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New Measures of Information Based Upon Measures of Central Tendency and Dispersion

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Abstract: In this paper, we have developed some probabilistic measures of information based upon measures of central tendency, namely, arithmetic mean, geometric mean and harmonic mean. We have also extended this theory for the development of measures of information involving measures of central tendency, dispersion and other means like power mean, exponential mean and quadratic mean etc.

1. Introduction

In 1948, Shannon [8] introduced the concept of information theoretic entropy associated with every probability distribution $P = (p_1, p_2, \ldots, p_n)$ where $p_1, p_2, \ldots, p_n$, are the probabilities of n outcomes. He laid down certain very plausible postulates which this measure should possess and found that the only function which satisfies these postulates is given by

$$H(P) = -\sum_{i=1}^{n} p_i \text{ In } p_i \tag{1.1}$$

After Shannon's [8] measure of entropy, Renyi [7] introduced entropy of order α, given by the following expression:

$$H_{\alpha}(P) = \frac{1}{1-\alpha} \text{In}\left(\sum_{i=1}^{n} p_i^{\alpha} \Bigg/ \sum_{i=1}^{n} p_i\right), \alpha \neq 1, \alpha > 0 \tag{1.2}$$

Havrada and Charvat [3] introduced another measure called the non-additive measure of entropy, given by:

$$H^{\alpha}(P) = \frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha} - 1\right), \alpha \neq 1, \alpha > 0. \tag{1.3}$$

Burg [1] developed his own non-parametric measure of entropy, given by

$$H^{1}(P) = \sum_{i=1}^{n} \log p_i \tag{1.4}$$

Many other probabilistic measures of entropy have been derived and discussed by Kapur [5,6], Chakrabarti [2], Herremoes [4], Sharma and Taneja [9], Singh and Kumar [11], Zadeh [13], Zuripov [14], Zyczkowski [15] etc.

In Biological systems, there exist many well-known measures of information which are frequently used for measuring diversity and equitability of different communities. Some of these measures are due to Shannon [8], Renyi [7], Simpson [10], Weiner [12] etc. Of course Shannon's [8] measure is most widely applicable and possesses many interesting properties. But, it has a limitation that it deals with exponential families only. In actual practice, there are distributions which are non-exponential. Thus there is a need for developing new measures to extend the scope of their applications. Such measures have been developed in this paper.

2. New Measures of Information and Their Properties

In this section, we introduce different measures of information which can find their applications to different fields of biological sciences.

I. Information Measure in Terms of Measures of Central Tendency

Let a random variable X takes values $x_1, x_2, \ldots, x_n$. Then geometric mean G, arithmetic mean M and harmonic mean H of these n observations are given by:

$$G = (x_1 x_2 x_3 \ldots x_n)^{\frac{1}{n}} \tag{2.1}$$

$$M = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{2.2}$$

$$H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} \tag{2.3}$$

Equations (2.1), (2.2) and (2.3) can be rewritten as

$$G = (p_1 p_2 p_3 \ldots p_n)^{\frac{1}{n}} \cdot \sum_{i=1}^{n} x_i \tag{2.4}$$

where $$p_i = \frac{x_i}{\sum_{i=1}^{n} x_i}$$

$$nM = \sum_{i=1}^{n} x_i \tag{2.5}$$

$$\frac{H}{M} = \frac{n^2}{\sum_{i=1}^{n} \frac{1}{p_i}} \tag{2.6}$$

Again equation (2.4) can be written as

$$\frac{G}{M} = n(p_1 p_2 p_3 \ldots p_n)^{\frac{1}{n}}$$

or

$$\frac{1}{2^n} \sum_{i=1}^{n} \log p_i = \frac{1}{2^n} n \log\left(\frac{G}{nM}\right) \tag{2.7}$$

Adding equations (2.6) and (2.7), we get

$$\frac{1}{2^n} n \log\left(\frac{G}{nM}\right) + \frac{H}{M} = \frac{1}{2^n} \sum_{i=1}^{n} \log p_i + \frac{n^2}{\sum_{i=1}^{n} \frac{1}{p_i}} \tag{2.8}$$

Now, we introduce an information theoretic measure depending upon geometric mean G, arithmetic mean M and harmonic mean H. This measure is given by

$$\phi_n(P) = \frac{1}{2^n} \sum_{i=1}^{n} \log p_i + \frac{n^2}{\sum_{i=1}^{n} \frac{1}{p_i}}, \; n \geq 2,\; 0 < p_i < 1 \tag{2.9}$$

We shall prove that the R.H.S. of equation (2.9) represents an information measure or entropy. To prove this, we now study the essential properties of an entropy function.

(i) $\phi_n(P)$ is permutationally symmetric as it doesn't change if $p_1, p_2, \ldots, p_n$ are rearranged among themselves. This property is desirable since the labeling of the outcome shouldn't affect the entropy.

(ii) Since $\frac{1}{p_i}$ is continuous function for $0 < p_i < 1$, $\phi_n(P)$ is also continuous everywhere in the same interval.

Thus $\phi_n(P)$ is a continuous function of $p_1, p_2, ..., p_n$ for all p_i lying between 0 and 1. It is highly desirable that $\phi_n(P)$ is a continuous function for all permissible probability distributions, since when the values of some probabilities are changed by small amounts, the entropy should also change by only a small amount.

(iii) Since $0 < p_i < 1$, $\phi_n(P)$ is never negative. This property is again desirable, since entropy should never be negative.

(iv) We have

$$\phi'_n(p) = \frac{1}{2^n p_i} + \frac{n^2}{(p_i)^2}\left(\sum_{i=1}^{n}\frac{1}{p_i}\right)^{-2}$$

and

$$\phi''_n(p) = -\left[\frac{1}{2^n p_i^2} + \frac{2n^2}{p_i^4\left(\sum_{i=1}^{n}\frac{1}{p_i}\right)^3}\left\{p_i\sum_{i=1}^{n}\frac{1}{p_i} - 1\right\}\right]$$

Now, since $p_i\sum_{i=1}^{n}\frac{1}{p_i} - 1 > 0$ always, we see that

$$\phi''_n(p) < 0$$

Thus $\phi_n(p)$ is a concave function of $p_1, p_2, ..., p_n$. This is very useful property since a local maximum will also be the global maximum for a concave function. In fact, in this case, a stationary point, if it exists, will give maximum rather than minimum.

Thus, we see that $\phi_n(P)$ introduced in equation (2.9) satisfies all the essential properties of an information measure. Hence, we conclude from equation (2.8) that with the known values of geometric mean G, arithmetic mean M and harmonic mean H, we can find the information content. Consequently, we have proved that $\phi_n(P)$ is a new measure of information. Next, on fixing the value $n = 2$ and with the set of different probabilities, we have computed different values of the measure $\phi_n(P)$ which are shown in the following Table 2.1:

Table 2.1

p_1	p_2	$\phi_n(P)$
0.10	0.90	0.0986
0.20	0.80	0.4410
0.30	0.70	0.6726
0.40	0.60	0.8565
0.50	0.50	0.8494
0.60	0.40	0.8565
0.70	0.30	0.6726
0.80	0.20	0.4410
0.90	0.10	0.0986

Next, we have presented the values of $\phi_n(P)$ and obtained the Fig. 2.1 which shows that the measure introduced in equation (2.9) is a concave function.

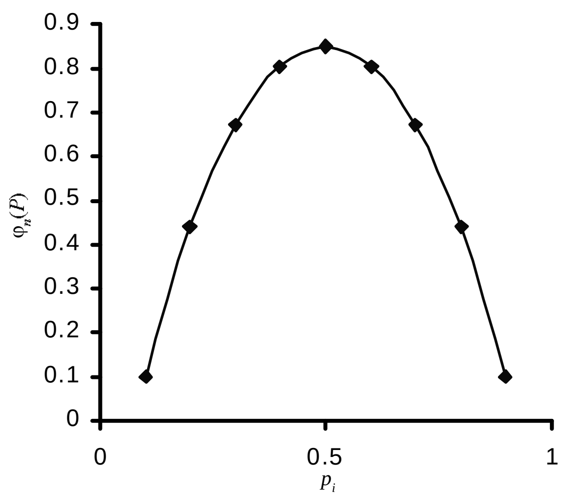

Figure 2.1

II. Information Measure in Terms of Measures of Dispersion

The variance of a discrete distribution of n observations $x_1, x_2, \ldots, x_n$ is defined as

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \tag{2.10}$$

The above equation (2.10) can be rewritten as

$$\sigma^2 = \frac{1}{n}\left[\left(p_1{}^2 + p_2{}^2 + \ldots + p_n{}^2\right) - \frac{1}{n}\right](nM)^2$$

Thus, we have

$$\frac{\sigma^2 + M^2}{2^n nM^2} = \frac{1}{2^n}\sum_{i=1}^{n} p_i{}^2 \tag{2.11}$$

Adding equations (2.6) and (2.11), we get

$$\frac{\sigma^2 + M^2}{2^n nM^2} + \frac{H}{M} = \frac{1}{2^n}\sum_{i=1}^{n} p_i{}^2 + \frac{n^2}{\sum_{i=1}^{n}\frac{1}{p_i}} \tag{2.12}$$

Now, we introduce an information theoretic measure depending upon arithmetic mean M, harmonic mean H and variance σ^2. This measure is given by

$$\psi_n(P) = \frac{1}{2^n}\sum_{i=1}^{n} p_i{}^2 + \frac{n^2}{\sum_{i=1}^{n}\frac{1}{p_i}}, \; n \geq 2,\; 0 \leq p_i \leq 1 \tag{2.13}$$

We shall prove that the R.H.S. of equation (2.13) is an information measure. To prove this, we study its following properties:

(i) Obviously $\psi_n(P) \geq 0$

(ii) $\psi_n(P)$ is continuous

(iii) $\psi_n(P)$ is symmetric

(iv) Concavity: To study its concavity, we have

$$\psi_n'(P) = 2^{1-n} p_i + \frac{n^2}{(p_i)^2}\left(\sum_{i=1}^{n}\frac{1}{p_i}\right)^{-2}$$

Also

$$\psi_n''(P) = -2\left[\frac{n^2}{p_i{}^4\left(\sum_{i=1}^{n}\frac{1}{p_i}\right)^3}\left\{p_i\sum_{i=1}^{n}\frac{1}{p_i} - 1\right\} - \frac{1}{2^n}\right] < 0$$

which shows that $\psi_n(P)$ is concave.

Thus, we see that the measure introduced in equation (2.13) satisfies all the essential properties of being an information measure. Hence, we conclude that $\psi_n(P)$ is another new measure of information.

Next, on fixing the value $n = 2$ and with the set of different probabilities, we have computed different values of the measure $\psi_n(P)$ and presented these values of $\psi_n(P)$ to obtain Fig. 2.2 which shows that the measure introduced in equation (2.13) is a concave function.

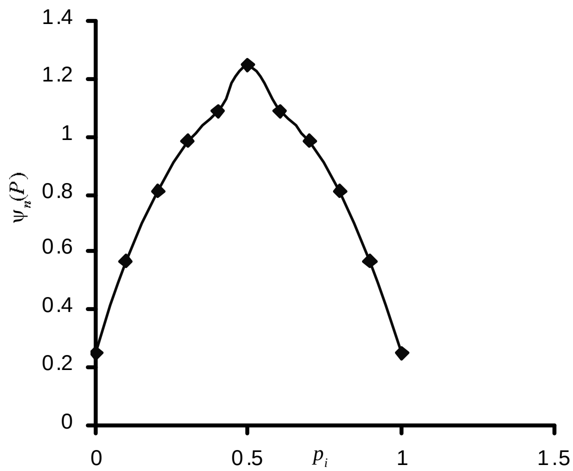

Figure 2.2

III. Information Measure in Terms of Power Mean and Quadratic Mean

We know that the power measure M_p for n observations $x_1, x_2, \ldots, x_n$ is given by

$$M_p = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^r\right]^{\frac{1}{r}} \tag{2.14}$$

$$= \left[\frac{1}{n}\left\{x_1^r + x_2^r + \ldots\ldots + x_n^r\right\}\right]^{\frac{1}{r}}$$

$$= \left[\frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^r \left(\sum_{i=1}^{n} p_i^r\right)\right]^{\frac{1}{r}}$$

Thus

$$M_p = \left[\frac{1}{n}(nM)^r\left(\sum_{i=1}^n p_i^r\right)\right]^{\frac{1}{r}}$$

or

$$2^n\frac{M_p^r}{M^r} = 2^n n^{r-1}\sum_{i=1}^n p_i^r \tag{2.15}$$

We also know that the quadratic mean of n observations $x_1, x_2, ..., x_n$ is defined as

$$M_q = \left[\frac{1}{n}\sum_{i=1}^n x_i^2\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{n}\left(x_1^2 + x_2^2 + ... + x_n^2\right)\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{n}\left\{p_1^2(nM)^2 + p_2^2(nM)^2 + ... + p_n^2(nM)^2\right\}\right]^{\frac{1}{2}}$$

$$= \left(nM^2\sum_{i=1}^n p_i^2\right)^{\frac{1}{2}}$$

$$= M\sqrt{n\sum_{i=1}^n p_i^2}$$

Thus $$\frac{M_q^2}{M^2} = n\sum_{i=1}^n p_i^2 \tag{2.16}$$

Subtracting equations (2.15) and (2.16), we get the following result:

$$2^n\frac{M_p^r}{M^r} - \frac{M_q^2}{M^2} = 2^n n^{r-1}\sum_{i=1}^n p_i^r - n\sum_{i=1}^n p_i^2 \tag{2.17}$$

We shall show that the R.H.S.of equation (2.17) is an information measure. Thus with the known values of arithmetic mean, power mean and quadratic mean, the information model can be constructed. We propose such a model by the following mathematical expression:

$$H_n(P) = 2^n n^{r-1}\sum_{i=1}^n p_i^r - n\sum_{i=1}^n p_i^2, \quad r<1, n\geq 2,\ 0\leq p_i\leq 1 \tag{2.18}$$

Next, we have the following properties:

(i) Obviously $H_n(P) \geq 0$

(ii) $H_n(P)$ is permutationally symmetric function of p_i

(iii) $H_n(P)$ is continuous function of p_i

(iv) Concavity: We have $\dfrac{\partial H_n(P)}{\partial p_i} = 2^n r n^{r-1} p_i^{r-1} - 2np_i$

Also $$\frac{\partial^2 H_n(P)}{\partial p_i^2} = 2^n r(r-1) n^{r-1} p_i^{r-2} - 2n < 0 \quad \text{as } r < 1$$

Thus $H_n(P)$ is a concave function.

Hence the measure proposed in (2.18) satisfies all the conditions of being a measure of information. Thus, we conclude that $H_n(P)$ is a correct information measure.

Next, to check the validity of the proposed measure, we have fixed the value of $n = 2$, $r = 1/2$, and with the set of different probabilities, we have computed different values of $H_n(P)$ which are presented graphically. Thus, we have obtained the following Fig. 2.3 which shows that the measure introduced in equation (2.18) is a concave function.

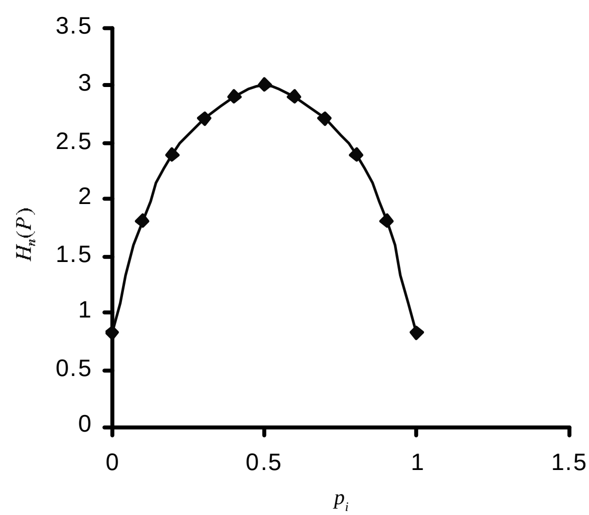

Figure 2.3

III. Information Measure in Terms of Exponential and Quadratic Mean

The exponential mean of n observations $x_1, x_2, \ldots, x_n$ is defined as

$$M_e = \log\left(\frac{e^{x_1}+e^{x_2}+\ldots+e^{x_n}}{n}\right)$$

$$= \log\left(\frac{\sum_{i=1}^{n}\lambda^{p_i}}{n}\right)$$

where $\lambda = e^{nM}$.

To develop a new information model, we make a simple adjustment in the value of exponential mean. This adjustment gives the following expression:

$$\log\left(\frac{\lambda+n-1}{n}\right) - M_e = \log\left(\frac{\lambda+n-1}{n}\right) - \log\left(\frac{\sum_{i=1}^{n}\lambda^{p_i}}{n}\right) \qquad (2.19)$$

Multiplying equation (2.19) both sides by 4^{n^2} and then subtracting equation (2.16) from the resulting equation, we get

$$4^{n^2}\left[\log\left(\frac{\lambda+n-1}{n}\right)-M_e\right] = 4^{n^2}\left[\log\left(\frac{\lambda+n-1}{n}\right)-\log\left(\frac{\sum_{i=1}^{n}\lambda^{p_i}}{n}\right)\right]-n\sum_{i=1}^{n}p_i^2 \qquad (2.20)$$

We shall show that the R.H.S. of equation (2.20) is an information measure. Thus with the known values of arithmetic mean, exponential mean and quadratic mean, the information model can be constructed. We propose such a model by the following mathematical expression:

$$\xi_n(P) = 4^{n^2}\left[\log\left(\frac{\lambda+n-1}{n}\right)-\log\left(\frac{\sum_{i=1}^{n}\lambda^{p_i}}{n}\right)\right]-n\sum_{i=1}^{n}p_i^2\,,$$

$$\lambda > 0,\ \lambda \neq 0,\ 1 \quad \text{and} \quad 0 < p_i < 1 \qquad (2.21)$$

Next, we study the desirable properties of the function $\xi_n(P)$.

(i) Obviously $\xi_n(P) \geq 0$

(ii) $\xi_n(P)$ is permutationally symmetric function of p_i

(iii) $\xi_n(P)$ is continuous function of p_i

(iv) Concavity: We have $\frac{\partial \xi_n(P)}{\partial p_i} = -4^{n^2} \frac{1}{\sum_{i=1}^{n} \lambda^{p_i}} \lambda^{p_i} \log \lambda - 2np_i$

Also $$\frac{\partial^2 \xi_n(P)}{\partial p_i^2} = -4^{n^2} \frac{\lambda^{p_i}\{\log\lambda\}^2}{\left\{\sum_{i=1}^{n} \lambda^{p_i}\right\}^2}\left[\sum_{i=1}^{n} \lambda^{p_i} - \lambda^{p_i}\right] - 2n < 0$$

Thus $\xi_n(P)$ is a concave function.

Hence the measure proposed in (2.21) satisfies all the conditions of being a measure of information. Thus, we conclude that $\xi_n(P)$ is a correct information measure. Next, to check the validity of the proposed measure, we have computed various values of $\xi_n(P)$, presented these values graphically and obtained the following Fig. 2.4 which shows that the measure introduced in equation (2.21) is a concave function.

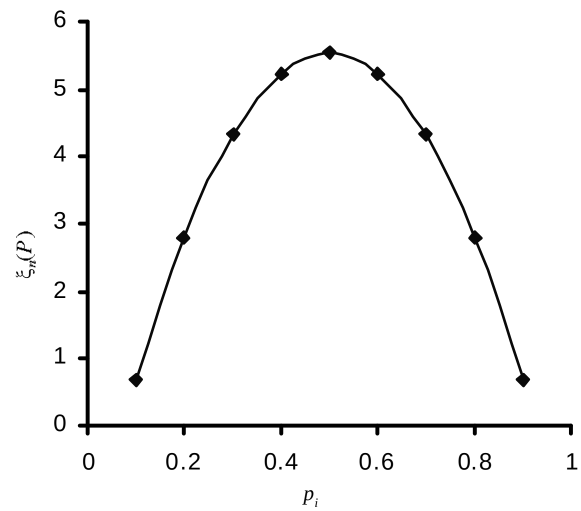

Figure 2.4

Concluding Remarks

In the existing literature of information theory, we find many probabilistic, non-probabilistic, parametric and non-parametric measures of entropy, each with its own merits, demerits and limitations. But, we have to develop those measures which can be successfully applied to a variety of disciplines so that one will be more flexible. Keeping this in mind, we have developed some measures in this paper. Thus, we conclude that for the known values of arithmetic mean, geometric mean, harmonic mean, power mean, exponential mean, quadratic mean and other measures of dispersion, the information content of a discrete frequency distribution can be calculated and consequently, new information measures can be developed.

REFERENCES

[1] Burg J. P., "The Relationship Between Maximum Entropy Spectra and Maximum Likelihood Spectra," *Mod. Spec. Ana.* (Ed. Childers), (1972), 130-131.

[2] Chakrabarti C. G., "Shannon Entropy: Axiomatic Characterization and Application", *Int. Jr. Math. Math.Sci.*, **17** (2005), 2847-2854.

[3] Havrada J. H., and Charvat F., "Quantification Methods of Classification Process: Concept of Structural a-Entropy", *Kybernetika*, **3** (1967), 30-35.

[4] Herremoes P., "Interpretations of Renyi Entropies and Divergences", *Phys. A.*, **365** (1) (2006), 57-62 .

[5] Kapur J. N., "Generalized Entropy of Order α and Type β", *Maths. Semi.*, **4** (1967), 79-84.

[6] Kapur J. N., "Non-Additive Measures of Entropy," *Univ. Manitoba, Research Report*, (1980).

[7] Renyi A., "On Measures of Entropy and Information", *Proc. 4th Ber. Symp. Math. Stat. and Prob.*, **1** (1961), 547-561.

[8] Shannon C. E., "A Mathematical Theory of Communication", *Bell. Sys. Tech. Jr.*, **27** (1948), 379-423, 623-659.

[9] Sharma B. D., and Taneja I. J., "Entropies of Type (a, b) and Other Generalized Measures of Information Theory", *Met.*, **22** (1975), 202-215.

[10] Simpson E. H., "Measuremant of Diversity", *Nature*, **163** (1949), 688.

[11] Singh R. P., and Kumar S., "Parametric R-Norm Information Measures", *Pure & App. Mathematika Sci.*,**35**(1-2) (2007), 41-61.

[12] Wiener N., "The Extrapolation, Interpolation and Smoothing of Stationary Time Series with Engineering Applications", John Wiley and Sons, New York, (1949).

[13] Zadeh L. A., "Towards a Generalized Theory of Uncertainty (GTU)- An Outline", *Information Sciences*, **172** (2005), 1-40.

[14] Zuripov R.G., "Trigonometric Entropies in the Thermodynamics of Nonextensive Systems", *Izv, Vyssh. Uchebn. Zaved. Fiz.,* **6** (2006), 60-66.

[15] Zyczkowski, K., "Renyi Extrapolation of Shannon Entropy", *Open Syst. Inf. Dyn.*, **10**(3) (2003), 297-310.

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