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STRONG STATIONARY TIMES AND THE FUNDAMENTAL MATRIX FOR RECURRENT MARKOV CHAINS

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Abstract. We show that for a finite state space Markov chain, the occupation-time matrix up to a strong stationary time coincides with fundamental matrix of Kemeny and Snell, when each matrix is viewed as operating on functions with mean zero with respect to the stationary distribution.

1. Introduction

My aim in this short note is to point out a connection between two well-studied objects in the theory of Markov chains that appears to have gone unnoticed. I will confine my attention to discrete-time Markov chains with finite state space, but there is little doubt that the analogous results hold for Markov chains in continuous time with countable state spaces.

Throughout, $X = (X_n)$ will be a discrete-time Markov chain with finite state space $E = \{1, 2, \ldots, N\}$ and one-step transition matrix P. We assume that X is irreducible and aperiodic. Let π denote the unique stationary distribution for X. That is, $\pi P = \pi$ and $\pi \cdot \mathbf{1} = 1$. (Here $\mathbf{1}$ is an $N \times 1$ column of 1s. Measures on E are row vectors; function are column vectors.) As is well known, $\lim_n P^n = \Pi$, the matrix with all rows equal to π.

The law of X started at $x \in E$ is $\mathbf{P}^x$, on the sample space Ω of all E-valued sequences $\omega = (\omega_n)_{n\geq 0}$. The symbol $\mathbf{P}^x$ will also be used for the associated expectation, and if μ is a probability measure on E then $\mathbf{P}^\mu := \sum_{x\in E} \mu(x)\mathbf{P}^x$ denotes the law (or expectation) of X under the initial distribution μ.

2. Fundamental Matrix

The *fundamental matrix* Z associated with X and P was introduced in [2] and generalized in [3]. We provide a bit of detail on the construction of Z for completeness.

Proposition 2.1. (a) *The matrix $I - P + \Pi$ is invertible, and $Z := (I - P + \Pi)^{-1}$ denotes its inverse.*
(b) $\pi Z = \pi$.
(c) $Z\mathbf{1} = \mathbf{1}$.

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Proof. (a) Let $f : E \to \mathbf{R}$ (viewed as a column vector) be such that $(I-P+\Pi)f = 0$. Clearly $\Pi f = c\mathbf{1}$, where $c = \pi(f) := \sum_{i\in E} \pi_i f_i$. Consequently,

$$f - Pf + c\mathbf{1} = 0. \tag{2.1}$$

Applying P on the left in (2.1) we obtain

$$Pf - P^2 f + c\mathbf{1} = 0, \tag{2.2}$$

and then, adding (2.1) to (2.2),

$$f - P^2 f + 2c\mathbf{1} = 0.$$

Proceeding recursively we find that

$$f - P^n f + nc\mathbf{1} = 0, \tag{2.3}$$

for $n = 1, 2, \ldots$. Because the entries of $P^n f$ remain bounded as $n \to \infty$, it must be that $c = 0$. In particular, (2.3) now tells us that $P^n f = f$ for all $n \geq 1$. But the only P-invariant functions are the constants, so

$$f_i = \pi(f) = c = 0, \qquad \forall i \in E, \tag{2.4}$$

which means that f is the zero function. This proves that $I - P + \Pi$ is invertible.

(b) Define $\nu := \pi Z$. Then $\nu(I - P + \Pi) = \pi$; that is (writing $\tilde{c}$ for $\sum_{i\in E} \nu_i$)

$$\nu - \nu P + \tilde{c}\pi = \pi,$$

and so

$$\nu - \nu P = (1 - \tilde{c})\pi. \tag{2.5}$$

Multiply (2.5) on the right by $\mathbf{1}$ to see that

$$\tilde{c} - \tilde{c} = (1 - \tilde{c}),$$

and so $\tilde{c} = 1$. This yields $\nu P = \nu$, and finally $\nu = \pi$.

(c) The proof that $Z\mathbf{1} = \mathbf{1}$ follows the pattern of the proof of part (a) and is therefore omitted. $\square$

3. Poisson Equation

In potential theoretic terms, the fundamental matrix Z is a recurrent potential operator for X, yielding solutions of the Poisson equation. More precisely, let $f : E \to \mathbf{R}$ be given. We seek a function $u : E \to \mathbf{R}$ such that $u - Pu = f$. Observe that a necessary condition for this equation to have a solution is that $\pi(f) = 0$. Moreover, if u is a solution, then so is $u + b\mathbf{1}$ for any real constant b.

Given $f : E \to \mathbf{R}$ define $u := Zf$. Then $u - Pu + c\mathbf{1} = f$ (where $c := \pi(u)$), and then recursively

$$u - P^n u = \sum_{k=0}^{n-1} \left(P^k f - c\mathbf{1}\right), \qquad n \geq 1,$$

so that

$$u - \pi(u)\mathbf{1} = \sum_{k=0}^{\infty} \left(P^k f - c\mathbf{1}\right).$$

As is well known, the entries of $P^k f$ converge to $\pi(f)$, at a geometric rate. It follows that $\pi(u) = c = \pi(f)$. Consequently,

$$u - \pi(u)\mathbf{1} = \sum_{k=0}^{\infty} \left(P^k f - \pi(f)\mathbf{1}\right).$$

In particular, if $\pi(f) = 0$, then

$$u = \sum_{k=0}^{\infty} P^k f, \tag{3.1}$$

the series in (3.1) converging absolutely. Thus $u = f + P(\sum_{k=0}^{\infty} P^k f) = f + Pu$, so u is a solution of the Poisson equation. Finally, the most general solution of the Poisson equation is $u_c := Zf + c\mathbf{1}$, where $c = \pi(u_c)$.

4. Strong Stationary Time

Fix an initial state $x \in E$. From the work of Aldous and Diaconis [1] we know that there are (randomized) stopping times S, so-called *strong stationary times*, such that

$$\mathbf{P}^x[X_S = i, S = k] = \pi_i \cdot \mathbf{P}^x[S = k], \qquad i \in E, k = 0, 1, 2, \ldots.$$

That is, under $\mathbf{P}^x$, X_S has law π and is independent of S, at least on $\{S < \infty\}$. This last caveat is rendered moot if we assume, as we may, that $\mathbf{P}^x[S] < \infty$. (Our chain X admits such strong stationary times, and only such times will be of interest here.) Strong stationary times play a crucial role in bounding the *separation distance* $s_x(n)$ between π and $\mathbf{P}^x[X_n = \cdot\,]$:

$$s_x(n) := \max_i \left(1 - \mathbf{P}^x[X_n = i]/\pi_i\right), \qquad n = 0, 1, 2, \ldots.$$

To wit,

$$s_x(n) \le \mathbf{P}^x[S > n], \qquad n = 0, 1, 2, \ldots, \tag{4.1}$$

provided S is a strong stationary time (under $\mathbf{P}^x$). See [1] (3.2); especially note that this bound is sharp in the sense that there is a strong stationary time for which equality holds in (4.1) for all $n \ge 0$. For more on these matters see [4, 5] and for the extension to continuous time see [6, 7].

Let's now suppose that a $\mathbf{P}^x$-strong stationary time S_x has been chosen for each $x \in E$. Define $S(\omega) := S_{X_0(\omega)}(\omega)$, $\omega \in \Omega$. Let μ be any initial distribution for X. Then

$$\begin{aligned}\mathbf{P}^\mu[X_S = i, S = k] &= \sum_{x \in E} \mathbf{P}^x[X_{S_x} = i, S_x = k]\mu(x) \\ &= \sum_{x \in E} \pi(i)\mathbf{P}^x[S_x = k]\mu(x) \\ &= \pi(i)\mathbf{P}^\mu[S = k].\end{aligned}$$

In other words, S is strongly stationary for each initial distribution. Moreover,

$$\mathbf{P}^\mu[S] = \sum_{x \in E} \mathbf{P}^x[S_x]\mu(x) \le \max_{x \in E} \mathbf{P}^x[S_x] < \infty.$$

In what follows S will always be such a "universal" strongly stationary time with finite expectation.

5. The Connection

Fix S as at the end of section 4, and define an operator $W = W_S$ ("mean occupation measure") by

$$Wf(x) := \mathbf{P}^x \sum_{k=0}^{S-1} f(X_k).$$

Clearly $|Wf(x)| \le \|f\|_\infty \cdot \mathbf{P}^x[S]$ for each x. Moreover, $W\mathbf{1}(x) = \mathbf{P}^x[S]$. The crucial observation is that by the simple Markov property, $W(Pf)(x) = Wf(x) - f(x) + \mathbf{P}^x[f(X_S)] = Wf(x) - f(x) + \pi(f)$. Here is our main result.

Theorem 5.1. *If $\pi(f) = 0$ then $Wf(x) = Zf(x)$ for all $x \in E$.*

Proof. Define $V_0 := \{f \in \mathbf{R}^E : \pi(f) = 0\}$, and note that $I - P : V_0 \to V_0$. In view of the discussion in section 3, $Z : V_0 \to V_0$ and $(I - P)Z = I$ on V_0. Also, by the computation preceding the statement of the theorem, $W(I - P) = I$ on V_0. (This is true even without the independence of X_S and S.) Now fix $\alpha > 0$, and write $U^\alpha := \sum_{k=0}^{\infty} e^{-k\alpha} P^k$ for the α-potential operator associated with P. We have, by the strong Markov property at time S, the independence of X_S and S, and the fact that X_S has law π,

$$U^\alpha f(x) = W^\alpha f(x) + \mathbf{P}^x[e^{-\alpha S}]\pi(U^\alpha f), \tag{5.1}$$

where

$$W^\alpha f(x) := \mathbf{P}^x \sum_{k=0}^{S-1} e^{-\alpha k} f(X_k).$$

But π is invariant, so $\pi U^\alpha = (1 - e^{-\alpha})^{-1}\pi$, and therefore $\pi(U^\alpha f) = 0$ provided $f \in V_0$. It now follows from (5.1) that $\pi(W^\alpha f) = 0$ for each $\alpha > 0$. Sending $\alpha \downarrow 0$ we find that $\pi(Wf) = 0$ provided $\pi(f) = 0$. That is, $W : V_0 \to V_0$ as well. We have identified left and right inverses of the restriction of $I - P$ to V_0. It follows that this restriction is invertible and of course the left and right inverses coincide. That is, $W = Z$ on V_0. □

Corollary 5.2. (a) *With S as above, but now for general $f : E \to \mathbf{R}$,*

$$Wf(x) = Zf(x) + \pi(f) \cdot [\mathbf{P}^x[S] - 1], \qquad \forall x \in E.$$

(b) *If R is a second strong stationary time, then*

$$W_S f(x) - W_R f(x) = \pi(f) \cdot [\mathbf{P}^x[S] - \mathbf{P}^x[R]].$$

It may be worth noting that if R and S are strong stationary times, then so is their concatenation $R + S\circ\theta_R$.

References

1. Aldous, D. and Diaconis, P.: Strong uniform times and finite random walks. *Adv. Appl. Math.* **8** (1987), 69–97.
2. Diaconis, P. and Fill, J. A.: Strong stationary times via a new form of duality. *Ann. Probab.* **18** (1990), 1483–1522.
3. Diaconis, P., and Fill, J. A.: Examples for the theory of strong stationary duality with countable state spaces. *Prob. Eng. Inform. Sci.* **4** (1990), 157–180.
4. Fill, J. A.: Time to stationarity for a continuous-time Markov chain. *Prob. Eng. Info. Sci.* **5** (1991), 45–70.
5. Fill, J. A.: Strong stationary duality for continuous-time Markov chains. Part I: Theory. *J. Theoret. Probab.* **5** (1992), 45–70.
6. Kemeny, J. G.: Generalization of a fundmental matrix. *Linear Algebra Appl.* **38** (1981), 193–206.
7. Kemeny, J. G. and Snell, J. L.: *Finite Markov Chains*, Van Nostrand, Princeton-Toronto-London-New York, 1960.

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