-1$, $\Re(\gamma)>0$ and $q \in (0,1) \cup \mathbb{N}$, then the solution of the fractional kinetic equation:*

$$N(t) - N_0 J_{(\nu),q}^{(\lambda),\gamma}(t) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{37}$$

is given by following formula

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\Gamma(\nu n+1)}{\Gamma(\lambda n+\nu+1)} \frac{(-t^\nu)^n}{n!} E_{\nu,\nu n+1}(-d^\nu t^\nu)^l. \tag{38}$$

If we choose $m = 1$, then v is replaced by $v-1$ and z by $-z$ then generalized Bessel-Maitland function reduces to well known Mittag-Leffler function i.e. $J_{\nu-1,q}^{\lambda,\gamma}(-z) = E_{\lambda,\nu}^{\gamma,q}(z)$. Using this concept all the Theorem 1, 2 and 3 reduces to new one involving generalized Mittag-Leffler function.

4. CONCLUSION

In this paper, we give a new fractional generalization of the standard kinetic equation and derived solution for the same. From the close relationship of the generalized Bessel-Maitland function with many special functions, we can easily construct various known and new fractional kinetic equations. From the graphical presentation, we conclude that $N(t) > 0$ for different values of the parameters and in different intervals of t.

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Mehar Chand, Rakesh Kumar and Daljeet Kaur
Department of Applied Sciences
Guru Kshi University, Bathinda-151302 (India)
E-mail: mehar.jallandhra@gmail.com