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On the Stability of the Quadratic Functional Equation of Pexider Type in Non-Archimedean Spaces

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Abstract: In this article, we prove the generalized Hyers-Ulam-Rassias stability of the pexiderial functional equation

$$f(ax + 2ay) + f(ax - 2ay) = 2ag(x) + 4ag(2y) - 8g(ay)$$

in non-Archimedean normed spaces.

Keywords: Hyers-Ulam-Rassias stability, Non-Archimedean space, pexiderial functional equation.

MSC: 39B82, 46S10, 12J25.

1. INTRODUCTION

In 1940 S. Ulam originally raised the stability problem of functional equation. He had posed the following question. Let $(G,.)$ be a group and $(B,.,d)$ be a metric group given >0 does there exist a $\delta > 0$ such that if a function $f : G \rightarrow B$ satisfies the inequality $d(f(xy), f(x)\, f(y)) \leq \delta$ for all $x, y \in G$ there exists a homomorphism $g : G \rightarrow B$ such that $d(f(x), g(x)) \preceq \epsilon$ for all $x \in G$. In 1941, D.H. Hyers[8] solved this question in the context of Banach spaces. This was the first step towards more studies in this domain of research. In 1978, T.M. Rassias [19] has generalized the Hyers theorem by considering an unbounded Cauchy difference.

Theorem 1.1. *Let $f : E \rightarrow F$ be a mapping from a norm vector space E into a Banach space F subject to the inequality $\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$ For all $x, y \in E$, where and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \rightarrow F$ such that $\| f(x) - T(x) \| \leq \frac{2\epsilon}{2-2^p} \| x \|^p$, For all $x \in E$. If $p < 0$ then inequality $f(xy) = f(x) + f(y)$ holds for all x, y6 =* 0. *During the last decade many mathematicians investigated several stability problems of functional equations.*

In 1897, Hensel [7] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [4], [8], [9], [15]).

The Hyers- Ulam stability of the functional equation

$$f(ax + 2ay) + f(ax - 2ay) = 2af(x) + 4af(2y) - 8f(ay)$$

was proved by G. ZamaniEskandani, H. Vaezi and Y. N. Dehghan [22] for mappings $f : X \rightarrow Y$, where X is a non-Archimedean Banach modules and Y is a unital Banach algebra. In this paper, we consider the following Pexider functional equation

$$f(ax + 2ay) + f(ax - 2ay) = 2ag(x) + 4ag(2y) - 8g(ay).$$

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2. PRELIMINARIES

In this section, we give some definitions and related lemmas for our main result.

Definition 2.1. A triangular norm (shorter t-norm) is a binary operation on the unit interval [0,1], i.e., a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following four axioms: For all *a,b,c*” [0,1]

(i) $T(a,b) = T(b,a)$ (commutativity),

(ii) $T(a,T(b,c)) = T(T(a,b),c)$ (associativity),

(iii) $T(a,1) = a$ (boundary condition),

(iv) $T(a,b) \leq T(a,c)$ whenever $b \leq c$ (monotonicity).

Basic examples are the L *ukasiewicz t*-norm T_L and the t-norms T_P and T_M where $T_L(a, b) := max\{a + b - 1,0\}$, $T_P(a,b) := ab$ and $T_M(a,b) := min\{a,b\}$.

Definition 2.2: Let K be a field. A non-Archimedean absolute value on K is a function

$|\ | : K \to [0, +\infty)$ such that, for any $a, b \in K$,

(i) $|a| \geq 0$ and the equality holds if and only if $a = 0$,

(ii) $|ab| = |a||b|$,

(iii) $|a + b| \leq max\{|a|,|b|\}$ (the strict triangle inequality).

Note that $|n| \leq 1$ for each integer n. We always assume, in addition, that$|\ |$ is non-trivial, i.e., there exists an $a_0 \in K$ such that $|a_0|$ 6 = 0,1.

Definition 2.3: Let X be a vector space over a scalar field K with a non-Archimedean non-trivial valuation |.| . A function k.k : $X \to R$ is a non- Archimedean norm (valuation) if it satisfies the following conditions:

(i) kxk = 0 if and only if $x = 0$,

(ii) krxk = $|r|$kxk($r \in K$; $x \in X$),

(iii) The strong triangle inequality (ultrametric); namely

$$\text{k}x + y\text{k} \leq max\{\text{k}x\text{k};\ \text{k}y\text{k} : x; y \in X\}.$$

Then (X; k. k) is called a non-Archimedean space. Due to the fact that

$$\text{k}x_n - x_m\text{k} \leq max\{\text{k}x_{j+1} - x_j\text{k}|m \leq j \leq n-1\} \qquad (n > m).$$

Definition 2.4: A sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

In this paper, we solve the stability problem for the pexiderial functional equations

$$f(ax + 2ay) + f(ax - 2ay) = 2ag(x) + 4ag(2y) - 8g(ay)$$

when the unknown functions are with values in a non-Archimedean space.

3. MAIN RESULTS

Throughout this section, we assume that H is an additive semigroup and X is a complete non-Archimedean space and $a \in N$.

Theorem 3.1: *Let* $\psi : H \times H \to [0, \infty]$ *be a function such that*

$$\lim_{n\to\infty} \frac{\psi(2^n x, 2^n y)}{|2|^n} = 0 \tag{3.1}$$

for all x, y ∈ H and let for each x ∈ H the limit

$$\psi(x) = \lim_{n\to\infty} \max\left\{\frac{\psi(2^k x, 0)}{|4|^k} \mid 0 \le k < n\right\} \tag{3.2}$$

exists. Suppose that f, g: H → X are mappings with f(0) = g(0) = 0 and satisfying the following inequality

$$\|f(ax + 2ay) + f(ax - 2ay) + 8g(ay) - 2ag(x) - 4ag(2y)\| \le \psi(x,y) \tag{3.3}$$

for all x ∈ X. Then there exists a mapping T : H → X such that

$$\| f(x) - T(x) \| \le \frac{1}{|4|}\Psi(x) \tag{3.4}$$

and

$$\| g(x) - T(x) \| \le \max\left\{\frac{\psi(x,0)}{|2|}, \frac{1}{|4|}\Psi(x)\right\} \tag{3.5}$$

for all x ∈ X. Moreover, if

$$\lim_{j\to\infty} \lim_{n\to\infty} \max\left\{\frac{\psi(2^k x, 0)}{|4|^k} \mid j \le k < n + j\right\} = 0$$

then T is the unique mapping satisfying (3.4) and (3.5).

Proof. Putting $y = 0$ and $a = 1$ in (3.3), we get

$$\| f(x) - g(x) \| \le \frac{\psi(x,0)}{|2|} \tag{3.6}$$

Substituting $y = 0$ and $a = 2$ in (3.3), we have

$$\left\| \frac{f(2x)}{2} - g(x) \right\| \le \frac{\psi(x,0)}{|4|} \tag{3.7}$$

so

$$\left\| \frac{f(2x)}{2} - f(x) \right\| \le \frac{\psi(x,0)}{|4|} \tag{3.8}$$

for all $x \in H$. Replacing x by $2^{n-1}x$ in (3.8) and dividing both sides by 2^{n-1}, then

$$\left\| \frac{f(2^n x)}{2^n} - \frac{f(2^{n-1} x)}{2^{n-1}} \right\| \le \frac{\psi(2^{n-1} x, 0)}{|4|^n} \tag{3.9}$$

It follows from (3.1) and (3.9) that the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}_n$ is a Cauchy sequence. Since X is complete, so $\left\{\frac{f(2^n x)}{2^n}\right\}$ is convergent. Set $T(x) := \lim_{n\to\infty} \frac{f(2x)}{2^n}$. Using induction we see that

$$\| \frac{f(2^n x)}{2^n} - f(x) \| \le \frac{1}{|4|} \max\left\{ \frac{\psi(2^k x, 0)}{|4|^k} \mid 0 \le k < n \right\} \tag{3.10}$$

It's clear that (3.10) holds for $n = 1$ by (3.9). Now, if (3.10) holds for every $0 \le k < n$, we obtain

$$\begin{aligned}
\left\| \frac{f(2^n x)}{2^n} - f(x) \right\| &= \left\| \frac{f(2^n x)}{2^n} - \frac{f(2^{n-1} x)}{2^{n-1}} + \frac{f(2^{n-1} x)}{2^{n-1}} - f(x) \right\| \\
&\le \max\left\{ \left\| \frac{f(2^n x)}{2^n} - \frac{f(2^{n-1} x)}{2^{n-1}} \right\|, \left\| \frac{f(2^{n-1} x)}{2^{n-1}} - f(x) \right\| \right\} \\
&\le \max\left\{ \frac{\psi(2^{n-1} x, 0)}{|4|^n}, \frac{1}{|4|} \max\left\{ \frac{\psi(2^k x, 0)}{|4|^k} \mid 0 \le k < n-1 \right\} \right\} \\
&\le \frac{1}{|4|} \max\left\{ \frac{\psi(2^k x, 0)}{|4|^k} \mid 0 \le k < n \right\}.
\end{aligned}$$

So for all $n \in N$ and all $x \in H$, (3.10) holds. By taking n to approach infinity in (3.10) and using (3.2) one obtains (3.4). On the other hand, by (3.7), we obtain

$$\begin{aligned}
\| g(x) - T(x) \| &\le \max\{ \| g(x) - f(x) \|, \| f(x) - T(x) \| \} \\
&\le \max\left\{ \frac{\psi(x, 0)}{|2|}, \frac{1}{|4|} \Psi(x) \right\}.
\end{aligned}$$

If S be another mapping satisfies (3.4) and (3.5), then for all $x \in H$, we get

$$\begin{aligned}
\| T(x) - S(x) \| &\le \lim \max_{j \to \infty} \left\{ \| T(x) - \frac{f(2^j x)}{2^j} \|, \| \frac{f(2^j x)}{2^j} - S(x) \| \right\} \\
&\le \frac{1}{|4|} \lim_{j \to \infty} \lim_{n \to \infty} \max\left\{ \frac{\psi(2^k x, 0)}{|4|^k} \mid j \le k < n + j \right\} = 0
\end{aligned}$$

Therefore $T = S$. This completes the proof.

Corollary 3.2. *Let* $\lambda : [0; \infty) \to [0; \infty)$ *be a function satisfying*

$$\lambda(|2|t) \le \lambda(|2|)\, \lambda(t) (t \ge 0); \ \lambda(|2|) < |4|$$

Let $\delta > 0$, *H be a normed space and let f; g*: $H \to X$ *are mappings with* $f(0) = g(0) = 0$ *and satisfying*

$$\|f(ax + 2ay) + f(ax - 2ay) - 2ag(x) - 4ag(2y) + 8g(ay)\| \le \delta(\lambda\|x\| + \lambda\|y\|)$$ *for all*

$x; y \in H$. *Then there exists a unique mapping* $T : H \to X$ *such that*

$$\| f(x) - T(x) \| \le \frac{1}{|4|} \delta\lambda(\| x \|)$$

and

$$\| g(x) - T(x) \| \leq \frac{\delta\lambda(\| x \|)}{|4|}$$

for all $x \in H$.

Proof. Defining $\psi : H^2 \to [0; \infty)$ by $\psi(x; y) := \delta(\lambda\text{k}x\text{k} + \lambda\text{k}y\text{k})$, then we have

$$\lim_{n\to\infty} \frac{\psi(2^k x, 0)}{|4|^k} \leq \lim_{n\to\infty} \left[\frac{\lambda(|2|)}{|4|}\right]^n \psi(x, y) = 0$$

for all $x, y \in H$. On the other hand

$$\Psi(x) = \lim_{n\to\infty} \max\left\{\frac{\psi(2^k x, 0)}{|4|^k} \,|\, 0 \leq k < n\right\} = \frac{\psi(x, 0)}{|4|}$$

exists for all $x \in H$. Also

$$\lim_{j\to\infty} \lim_{n\to\infty} \max\left\{\frac{\psi(2^k x, 0)}{|4|^k} \,|\, j \leq k < n + j\right\} = 0..$$

Applying Theorem (3.1), we conclude desired result.

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