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ELLIPTIC VARIATIONAL INEQUALITIES

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ABSTRACT

In this Paper we shall present an introductory treatment of the theory variational inequationlities of stationary type. Since its inception in the work of Lions and stamacchla [60]. This has been one of the principal fields of applications of the methods and results of nonlinear analysis. The main motivation for end interest of this theory stem from its relevance to the study of free boundary problems. These are boundary value problems involving partial differential equations and must be found as a component of solution.

1. ABSTRACTEXISTENCE RESULTS

Throughout this section V and H are real Hilbert spaces such that V is dense in H and the injection V into H is continous. The norms of V and H will be denoted by ||.|| and |.| respectively. H is identified with its own dual, and its then identified with a subspace of the dual V of V. Hence $V \subset H \subset V$, algebraically and topologically. For v e V and V e V denoted by (v at v) the value of v at v. We shall denoted by ||.|| the norm of V.

Let A ε L (V.V) be such that for lim v > 0,

$$(Av. v) > v\|v\|2 \text{ for all } v \,\varepsilon\, V \tag{3.1}$$

The operator A is often defined by the equation

$$(u, Av) = a(u, v), \text{ for all } u, v, \,\varepsilon\, V \tag{3.2}$$

where $a : V$ ε V → R is a bilinear continous functional and

$$a(v, v) > v\|v\|2 \text{ for all } v \,\varepsilon\, V \tag{3.3}$$

Let $\phi : v \rightarrow R$ be a lower semi continous convex function. If f is a given element of V. Consider the following problem:

Find $y \,\varepsilon\, v$ such that

$$a(y, y\text{–}z) + \phi(y) - \phi(z) < (y\text{–}z, f) \text{ for all } z \,\varepsilon\, V \tag{3.3}$$

where a is the billinear form defined by (3.2).

This is an abstract elliptic variational inequality associated elliptic variational inequality associated with the operator Aand the function ϕ.

Eq. (3.3) can be rewritten in the form

$$Ay + \partial\,\varphi(\,y)\,'\,f.... \tag{3.3}$$

where $\partial\,\varphi : V \rightarrow V$ is the subdifferential of f.

In the special case where ϕ is the indicator function IK of a closed convex subset k of v_0 i.e.,

$$I_k(x) = 0 \text{ if } x\ \varepsilon\ k,\ I_k(x) = +\infty \text{if } x\ \varepsilon\ K \tag{3.4}$$

then problem (3.3) becomes:

Find $y\ \varepsilon\ K$ such that

$$a(y, y - z) < (y - z, f) \text{ for all } z\ \varepsilon\ K \tag{3.5}$$

If the operator A is symmetric i.e.,

$a(y, z) = a(z, y)$ for all z, y e V, then the variational inequality (3.3) is equivalent to the following minimization problem, the Districhlet principle:

$$\frac{1}{2}\underset{\rho}{a(z\ \ z)} + \phi(z)(z, f) : z\,\varepsilon V \tag{3.6}$$

Thus every solution y to (3.3) solves problem (3.6). Conversely, if y is a minimum point for the functional

$$\psi\,(z) = \frac{1}{2}a(z,\ z) + \phi(z) - (z, f)$$

then $0\varepsilon\partial\psi(\,y)$. Since $A + \partial\phi$ is maximal monotone and therefore $\partial\psi = A + \partial\phi - f$. we may conclude that y is a solution to (3.3) (or) (3.3).

In applications to partial differential equations V is usually a Sobolev space on an open subset Ω of R^N and A is an elliptic differential operator on Ω. The space V and the function f or the subset K of V incorporate various conditions on the boundary or in Ω.

Theory 1: Let A be a linear continuous operator from V to V satisfying condition (3.1) and let $\phi: V \rightarrow R$ be a lower semi continuous convex function. Then for every $f\ \varepsilon\ v'$ the variational inequality (3.3) has a unique solution $y\ \varepsilon$ V. Moreover, the mapping $f \rightarrow y$ is Lipschitzien from v to v.

Proof: The operator $A + \partial\,\phi$ is maximal monotone in V × V'. By condition (3.1), using the definition of $\partial\,\phi$. we have

$$(Au + v_0 u - u_0) \geq v \,\|\, u \,\|^2 - \|\, A \,\|\,\|\, u \,\| + \phi(u) - \phi(u_0)$$
$$\text{for all } (u, v)\ \ \varepsilon\partial\,\phi \tag{3.7}$$

Hence $A + \partial \phi$ is coercive and surjective.

Bycondition (3.1) it follows that the solution yto (3.3) is unique

$$\| y \| < w^{-1} \| f \|_e \tag{3.8}$$

For f = I_K, defined by (3.4), we have

Corrollary 1: Let $A : V \to V'$ be a linear continous operator satisfying assumption (3.1). Then for every $f \varepsilon V'$ the variational inequality (3.5) has a unique solution, y ε K.

Remark 1: For existence, the coercivity condition (3.1) is too restrictive. For existence in variational inequality (3.5) it is sufficient to assume that for some v_0 e K.

$$\| v \| \lim_{\substack{x \to \infty \\ V \varepsilon K}} \left(Av_0\, v - v_0 \right) \| v \|^{-1} = +\infty.$$

Nowlet $\{\phi^\varepsilon\}$ beafinallyoffrechet differentiable convexfunctions

$$\phi^\varepsilon(y) \geq -c\,(\| y \| + 1) \text{ for all } > 0 \text{ and } y \varepsilon v \tag{3.9}$$

where C is independent of e and y.

$$\lim_{\substack{x \to \infty \\ \varepsilon \to 0}} \phi^\varepsilon = \phi(y) \text{ for all } y \varepsilon v \tag{3.10}$$

$$\liminf_{\substack{x \to \infty \\ \varepsilon \to 0}} \phi^\varepsilon(y_e) \geq \phi(y) \tag{3.11}$$

for all y ε V and every sequence $\{y_\varepsilon\} \subset V$ weakly convergent in V to y:

Let $\{y_\varepsilon\} \subset V$ be such that for $\varepsilon \to 0$

$$f_\varepsilon \to f \text{ strongly in V} \tag{3.12}$$

consider the equation

$$Ay_\varepsilon + \nabla\phi^\varepsilon(y) = f_e \tag{3.13}$$

where $\nabla\phi^\varepsilon : V \to V'$ is the gradient of ϕ^ε .

By theorem 1 for every $\varepsilon > 0$, (3.13) has a unique solution $y_\varepsilon\, \varepsilon V$.

Theorem 2: Let A ε L (V, V') be a symmetric operator satisfying condition (3.1). Then under assumption (3.9) to (3.12), for $c \to 0$.

$$y_\varepsilon \to y^\varepsilon \text{ weekly in V} \tag{3.14}$$

where y^{ε} is the solution to (3.3). Further assume that

$$\left(\nabla \phi^{\varepsilon}(y) - \nabla \phi^{\lambda}(z), y - z \geq -c(1 + (\| \nabla \phi^{\varepsilon}(y) K \|^2). \quad \| \nabla \phi^{\cdot\cdot}(z) \|^{-} .))(\in + \lambda) \right)$$

$$\text{for all } \varepsilon_0 \ \lambda > 0 \text{ and } y_0 \ E \ \varepsilon V \tag{3.15}$$

$$\text{Then } y_{\varepsilon} \to y^0 \text{ strongly in V} \tag{3.16}$$

Proof: Let z be arbitrary but fixed in D(ϕ). By (3.13) and the definition of gradient we have

$$(y_{\varepsilon} - z, Ay_{\varepsilon}) + \phi^{\varepsilon}(y_{\varepsilon}) - \phi^{\varepsilon}(z) \leq (f_{\varepsilon}, y_{\varepsilon} - z), Vz\varepsilon V \tag{3.17}$$

Then by (3.1), (3.9) and (3.10) we see that $\{\|y_{\varepsilon}\}$ is bounded for $\varepsilon \to 0$.

Hence there exists $y^0 \ \varepsilon \ V$ and a sequence $\varepsilon_n \to 0$ for $n \to \infty$ such that

$$\begin{aligned} y_{e_n} &\to y^* \text{ weekly in V} \\ A_{y_{e_n}} &\to Ay^* \text{ weekly in V'} \end{aligned} \tag{3.18}$$

Since the function $y \to (y, Ay)$ is convex and continous on v. it is weakly lower semi continous.

Hence $\liminf_{n\to\infty} \left(y_{e_n} . Ay_{e_n} \right) \geq (y^*, Ay^*)$

Together with (3.11), (3.17) and (3.18) this yields $(Ay^*, y^* - z) + \phi(y^*)$ $\leq \phi(z) + (f, y^* - z)$ for all $z \ \varepsilon V$. Hence y^* is the solution to (3.3). Since the limit is unique we conclude that (3.14) holds.

Now assume that condition (3.15) is satisfied. Since as seen above

$\| \nabla \phi^{\varepsilon}(y_{\varepsilon}) \|_e \leq \| f_{\varepsilon} \|_e + \| Ay_{\varepsilon} \|_e \leq c \ V \varepsilon > 0,$

if follows by (3.1), (3.13) and (3.15) that

$w \| y_{\varepsilon} - y_{\lambda} \| \leq c(\varepsilon + \lambda) + \| f_e - f_{\lambda} \|_* \| y_{\varepsilon} - y_{\lambda} \|$

for all $\varepsilon.\lambda > 0$. This gives (3.16). //

$\nabla \phi^{\varepsilon}$ is an operator associated with the variational inequality (3.3). A possible choice for ϕ^{ε} is

$$\phi^{\varepsilon} - \phi_{\varepsilon}(z) - \inf\{(2\varepsilon)^{-1} \| z - v \|^2 + \phi(v); v \ \varepsilon V\} \tag{3.19}$$

3.2 AREGULARITYRESULT

We shall denote $A_H : H \to H$ the operator $A_H y - Ay$ for all $y \varepsilon D(A) - \{v \varepsilon V : Av \varepsilon H\}$. The operator A_H is positive definite in H and R(I + AH) = H because the operature $I + A; V \to V'$ is surjective (I is the unit operator in H). Hence AH is maximal monotone in H × H.

Theorem 3: Under assumption (3.1), suppose in addition that there exist $h \varepsilon H$ *and* $c \varepsilon R$ such that

$$\phi(I + \lambda A_H)^{-1}(y + \lambda h) \leq \phi(y) + 0\phi \text{ for all } \lambda > 0 \text{ and } y \varepsilon V \tag{3.20}$$

Then for every $f \varepsilon H$ the solution y^* to (3.3) belongs to $D(A_H)$ and

$$|Ay^*| < 0\ (1 + |f|) \text{ for all } f \varepsilon H \tag{3.21}$$

Proof: Let A_λ e L (H, H) be the operator defined by $A_\lambda = \lambda^{-1}(I - J_\lambda) = AJ_\lambda$. $\lambda > 0$

where $J_\lambda = (I + \lambda A_H)^{-1}$.

Let y* ε V be the solution to (3.3), we have

$$(Ay^*, A_\lambda y^*) + (Ay^*, h - J_\lambda h) + \lambda^{-1}(\partial \phi(y^*), y^* + \lambda h - J_\lambda)$$
$$(y^* + \lambda h)) = (f, A_\lambda(y^* + \lambda h).$$

Now from $(Ay, A_\lambda y) \geq |A_\lambda y|^2$ for all $y \varepsilon V$ and bycondition (3.20)

$$(\partial \phi(y^*), y^* - J_\lambda(y^* + \lambda h) \geq \phi(y^*) - \phi(J_\lambda(y^* + \partial h) \geq - o\lambda,$$

We conclude that

$|A_\lambda y^*|^2 \leq c + |f|(|A_\lambda y^*| + C)$ for all $\lambda > 0$ Thus $\{|A_\lambda y^*|\}$ is bounded and $|A_H y^*| \leq c(1 + |f|)$ where $y^* e\ D(A_H)$. //

Corollary 2: Let A be a linear continous operator from V to V satisfying condition (3.1) and K be a closed convex subset of V having the property that, for some $h \varepsilon H$.

$$(I + \lambda A_H)^{-1}(y + \lambda h)\ \varepsilon \text{ for all } \lambda > 0 \text{ and all } y \varepsilon K \tag{3.22}$$

Thus for every F ε H, the variational inequality (3.5) has a unique solution $y^* \varepsilon K \cap D(A_H)$ which satisfies (3.21).

Now we shall prove an approximating result similar to theorem 2 in case where $\{\phi^\varepsilon\}$ is a family of convex Frechet differentiable functions on H satisfying the following conditions:

$$\phi^{\varepsilon}(y) \leq -0(|y|+1) \text{ for all y e H and e} > 0 \quad (3.23)$$

$$\lim_{\varepsilon \to 0} \phi^{\varepsilon}(y) = \phi(y) \text{ for all } y \ \varepsilon \ D(\phi) \quad (3.24)$$

$$\liminf_{\varepsilon \to 0} \phi^{\varepsilon}(y_{\varepsilon}) \geq \phi(y) \text{ if } y_{\varepsilon} \to y \text{ strongly in V} \quad (3.25)$$

$$(Ay.\nabla\phi^{\varepsilon}(y)) \geq -c(1+|\nabla\phi^{2}(y)|+|ay|) \text{ for all } y \ \varepsilon \ D(A_H) \text{ and } \varepsilon > 0 \quad (3.26)$$

$$(\nabla\phi^{2}(y) - \nabla\phi^{\lambda}(z), y-z) \geq -c(c+\lambda)(|\nabla\phi^{\varepsilon}(yy)|^{2} + |\nabla\phi^{\lambda}(z)|^{2} + 1$$

for all $\varepsilon. \lambda > 0$ and $y, z \ \varepsilon \ H$ (3.27)

Here $\nabla\phi^{\varepsilon} : H \to H$ is the gradient of ϕ^{ε}.

Let $y^{\varepsilon} \ \varepsilon$ D (A_H) be the solution to (3.13) *i.e.*,

$$Ay^{\varepsilon} + \nabla\phi^{\varepsilon}(y^{\varepsilon}) - f^{\varepsilon} \quad (3.28)$$

where $\{f^{\varepsilon}\} \subset H$ is such that

$f^{\varepsilon} \to f$ strongly in H (3.29)

Theorem 4: Under assumption (3.23) to (3.27) for $\varepsilon \to 0$

$$y^{\varepsilon} \to y^{*} \text{ strongly in V} \quad (3.30)$$

$A_H y^{\varepsilon} \to Ay^{*}$ weakly in H (3.31)

$\nabla \phi^{\varepsilon}(y^{\varepsilon}) \to \varepsilon \in \partial \phi(y^{*})$ weakly in H (3.32)

Proof: Taking the scalar product of (3.28) by y^{ε} using (3.1), (3.23) and (3.26), we see that

$$|\nabla\phi^{\varepsilon}(y^{\varepsilon})|^{2} + \|y^{\varepsilon}\|^{2} + |A_H y^{\varepsilon}|^{2} \leq 0$$

for all $\varepsilon > 0$. Then using conditions (3.1) and (2.27) it follows by (3.29) that

$\|y^{\varepsilon} - y^{\lambda}\|^{2} \leq 0(\varepsilon + \lambda)$ for all $\varepsilon.\lambda > 0$. Hence for $\varepsilon \to 0$.

$\nabla\phi^{\varepsilon}(y^{\varepsilon}) \to E = f - A_H \ y^{*}$ weakly in H. Then taking $\varepsilon \to 0$ in inequality

$$\phi^{\varepsilon}(y^{\varepsilon}) - \phi^{\varepsilon}(z) \leq (\nabla \phi^{\varepsilon}(y^{\varepsilon}), y^{\varepsilon} - z) + z \ \varepsilon \ H,$$

It follows by (3.24) and (3.25) that

$$\phi(y^{\varepsilon}) \leq \phi(z) + (\varepsilon, y^{*} - z) + z \in H. //$$

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