ALTERNATIVE TERM STRUCTURE MODELS FOR REVIEWING EXPECTATIONS PUZZLES

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ABSTRACT

According to the expectations hypothesis, the forward rate is equal to the expected future short rate, an argument that is not supported by most empirical studies that demonstrate the existence of term premiums. An alternative arbitrage-free term structure model for reviewing the expectations hypothesis is presented and tractable expressions for time-varying term premiums are obtained. The model is constructed under the real-world probability measure and depends on two stochastic factors: the short rate and the market price of risk. The model suggests that for short maturities the short rate contribution determines the term premiums, while for longer maturities, the contribution of the market price of risk dominates.

JEL Classification: G13.

Keywords: Expectations hypothesis, time-varying term premiums, real-world probability measure, market price of risk.

1. INTRODUCTION

The Expectations Hypothesis (EH) plays an important role in the common understanding of continuous time term structure models as it relates equilibrium conditions, market prices of risk and associated premiums, see Ingersoll (1987) and Cochrane (2001). The appeal of this hypothesis is unsurprising since it can provide valuable views on future spot interest rates by using only the current term structure of interest rates. It provides an effortless forecast about market movements to practitioners. Furthermore, it has been widely used in many trading strategies in several markets such as currency and bond markets, see Chance and Rich (2001).

The main notion of the pure EH is that the forward rate is an unbiased predictor of the future spot rate. Originated by Fisher (1896), and followed by Hicks (1939), Lutz

(1940) and many others, the EH of the term structure of interest rates has been studied extensively over the years. Cox, Ingersoll and Ross (1981) carried out a thorough theoretical analysis of the pure expectations hypotheses demonstrating that it is not compatible with other EH. Prime early works by Fama (1984), Campbell (1986), Fama and Bliss (1987), Fama (1990), Campbell and Shiller (1991) with more recent works by Sarno, Thornton and Valente (2007), Della Corte, Sarno and Thornton (2008) provide compelling evidence on the empirical failure of the pure EH.

The first attempt to reconcile empirical properties of the term structure of interest rates and the EH, involved the inclusion of a constant or maturity dependent (but time invariant) term premium, see for instance Campbell (1986) and Fama and Bliss (1987). Further, Campbell and Shiller (1991) and Harris (1998), among others, demonstrated that the premium should be time-varying. However, the EH which accounts for an additive term premium, is also rejected. This version of the EH cannot explain the empirically observed feature of falling long rates when yield spreads are high. As supported by the ARCH literature, the interest rate volatilities evolve stochastically. By using the Cox, Ingersoll and Ross (1985) model for interest rates, Frachot and Lesne (1993) demonstrate that it is essential to additionally multiply by a term premium, which arises as a result of this stochastic volatility requirement. Nevertheless, this specification fails again, in the sense that, it does not work very well for longer maturities. Using a CIR model for the short rate, we encounter the issue of variances falling to zero for long maturities, thus the short rate is regressed with itself.

The ability of dynamic term structure models to account for the empirical features of the interest rate term structure, in terms of providing an explanatory basis to the empirical failure of the EH, has been studied by Musiela and Sondermann (1993), Frachot and Lesne (1993), Backus, Foresi, Mozumdar and Wu (2001) and Dai and Singleton (2002). More specifically, Backus et al. (2001) show that one-factor CIR models cannot match term premium elements and the average upward slope of the yield curve, while affine models with negative factors perform better, especially in the long run. Dai and Singleton (2002) re-examine the EH and show that one of the key factors to matching dynamic term structure models to empirically observed features is the market price of risk specification. They consider a state dependent market price of risk within a family of Gaussian affine term structure models and demonstrate that it fully matches the (sample-based linear projection) coefficients in yield regressions. We propose an alternative dynamic term structure model and examine whether it is able to accommodate the empirically observed features of interest rates, as a basis for explaining expectations puzzles.

The proposed model for the term structure of interest rates incorporates stochastic interest rates, stochastic interest rate volatilities and time-varying market prices of risk. By using the growth optimal portfolio (GOP) as numeraire, a term structure model is established under the real-world probability measure, see Platen and Heath (2006). The dynamics of the GOP are determined by the Markovian market price of risk and risk-free short rate. Consequently, the forward rates, yields to maturity and

the corresponding term premiums can be expressed in terms of these two factors. Under the proposed model, there are two contributions to the (additive) term premium: the contribution of the stochastic interest rate and the contribution of the stochastic market price of risk. The market price of risk contribution is minimal for short maturities (up to five to ten years, depending on the parameter specifications). It determines the shape of the forward rate (and yield) for longer maturities. The short rate contribution is the central determinant of the shape of the forward (yield) curve for short maturities. By using two well known short rate models, the Vasicek (1977) and the Cox et al. (1985) we gauge the effect of the short rate contribution to the forward curve and yield curve. Thus, the model has the potential to fit both the short and the long end of the forward (and yield) curve. Additionally, as the model can also have stochastic volatility specifications, under the Cox *et al.* (1985) model assumption, the (multiplicative) term premium still remains.

The key advantage of the proposed model is that it does not require the existence of an equivalent risk-neutral probability measure. Traditional models of the term structure of interest rates are usually specified under an equivalent risk neutral probability measure. Previous theoretical studies of the EH separately model the change of measure from these equivalent probability measures to the real-world (objective) probability measure, see Frachot and Lesne (1993) and Musiela and Sondermann (1993). The conclusions of these studies heavily depend on how this change of measure is specified. In our approach, we sidestep these complications as we model directly the term structure under the real-world probability measure.

Furthermore, real-world pricing recognises trends in the long run, in particular, the presence of an equity premium, which characterises the superior expected long term return of equities over the expected short rate. In the long run, the bias between the expected short rate and the forward rate is dominated by the market price of risk. However, in the short run, the bias depends largely on the specifications of the short rate model.

The paper is structured as follows. Section 2 introduces a dynamic term structure model under the real-world probability measure which depends on two factors: the market price of risk and the short rate. The forward rate and yield to maturity are completely determined by these two factors. The effect of these factors on the forward curve and yield curve is examined for a range of parameter specifications. By employing this model, we review the EH in Section 3, derive the related term premiums and study their properties. The suitability of the proposed model to explain expectations puzzles is also examined. Section 4 concludes.

2. ALTERNATIVE MODELS FOR THE TERM STRUCTURE OF INTEREST RATES

We assume a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, \mathbb{P})$, $T \in [0, \infty)$ with $\mathcal{A} = (\mathcal{A}_t)_{t \in [0,T]}$, satisfying the usual conditions. The continuous uncertainty is modeled as an

$\mathcal{A}$-adapted Wiener process $W = \{W_t,\, t \in [0, T]\}$ under the real-world probability measure $\mathbb{P}$.

Let $P(t, T)$ be the price at time $t \in [0, T]$ of a zero-coupon bond with maturity T. For all $t \in [0, T]$ we define the following quantities.

Definition 2.1: *The yield to maturity $Y(t, T)$ for the period $[t, T]$ is defined as*

$$Y(t,T) := -\frac{\ln P(t,T)}{T-t}. \tag{2.1}$$

The instantaneous forward rate f(t, T) with maturity T contracted at time t is defined as

$$f(t,T) := -\frac{\partial}{\partial T} \ln P(t, T). \tag{2.2}$$

The instantaneous short rate r_t at time t is then defined as

$$r_t := f(t, t). \tag{2.3}$$

We propose an alternative model for the term structure of interest rates under the realworld probability measure. The central element of this alternative model is the growth optimal portfolio (GOP). The portfolio which maximises the expected logarithm of terminal wealth for all times $t \in [0, T]$ represents the GOP, see Karatzas and Shreve (1998) and Platen (2002) building on the early results by Long (1990). The GOP is the portfolio that almost surely provides the best outcome in the long run. It has the long term growth rate that is almost surely greater than or equal to the long term growth rate of any other strictly positive portfolio. Yet, the GOP is a tradable portfolio, as it can be approximated by a well diversified global accumulation index, see Platen and Heath (2006).

Assume that S_t^0 is the value of the locally riskless savings account at time t, which continuously accrues the short rate r_t, thus

$$S_t^0 = \exp\left\{\int_0^t r_s\, ds\right\}, \tag{2.4}$$

for $t \in [0, \infty)$, where $r = \{r_t,\, t \in [0, T]\}$ denotes the adapted short rate process. When the total market price of Wiener process risk follows the predictable vector process $\Theta = \{\theta_t,\, t \in [0, T]\}$, then the unique GOP, S_t, satisfies the stochastic differential equation (SDE)

$$dS_t = S_t\, (r_t dt + \theta_t(\theta_t dt + dW_t)), \tag{2.5}$$

for all $t \in [0, T]$, with $S_0 = 1$. We recall that $W = \{W_t,\, t \in [0, T]\}$ is a standard Wiener process under the real-world probability measure $\mathbb{P}$. It is noted that the dynamics of the GOP are determined solely by the short rate r_t and the market price of Wiener process risk θ_t.

The benchmarked value of an asset is the value of the asset which is expressed in units of the GOP. When a benchmarked price process forms a martingale, then the price process is called fair. Platen and Heath (2006) show that any nonnegative benchmarked portfolio forms an $(\mathcal{A}, P)$-supermartingale.[1] Therefore, when benchmarked asset prices are fair, then their pricing is performed under the real-world probability measure with the GOP as numeraire. Thus, the price of the fair zero-coupon bond $P(t, T)$ under the real-world probability measure is evaluated by the real-world pricing formula as

$$P(t,T) = S_t \mathbb{E}\left(\frac{1}{S_T} \mid \mathcal{A}_t \right),$$ (2.6)

for $t \in [0, T]$. This implies that the benchmarked zero-coupon bond

$$\hat{P}(t,T) = \frac{P(t,T)}{S_t},$$ (2.7)

is an $(\mathcal{A}, P)$-martingale. By using the savings account (2.4), the zero-coupon bond price $P(t, T)$, as evaluated in (2.6), can be expressed in terms of the discounted GOP, $\bar{S}_t = \frac{S_t}{S_t^0}$, as

$$P(t,T) = \mathbb{E}\left(\frac{S_t}{S_T} \mid \mathcal{A}_t \right) = \mathbb{E}\left(\frac{\bar{S}_t}{\bar{S}_T} \exp\left\{ -\int_t^T r_s ds \right\} \mid \mathcal{A}_t \right),$$ (2.8)

for $t \in [0, T]$, $T \in [0, \infty)$. Using (2.5) and Ito's lemma, the discounted GOP, $\bar{S}_t = S_t / S_t^0$, is found to satisfy the SDE

$$d\bar{S}_t = \bar{S}_t \left(|\theta_t|^2 \, dt + \theta_t dW_t \right),$$ (2.9)

for all $t \in [0, T]$. In the general case, the short rate process and the discounted GOP process are correlated which leads to pricing relationships that typically have to be handled numerically. For the sake of tractability, we assume here independence between the short rate process and the discounted GOP process. Later we will see that the short rate impacts mainly the short term dynamics of a fair zero-coupon bond, whereas the market price of risk governs its long term dynamics and independence is not really required. Then (2.8) yields

$$P(t,T) = M_t(T, \bar{S}_t) \, \mathbb{E}\left(\exp\left\{ \int_t^T r_s ds \right\} \mid \mathcal{A}_t \right),$$ (2.10)

where $M_t(T, \bar{S}_t)$ is the market price of risk contribution to the bond price, namely

$$M_t\left(T, \bar{S}_t\right) = \mathbb{E}\left(\frac{\bar{S}_t}{\bar{S}_T} \mid \mathcal{A}_t \right).$$ (2.11)

The second factor, in (2.10), is the short rate contribution to the bond price. By substituting (2.10) into the definition (2.2), the forward rate is expressed as

$$f(t, T) = n_f(t, T) + e_f(t, T), \qquad (2.12)$$

where $n_f(t, T)$ is the market price of risk contribution to the forward rate

$$n_f(t,T) = -\frac{\partial}{\partial T} \ln\left[M_t\left(T, \bar{S}_t\right)\right], \qquad (2.13)$$

and $e_f(t, T)$ is the *short rate contribution* to the forward rate

$$e_f(t,T) = -\frac{\partial}{\partial T} \ln\left[\mathbb{E}\left(\exp\left\{-\int_t^T r_s ds\right\} \mid \mathcal{A}_t\right)\right]. \qquad (2.14)$$

By substituting (2.10) into the definition (2.1), the yield to maturity is expressed as

$$Y(t, T) = n_y(t, T) + e_y(t, T), \qquad (2.15)$$

where $e_y(t, T)$ is the *market price of risk contribution* to the yield to maturity

$$n_y(t,T) = -\frac{\ln\left[M_t\left(T, \bar{S}_t\right)\right]}{T - t}, \qquad (2.16)$$

and $e_y(t, T)$ is the *short rate contribution* to the yield to maturity

$$e_y(t,T) = -\frac{1}{T-t} \ln\left[\mathbb{E}\left(\exp-\left\{\int_t^T r_s ds\right\} \mid \mathcal{A}_t\right)\right]. \qquad (2.17)$$

The yield to maturity and the forward rate depend on two factors: the discounted GOP (or more directly the market price of risk θ_t) and the short rate r_t.

2.1. Market Price of Risk Contribution

To model the market price of risk contribution function, we adopt a particular model to specify the discounted GOP, the Minimal Market Model (MMM), as discussed in Platen (2001) and Platen (2002).

Assumption 2.1: *Assume that the drift α_t of the discounted* GOP, $\bar{S}_t$, *is of the form*

$$\alpha_t := \bar{S}_t \mid \theta_t \mid^2 = \alpha_0 \exp\{\eta t\}, \qquad (2.18)$$

for $t \in [0, T]$, where $\eta > 0$ is the constant net growth *rate and $\alpha_0 > 0$ is an initial parameter.*

Empirical observations demonstrate that in the long term the world economy has been growing exponentially, which suggests that the discounted GOP should also grow in a similar manner. Under this assumption we obtain the MMM, where the market price of risk contribution function is explicitly obtained as follows.

Proposition 2.1: *Under Assumption 2.1, the market price of risk contribution function to the bond* $M_t(T, \bar{S}_t)$*, defined in (2.11), is given by*

$$M_t(T, \bar{S}_t) = 1 - \exp\left\{ -\frac{2R(t,T)\bar{S}_t}{\alpha_t} \right\}, \tag{2.19}$$

where

$$R(t,T) = \frac{\eta}{\exp\{\eta(T-t)-\}}. $$

Furthermore, the market price of risk contribution function $n_f(t, T)$ *to the forward rate, see (2.13), is given by*

$$n_f(t,T) := -\frac{\partial}{\partial T} \ln\left[M_t(T, \bar{S}_t) \right] = \frac{2R(t,T)(\eta + R(t,T))\bar{S}_t}{\alpha_t \left[\exp\left\{ \dfrac{2R(t,T)\bar{S}_t}{\alpha_t} \right\} - 1 \right]}, \tag{2.20}$$

and the market price of risk contribution function $n_y(t, T)$ *to the yield, see (2.16), will be*

$$n_y(t,T) = -\frac{1}{T-t} \ln\left[1 - \exp\left\{ -\frac{2R(t,T)\bar{S}_t}{\alpha_t} \right\} \right]. \tag{2.21}$$

Proof: The proof of the above result is presented in Appendix A.

Note that assumption (2.18) implies that the market price of risk contribution functions (2.20) and (2.21) are fully determined by the total market price of risk process $\Theta = \{\theta_t = \sqrt{\alpha_t / \bar{S}_t}, t \in [0, T]\}$.

In Figure 1, the market price of risk contributions, $n_f(t, T)$ and $n_y(t, T)$, see (2.20) and (2.21) respectively, are displayed as functions of maturity $\check{T} \in [0, 30]$ and net growth rate $\eta \in [0.01, 0.25]$. The initial market price of risk is here set to the value $\theta_0 = \sqrt{\alpha_0 / \bar{S}_0} = 0.2$, as proposed in Le and Platen (2006).

The market price of risk contribution $n_f(t, T)$ to the forward rate is approximately zero for short maturities and approaches asymptotically the net growth rate, for very long maturities. The contribution $n_y(t, T)$ of the market price of risk to the yield to maturity has similar features. It also approaches asymptotically, for extremely long maturities but with a lower rate of convergence.

2.2. Short Rate Contribution

To model the short rate contribution functions in a particular case, we assume that the short rate process r_t follows a process leading to affine bond prices. Then there exist unique deterministic functions of time $A(t, T)$ and $B(t, T)$ such that

$$\mathbb{E}\left(\exp\left\{-\int_t^T r_s ds\right\} \mid \mathcal{A}_t\right) = \exp\{A(t,T) - B(t,T)r_t\},$$ (2.22)

Figure 1
Market Price of Risk Contributions, $n_f(t,T)$ (in the First Panel) and $n_y(t,T)$ (in the Second Panel)
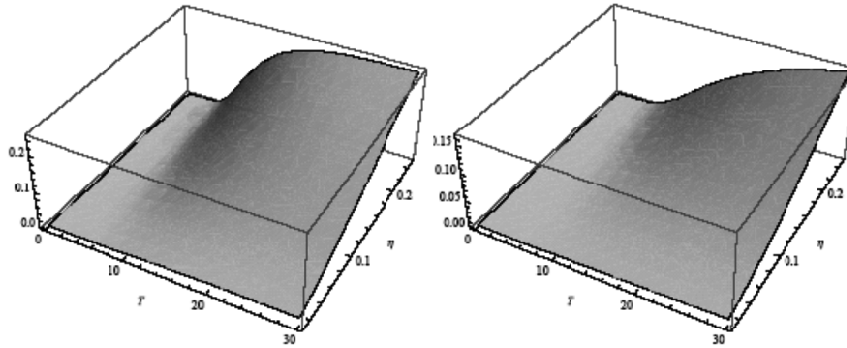
where Appendix B provides the technical details. Note that the above expectation is taken under the real-world probability measure. By using the expression (2.22), the short rate contributions, (2.14) and (2.17), will be obtained as

$$e_f(t,T) = -\frac{\partial A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T}r_t,$$ (2.23)

$$e_f(t,T) = -\frac{A(t,T)}{T-t} + \frac{B(t,T)}{T-t}r_t,$$ (2.24)

respectively. For further illustration, we consider below two examples of affine term structure models for the short rate process.

1. The Vasicek (1977) model: The short rate dynamics are specified by the SDE

$$dr_t = \kappa(\bar{r} - r_t)\, dt + \sigma dW_t,$$ (2.25)

where κ, $\bar{r}$, and σ are positive constants. Then $B(t, T)$ and $A(t, T)$ are given by

$$B(t,T) = \frac{1 - e^{-\kappa(T-t)}}{\kappa},$$ (2.26)

and

$$A(t,T) = \left(\bar{r} - \frac{\sigma^2}{2\kappa^2}\right)(B(t,T) - (T-t)) - \frac{\sigma^2}{4\kappa}B(t,T)^2,$$ (2.27)

respectively.

2. The Cox *et al.* (1985) model: the short rate dynamics are specified by the SDE

$$dr_t = \kappa(\bar{r} - r_t)\, dt + \sigma\, \sqrt{r_t}\, dW_t, \tag{2.28}$$

where κ, $\bar{r}$, σ are positive constants with $2\kappa\bar{r} > \sigma^2$, which ensures strictly positive solutions to (2.28). Then $B(t, T)$ and $A(t, T)$ are given by
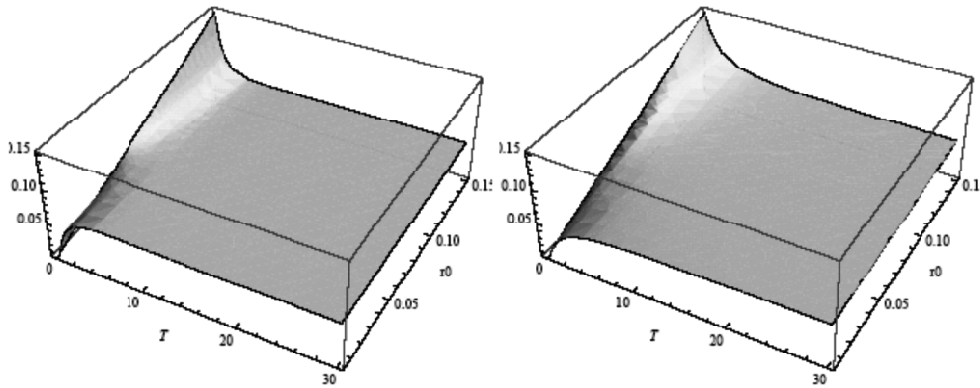
$$B(t,T) = \frac{L_1(T-t)}{L_2(T-t)}, \tag{2.29}$$

and

$$A(t,T) = \frac{2\kappa\bar{r}}{\sigma^2} \ln \frac{L_3(T-t)}{L_2(T-t)}, \tag{2.30}$$

respectively, where

$$L_1(x) = 2(e^{\varpi_1 x} - 1),$$
$$L_2(x) = \varpi1(e^{\varpi_1 x} + 1) + \kappa\,(e^{\varpi_1 x} - 1), \tag{2.31}$$
$$L_3(x) = 2\varpi_1 e^{(\varpi_1 + \kappa)x/2},$$

with $\varpi_1 = \sqrt{\kappa^2 + 2\sigma^2}$.

Figure 2
Short Rate Contributions, $e_f(t,T)$ (in the First Panel) and $e_y(t,T)$ (in the Second Panel) under the CIR Model

For illustrative purposes, Figure 2 presents the short rate contribution functions $e_f(t, T)$ and $e_y(t, T)$, (see (2.23) and (2.24), respectively) under the CIR model, as functions of maturity $T \in [0, 30]$ and $r_0 \in [0.005, 0.15]$. The parameter specifications are $\sigma = 0.10$, $\kappa = 1$ and $\bar{r} = 0.06$. The contributions $e_f(t, T)$ and $e_y(t, T)$ of the short rate to the forward rate and to the yield to maturity, respectively, are very pronounced for short maturities up to 10 years (subject to parameter values) while they remain minor for longer maturities. The short rate contributions under the Vasicek model are similar and not given here to save space.

Figure 3 displays the forward rate (2.12) under the MMM for the market price of risk and the CIR short rate model. In the first panel, the forward rate is displayed as a function of maturity $T \in [0, 30]$ and net growth rate $\eta \in [0.01, 0.25]$. In the second panel, the forward rate is displayed as a function of maturity $T \in [0, 30]$ and initial short rate $r_0 \in [0.005, 0.15]$. The parameter specifications are $\sigma = 0.10$, $\eta = 0.1$, $\theta_0 = 0.2$, $\bar{r} = r_0 = 0.05$ and $\kappa = 1$.

Figure 3
Short Forward Rate Surface $f(t,T) = n_f(t,T) + e_f(t, T)$ under the CIR Model, in the First Panel as a Functions of η, in the Second Panel as a Function of r_0
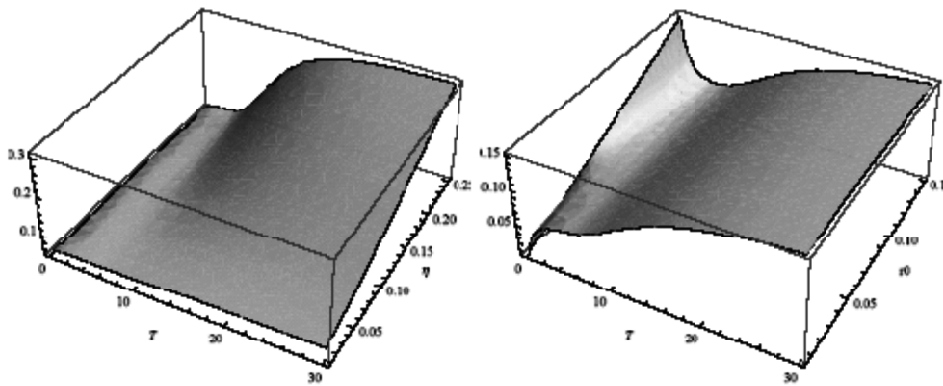

Figure 3 illustrates that the forward rate does not only depend on the evolution of the short rate. An additional factor is required, especially for longer maturities, which depends on the market price of risk. For longer maturities, the impact of the short rate is rather limited. Furthermore, the model has the ability to generate a variety of forward curves. These may be, for instance, increasing, decreasing or humped. Note that similar patterns are also obtained for the yield to maturity. Motivated by the above observations arising from these particular models, we propose an alternative formulation for the EH.

3. REVIEWING THE EXPECTATIONS HYPOTHESIS

For completeness, we firstly present the classical formulations of the EH presented by Cox *et al.* (1981). By using standard arbitrage pricing theory these authors have classified various expectations hypotheses for interest rates, the so-called "pure" EH, and studied their properties. We summarise their main results as follows:

1. The Unbiased Expectations Hypothesis (U-EH): The forward rate is assumed to be an unbiased estimate of the future short rate, namely

$$f(t, T) = \mathbb{E}\ (r_T \mid \mathcal{A}_t) \tag{3.1}$$

Then from (2.2), the value of a zero-coupon bond can be expressed as

$$p(t,T) := \exp\left\{-\int_t^T f(t,s)\right\} = \exp\left\{-\int_t^T \mathbb{E}(r_s \mid \mathcal{A}_t)ds\right\}. \tag{3.2}$$

This hypothesis holds only under the so-called T – forward measure, see for instance Björk (2004) (Lemma 24.10 p. 357), and not under the real-world probability\ measure.

2. The Yield To Maturity Expectations Hypothesis (YTM-EH): From definitions (2.1) and (2.2), and integration of (3.1) the yield to maturity is

$$Y(t,T) = \frac{1}{T-t}\mathbb{E}\left(\int_t^T r_s ds \mid \mathcal{A}_t\right) \tag{3.3}$$

By (2.1) this is equivalent to

$$P(t,T) = \exp\left\{-\mathbb{E}\left(\int_t^T r_s ds \mid \mathcal{A}_t\right)\right\}. \tag{3.4}$$

From (3.2) and (3.4), it is easy to conclude that the (U-EH) and (YTM-EH) are equivalent.

3. The Local Expectations Hypothesis (L-EH): The expected instantaneous return from holding a zero coupon bond is assumed to equal the short rate. Under standard arbitrage pricing theory this hypothesis is always true under an assumed risk neutral probability measure $\mathbb{Q}$, that is

$$\frac{d}{dt}\mathbb{E}^{\mathbb{Q}}\left(\frac{dP(t,T)}{P(t,T)} \mid \mathcal{A}_t\right) = r_t. \tag{3.5}$$

Empirical work has rejected the classification of the expectations hypotheses presented in Cox *et al.* (1981) as it ignores the important impact of observed term premiums, see Fama (1984) and Campbell (1986), Fama and Bliss (1987), Fama (1990), and Campbell and Shiller (1991) to name just a few relevant papers in this direction. Reviewing the above "pure" EH, the U-EH asserts a zero term premium while the LEH asserts a maturity dependent term premium. Empirical literature has rejected both these two hypotheses and shows evidence of time-dependent term premiums, see Backus *et al.* (2001), Sarno *et al.* (2007) and literature referred to therein. It will be our aim to formulate an alternative EH that will allow us to accommodate this fact in a very general market setting.

Dynamic term structure models can capture empirical features of the interest rate term structure. More specifically, a time-dependent (additive) term premium seems to be needed and arises from the assumption for the stochastic evolution of interest rates, while a time-dependent (multiplicative) term premium seems to be relevant for the requirement of modelling stochastic volatility of interest rates, as Frachot and Lesne (1993) have indicated. It is important to note that these term premiums fail to capture features of the interest rate term structure for longer maturities. This is mainly

a consequence of relying on the classical arbitrage pricing theory. The current paper allows us to go beyond the classical framework by employing an alternative approach as discussed in Section 2. Longstaff and Schwartz (1992) and Backus *et al.* (2001) provide empirical evidence that multi-factor term structure models perform better. Dai and Singleton (2002) demonstrate the importance of integrating the market price of risk into the model when fitting observed interest rate term structures. These authors consider a time-varying term premium that depends on both the short rate and the market price of risk. They show that there is a large subclass of dynamic term structure models, such as affine and quadratic-Gaussian models, which are consistent with the key empirical findings presented in Fama and Bliss (1987) and Campbell and Shiller (1991).

Motivated by the above mentioned empirical findings, we study an alternative term structure model to review the expectations hypotheses. By using real-world pricing for the specific tractable model class described in Section 2, we employ expectations under the real-world probability measure, and provide expressions for time-varying term premiums that depend on the market price of risk and the short rate. The proposed real-world pricing model does not require the existence of an equivalent risk-neutral probability measure, thus, our results do not rely on the specifications of the probability measure change, as it has been treated by most of the above mentioned studies. Note also that traditional affine term structure models, as well as models with stochastic interest rate volatilities, can also be accommodated in our suggested approach. What this paper essentially suggests is to use real-world pricing for a reasonably realistic market model and the typically time-varying premiums will automatically emerge in a manner consistent with what is empirically observed. We call this approach the Alternative Expectation Hypothesis (AEH).

3.1. Alternative Expectation Hypothesis

By using the type of term structure model described in Section 2, and applying real-world pricing, a particular relationship between the forward rate and the expected short rate is obtained. This relationship can be interpreted as being representative of what the EH literature aims to capture.

Proposition 3.1: *Under the model specifications of Section 2, the forward rate can be expressed as*

$$f(t, T) = c_f(t, T)\mathbb{E}(r_T \mid \mathcal{A}_t) + \gamma_f(t, T) + n_f(t, T), \tag{3.6}$$

where

$$c_f(t,T) = \frac{\partial B(t,T)}{\partial T} e^{\kappa(T-t)}, \tag{3.7}$$

$$\gamma_f(t,T) = -\frac{\partial A(t,T)}{\partial T} - \overline{r}\,\frac{\partial B(t,T)}{\partial T}(e^{\kappa(T-t)} - 1), \tag{3.8}$$

and $n_f(t, T)$ is specified in (2.20).

Proof: See Appendix C.

This relationship demonstrates that forward rates are biased predictors of future short rates. There is a multiplicative risk premium and an additive risk premium present.

Corollary 3.2: Under the Vasicek (1977) model for the short rate one obtains

$$c_f(t, T) = 1. \tag{3.9}$$

Proof: Substitute $B(t, T)$ as evaluated by (2.26) in (3.7).

Corollary 3.3: *Under the Cox et al. (1985) (CIR) model for the short rate one obtains*

$$c_f(t,T) = \frac{4\left(\kappa^2 + 2\sigma^2\right)e^{\left(\kappa+\sqrt{\kappa^2+2\sigma^2}\right)(T-t)}}{\left[\kappa\left(e\sqrt{\kappa^2+2\sigma^{2(T-t)}} - 1\right)+\left(e\sqrt{\kappa^2+2\sigma^{2(T-t)}} + 1\right)\sqrt{\kappa^{2+}2\sigma^2}\right]^2}. \tag{3.10}$$

Proof: Substitute $B(t, T)$, as evaluated by (2.29), in (3.7).

It is easy to confirm that, under the CIR model, $c_y(t, T)$ takes non-negative values and for very short maturities, $c_y(t, T)$ converges to 1. One has, $0 < c_y(t, T) < 1$, which is consistent with empirical evidence. Next we derive a relationship between the yield to maturity and the expected short rate.

Proposition 3.4: *Under the model specifications of Section 2, the yield to maturity can be expressed as*

$$Y(t,T) = \frac{c_y(t,T)}{T-t}\mathbb{E}\left(\int_t^T r_s ds \mid \mathcal{A}_t\right) + \gamma_y(t,T) + \delta_y(t,T), \tag{3.11}$$

where

$$c_y(t,T) = \frac{\kappa B(t,T)}{1-e^{-\kappa(T-t)}} \tag{3.12}$$

$$\gamma_y(t,T) = -\frac{A(t,T)}{T-t} - \frac{1}{T-t}\frac{\kappa\overline{r}\left[(T-t)-\frac{1-e^{-\kappa(T-t)}}{\kappa}\right]B(t,T)}{1-e^{-\kappa(T-t)}} \tag{3.13}$$

$$\delta_y(t,T) = -\frac{1}{T-t}\ln\left[1-\exp\left\{-\frac{2R(t,T)\overline{S}_t}{\alpha_t}\right\}\right] \tag{3.14}$$

Proof: See Appendix D.

Corollary 3.5: *Under the Vasicek (1977) model for the short rate it follows*

$$c_y(t, T) = 1. \tag{3.15}$$

Proof: Substitute $B(t, T)$ as evaluated by (2.26) in (3.12).

A term structure model, such as the Vasicek model, which ignores the stochastic nature of the volatility of the interest rates has no multiplicative term premium, see also Frachot and Lesne (1993).

Corollary 3.6: *Under the Cox et al. (1985) model for the short rate one has*

$$c_y(t,T) = \frac{\kappa L_1(T-t)}{L_2(T-t)(1-e^{-\kappa(T-t)})} , \qquad (3.16)$$

where $L_1(T-t)$ and $L_2(T-t)$ are defined in (2.31).

Proof: Substitute $B(t, T)$ as evaluated by (2.29) in (3.12).

Figure 4 plots $c_y(t, T)$ under the CIR interest rate model as a function of $T - t$. The parameter specifications are $\sigma = 0.10$ and $\kappa = 0.1$. Note that $c_y(t, T) < 1$ and for long

maturities it can also take negative values, which is consistent with empirical literature, see for instance Campbell and Shiller (1991).

Figure 4
$c_f(t,T)$ **as a Function of** T **-** t
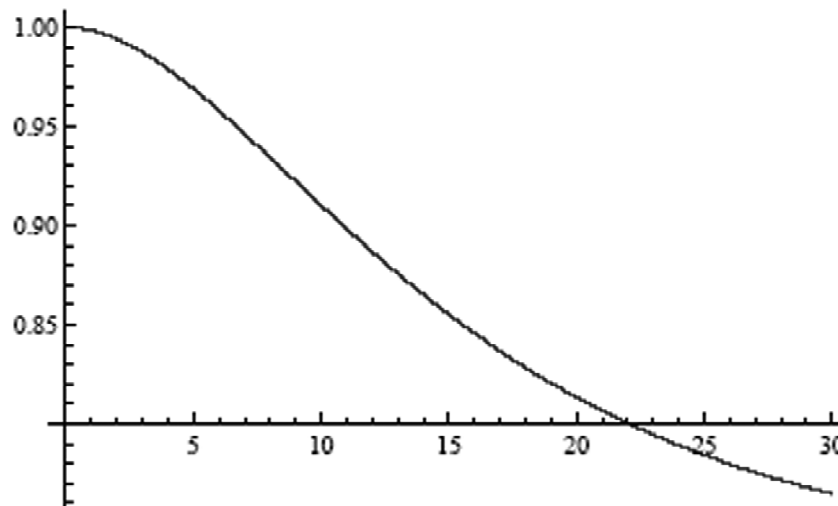
Campbell and Shiller (1991) provide empirical evidence that long-term interest rates under react to short-term interest rates, which implies that $c_y(t, T) < 1$. Under our model specifications, when the short rate volatility is stochastic (as in the CIR model) $c_y(t, T)$ is less than 1, and negative for longer maturities, which is again supported by the empirical literature, see the yields regressions of Campbell and Shiller (1991).

3.2. Term Premiums

From a different point of view, particular attention has also been given to the term premiums alone. We introduce two related notions of term premiums that have been extensively studied in the literature.

Definition 3.1: *The forward term premium* $\Psi\,(t, T)$ *is*

$$\Psi(t, T) = f(t, T) - \mathbb{E}(r_T \,|\, \mathcal{A}_t), \tag{3.17}$$

and the yield term premium $\Phi\,(t, T)$ *is*

$$\Phi(t,T) = Y(t,T) - \frac{1}{T-t}\mathbb{E}\left(\int_t^T r_s ds \,|\, \mathcal{A}_t\right). \tag{3.18}$$

By employing the term structure model presented in Section 2 we derive the following results.

Corollary 3.7: *The forward term premium* $\Psi\,(t, T)$ *is expressed as*

$$\Psi\,(t, T) = (c_f(t, T) - 1)\mathbb{E}(r_T \,|\, \mathcal{A}_t) + \gamma_f(t, T) + n_f(t, T), \tag{3.19}$$

where $c_f(t, T)$, $\gamma_f(t, T)$ and $n_f(t, T)$ are given in Proposition 3.1. The yield term premium Φ (t, T) is expressed as

$$\Phi(t,T) = (c_y(t,T) - 1)\frac{1}{T-t}\mathbb{E}\left(\int_t^T r_s ds \,|\, \mathcal{A}_t\right) + \gamma_y(t,T) + \delta_y(t,T), \tag{3.20}$$

where $c_y(t, T)$, $\gamma_y(t, T)$ *and* $\delta_y(t, T)$ *are given in Proposition 3.4.*

Proof: The definitions (3.17) and (3.18) together with (3.6) and (3.11), respectively, provide the result.

Recall that under the Vasicek (1977) short rate model, one has $c_f(t, T) = c_y(t, T) = 1$. Under Cox *et al*. (1985) it follows that $c_f(t, T) < 1$ and $c_y(t, T) < 1$, which coincides with the empirical results of the regressions in Campbell and Shiller (1991). Thus, the stochastic volatility of interest rates contributes to the term premiums and can explain some empirical findings. Additionally, this specification breaks down into two separate contributions to the term premiums; the short rate contribution and the market price of risk contribution.

In particular, note that under the Vasicek (1977) short rate model, by using (2.27) and (2.26), the forward term premium $\Psi(t, T)$, (3.19), is reduced to

$$\Psi(t,T) = -\frac{\sigma^2}{2\kappa^2}(e^{-\kappa(T-t)} - 1)^2 + n_f(t,T), \tag{3.21}$$

where $n_f\,(t, T)$ is given by (2.20). Figure 5 displays the short rate contribution, the market price of risk contribution $y(t, T)$ and the combined forward term premium as a function of maturity T. The first panel displays the short rate contribution for the

Vasicek model and the second panel the CIR model. The parameter specifications are $\sigma = 0.10$, $\kappa = 0.5$, $\bar{r} = 0.05$, $\eta = 0.1$ and $\theta = 0.2$.

Figure 5
Forward Term Premium $\Psi(t,T)$ a) Vasicek Model. b) CIR Model
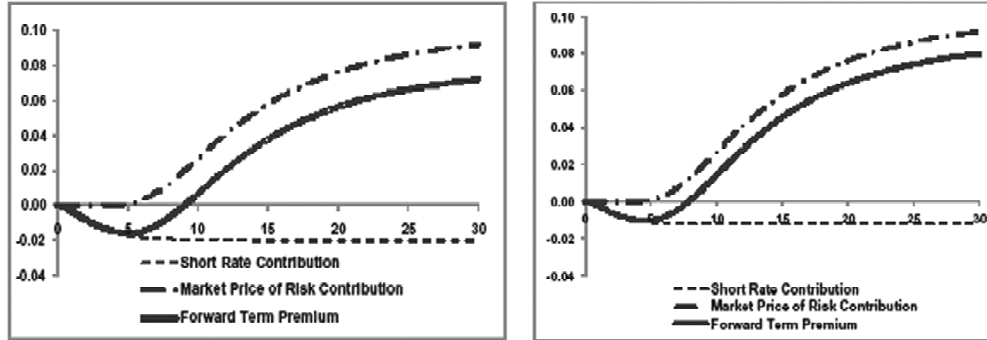
The short rate contribution to the forward term premium is always non positive and for the Vasicek model does not depend on the long term mean short rate $\bar{r}$, see (3.21). Figure 5 offers an interesting profile for the forward term premium (3.19). The market price of risk contribution to the term premium is negligible at the short end, so that only the contribution of the short rate is observed. At the long end of the forward term premium curve, there is a positive contribution of the market price of risk which asymptotically approaches the net growth rate of the equity market, $\eta = 0.10$ in our example. In addition, the short rate contribution converges to an asymptotic level as well. In general for long maturities, the forward term premium is dominated by the market price of risk contribution.

Finally, by making the simplistic assumption that in the long run

$$\lim_{T \to \infty} \mathbb{E}(r_T \mid \mathcal{A}_t) \to \bar{r}$$

from (3.19) and for the Vasicek model, we obtain

$$\lim_{T \to \infty} \Psi(t,T) \to -\frac{\sigma^2}{2k^2} + \eta, \tag{3.22}$$

while for the CIR model, we have that

$$\lim_{T \to \infty} \Psi(t,T) \to -\bar{r}\,\frac{\sqrt{k^2 + 2\sigma^2} - \kappa}{\sqrt{k^2 + 2\sigma^2} + k} + \eta. \tag{3.23}$$

One notes that the proposed alternative approach recognises real-world trends in the long run. More specifically, the expected long term return over the expected short

rate takes into account the average equity premium, which is expected to be the net growth rate.

3.3. An Empirical Illustration

The estimated coefficients for forward regressions, as presented in the empirical study by Backus *et al.* (2001), are displayed in Table 1.

Table 1

Maturity n months	Forward Regressions $\hat{a}_n^f$
3	0.7308
	(0.0916)
6	0.7971
	(0.0570)
12	0.8913
	(0.0393)
36	0.9576
	(0.0172)
60	0.9635
	(0.0124)
120	0.9634
	(0.0102)

Table 1: The estimated coefficients $\hat{a}_n^f$ of the forward regressions $f_{t+1}^{n-1} - r_t = \hat{a}_n^f \left(f_t^n - r_t \right) +$ constant + residual, where fn is the continuously compounded 1–month forward rate and $r_t = f_t^0$ is the 1–month short rate. The smoothed Fama-Bliss methods with monthly data from January 1970 to November 1995 have been used. The numbers in parentheses are the Newey-West standard errors.

We calculate the theoretical regression coefficients implied by the proposed model in Section 2 for the forward regression and evaluate these for a range of parameter values. We demonstrate that the theoretical regression coefficients can take values similar to the ones empirically observed, subject to the parameter values, as presented in Table 1.

We consider the "forward rate" regression as proposed by Backus et al. (2001), Dai and Singleton (2002) and Christiansen (2003) namely

$$f(t + 1, n - 1) - r(t, 1) = a_n^f [f(t, n) - r(t, 1)] + constant + residual. \qquad (3.24)$$

Coefficient values that are different from one indicate that the term premia are time dependent. This regression does not involve truncation, thus, it allows us to test the hypothesis of constant term premium for long maturities. The theoretical coefficients implied by the proposed model for the forward rate regression are (see Appendix E).

$$a_n^f = \frac{B(t+1,n)e^{-\kappa} - B(t+1,n-1)e^{-\kappa} - 1}{B(t,n+1) - B(t,n) - 1}, \tag{3.25}$$

and additionally

$$a_n^f = \frac{\ln M_{t+1}(t+n-1) - \ln M_{t+1}(t+n)}{\ln M_t(t+n) - \ln M_t(t+n-1)}, \tag{3.26}$$

where $M_t(T) = M_t(T, \bar{S}_t)$ for notational convenience, see (2.19). Figure 6 displays the theoretical regression coefficient (3.25) for $k \in [0.001, 2]$ and the theoretical regression coefficient (3.26) for $\eta \in [0.01, 0.25]$ as a function of $n \in [1, 20]$. Empirical studies typically find that this coefficient should be less than 1, which is satisfied by the values of the theoretical regression coefficient for a wide range of model parameter values. Indeed, the theoretical regression coefficient of the forward regression approaches 1 as maturity increases.

Figure 6
Theoretical Regression Coefficients α_n^f for Forward Regressions, see (3.25) and (3.26) Respectively
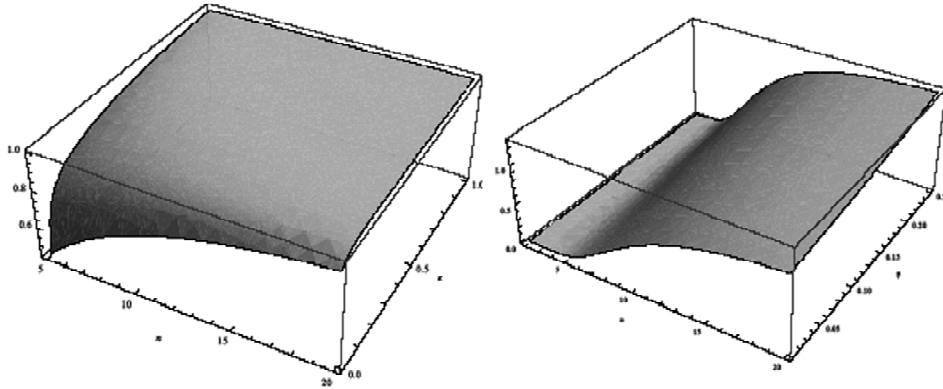
4. CONCLUSION

A simple term structure model is presented with flexibility to match empirical features of the term structure of interest rates and thus the potential to explain expectations puzzles. The model factors are the short rate and the market price of risk and the model accommodates stochastic volatility and a market price of risk. The term premiums implied from this formulation are time-varying and depend on these two factors. A key feature of the model is that the short rate contribution determines the term premia for short maturities. For longer maturities, the main determinant of the term premia is the market price of risk.

A subject of further research is the estimation of the model parameters, and the examination of the extent to which the model explains empirical findings.

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NOTE

1. An $\underline{\mathcal{A}}$-adapted process $X = \{X_t, t \in [0,\infty)\}$ is an $(\underline{\mathcal{A}}, P)$-supermartingale when $X_s \geq \mathbb{E}(X_t \mid \mathcal{A}_s)$ for $0 \leq s \leq t$. Supermartingales are important for financial market modelling as it has been shown empirically that the savings account expressed in units of GOP resembles a supermartingale, see Platen (2004). Classical risk-neutralmodelling implies that the savings account expressed in units of the market index is a martingale.

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A. APPENDIX: MINIMAL MARKET MODEL

The discounted GOP $\bar{S}_t$ is a time transformed squared Bessel process of dimension four with deterministic transformed time, see Revuz and Yor (1999). Thus, the total market price of risk, which is inversely proportional to the square root of the discounted GOP, see (2.18), is given by

$$|\theta_t| = \sqrt{\frac{\alpha_t}{\bar{S}_t}}, \tag{A.1}$$

and satisfies the SDE

$$\frac{d|\theta_t|^2}{|\theta_t|^2} = \eta dt - |\theta_t| \, dW_t. \tag{A.2}$$

Furthermore, by applying the explicit transition density of $\bar{S}_t$, the market price of risk contribution to the bond price, see (2.11), is obtained by the formula

$$M_t(T, \bar{S}_t) = \mathbb{E}\left(\frac{\bar{S}_t}{\bar{S}_T} \,\Big|\, \mathcal{A}_t \right) = 1 - \exp\left\{ -\frac{2R(t,T)\bar{S}_t}{\alpha_t} \right\}, \tag{A.3}$$

with

$$R(t,T) = \frac{\eta}{\exp\{\eta(T-t)\} - 1}.$$

Platen (2002) provides all the technical details. The market price of risk contribution function to the forward rate (2.13) follows then as in (2.20). In addition, by substituting (2.19) into (2.16), (2.21) is derived.

B. APPENDIX: AFFINE TERM STRUCTURE

Consider the one-dimensional short rate process

$$dr_t = (\alpha_1(t) + \alpha_2(t)r_t)dt + \sqrt{\beta_1(t) + \beta_2(t)r_t d} \, dW_t. \tag{B.1}$$

Then by the Feynman-Kac theorem it follows that the functional

$$u(t,x) := E\left(\exp-\left\{ \int_t^T r_s ds \right\} r_t = x \right), \tag{B.2}$$

satisfies the partial differential equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) - u(t,x)x = 0, \\ u(T,x) = 1, \end{cases} \tag{B.3}$$

for $(t, x) \in [0, T] \times \mathbb{R}$, with

$$\mathcal{L}u(t,x) = (\alpha_1(t) + \alpha_2(t)x)\frac{\partial u(t,x)}{\partial x} + \frac{\beta_1(t) + \beta_2(t)x}{2}\frac{\partial^2 u(t,x)}{\partial x^2}. \tag{B.4}$$

The functions $A(t,T)$ and $B(t,T)$ solve the system of ordinary differential equations

$$\frac{\partial B(t,T)}{\partial t} + \alpha_2(t)B(t,T) - \frac{1}{2}\beta_2(t)B^2(t,T) = -1, \tag{B.5}$$

$$B(T,T) = 0,$$

$$\frac{\partial A(t,T)}{\partial t} - \alpha_1(t)B(t,T) + \frac{1}{2}\beta_1(t)B^2(t,T) = 0, \tag{B.6}$$

$$A(T,T) = 0.$$

It is straightforward to show that the functional (2.22) satisfies (B.3), and thus generates an affine term structure. We emphasize that the expectation in (2.22) is taken under the real-world probability measure. In the traditional literature the expectation is taken under a risk neutral probability measure.

C. APPENDIX: PROPOSITION 3.1.

By substituting the specification (2.23) for the short rate contribution into (2.12) we obtain

$$f(t,T) = \eta_f(t,T) - \frac{\partial T(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T}r_t. \tag{C.1}$$

Under both the Vasicek (1977) short rate model (2.25), or the Cox *et al.* (1985) short rate model (2.28), it follows that

$$\mathbb{E}(r_T \mid \mathcal{A}_t) = r_t e^{-\kappa(T-t)} + \bar{r}\left(1 - e^{-\kappa(T-t)}\right). \tag{C.2}$$

Thus, by rearranging (C.2), the short rate satisfies the relation

$$r_t = \mathbb{E}(r_T \mid \mathcal{A}_t)e^{\kappa(T-t)} - \bar{r}\left(e^{\kappa(T-t)} - 1\right) \tag{C.3}$$

By substituting (C.3) in (C.1) and by performing some basic algebraic manipulations, the forward rate can be expressed as

$$f(t,T) = \frac{\partial B(t,T)}{\partial T}e^{\kappa(T-t)}\mathbb{E}\left(r_T \mid \mathcal{A}_t\right) - \bar{r}\,\frac{\partial B(t,T)}{\partial T}\left(e^{\kappa(T-t)} - 1\right) - \frac{\partial A(t,T)}{\partial T} + n_f(t,T),$$

from which (3.6) is derived.

D. APPENDIX: PROPOSITION 3.4

By (2.16) and (2.17), the yield to maturity (2.15) is expressed as

$$Y(t,T) = -\frac{1}{T-t}\ln M_t\left(T,\bar{S}_t\right) - \frac{1}{T-t}\ln \mathbb{E}\left(\exp\left\{-\int_t^T r_s ds\right\} + \mathcal{A}_t\right) \tag{D.1}$$

For the selected short rate model yielding (2.22) and the market price of risk modelled as in Section 2.1, see (2.21), one obtains

$$(T-t)Y(t,T) = -\ln\left[1 - \exp\left\{-\frac{2R(t,T)\bar{S}_t^{d*}}{\alpha_t}\right\}\right] - \left[\mathcal{A}(t,T) - B(t,T)r_t\right]. \tag{D.2}$$

Next we evaluate

$$\mathbb{E}\left(\int_t^T r_s ds \mid \mathcal{A}_t\right).$$

For a mean-reverting short rate, as modelled by the Vasicek (1977) model, see (2.25), or the Cox et al. (1985) model, see (2.28), we have that

$$\mathbb{E}\left(r_T \mid \mathcal{A}_t\right) = r_t e^{-\kappa(T-t)} + \bar{r}\left(1 - e^{-k(T-t)}\right), \tag{D.3}$$

and thus

$$\begin{aligned}
\mathbb{E}\left(\int_t^T r_s ds \mid \mathcal{A}_t\right) &= \int_t^T \mathbb{E}(r_s \mid \mathcal{A}_t) ds \\
&= \int_t^T (r_t e^{-\kappa(s-t)} + \bar{r}(1 - e^{-k(s-t)})) ds \\
&= r_t \frac{1 - e^{-\kappa(T-t)}}{\kappa} + \bar{r}\left[(T-t) - \frac{1 - e^{-k(T-t)}}{\kappa}\right].
\end{aligned} \tag{D.4}$$

By eliminating rt from (D.2) and (D.4), the relationship (3.11) between the yield to maturity and the expected value of the integral of the short rate is obtained.

E. APPENDIX: REGRESSION COEFFICIENTS

We denote as f_t^n t-time 1-month forward rate commencing in n periods (or commencing at date $t + n$) and $r_t = f_t^0$ is the 1"month short rate. Then

$$f_t^n \equiv \ln\left(\frac{P(t, t+n)}{P(t, t+n+1)}\right).$$

By using the expression (2.10) for the bond price, and setting $M_t(T) = M_t(T, \bar{S}_t)$ and $\rho(t, T) = \mathbb{E}\left(e^{-\int_t^T r_s ds} \mid \mathcal{A}_t\right)$ to ease the notation, then

$$f_t^n = \ln\left(\frac{M_t(t+n)}{M_t(t+n+1)}\right) = \ln\left(\frac{\rho(t, t+n)}{\pi(t, t+n+1)}\right).$$

Next substitute the specification (2.22) to obtain

$$f_t^n = \ln\left(\frac{M_t(t+n)}{M_t(t+n+1)}\right) + A(t, t+n) - A(t, t+n+1) = [B(t, t+n+1-) - B(t, t+n)]r_t.$$

The forward rate regression is given by

$$f_{t+1}^{n-1} - r_t = a_n^f\left[f_t^n - r_t\right] + constant + residual. \tag{E.1}$$

Note that, by assuming $r_{t+m} = r_t e^{-km}$, the left-hand side of the regression (E.1) is expressed as

$$f_{t+1}^{n+1} - r_t = \ln \frac{M_{t+1}(t+n-1)}{M_{t+1}(t+n)} + A(t+1, T = n-1) - A(t+1, t+n) \tag{E.2}$$

and the regressed term of the right-hand side is given by

$$a_n^f \left[f_t^n - r_t \right] \tag{E.3}$$

$$= a_n^f \left[\ln \frac{M_t(t+n)}{M_t)t+n+1)} + A(t,t+n) - A(t,t+n+1) + (B(t,t+n+1) - B(t,t+n) - 1)r_t \right]. \tag{E.4}$$

Based on the assumption that short rate and discounted GOP (market price of risk) are independent, the theoretical regression coefficient can be expressed as

$$a_n^f = \frac{B(t+1,t+n)e^{-\kappa} - B(t+1,t+n-1)e^{-\kappa} - 1}{B(t,t+n+1) - B(t,t+n) - 1}, \tag{E.5}$$

and additionally

$$a_n^f = \frac{\ln M_{t+1}(t+n-1) - \ln(M_{t+1}t+n)}{\ln M_t(t+n) - \ln M_t(t+n+1)}. \tag{E.6}$$