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APPROXIMATE CONTROLLABILITY OF SEMILINEAR CONTROL SYSTEMS WITH BOUNDED DELAY

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Abstract

In this paper approximate controllability of an abstract semilinear delay control system of the form $\frac{dx(t)}{dt} = Ax(t) + u(t) + f(t, x(t+\theta), u(t))$, where $0 \le t \le T$ and $-h \le \theta \le 0$, is proved under simple sufficient conditions on the system operators *A, B* and f. First we consider a system without delay, and then we extend the result for delay system. In last section result is illustrated with semilinear controlled heat equation.

1. INTRODUCTION

Let V be a Hilbert spaces. Let C be the Banach space of all continuous functions from an interval $[-h, 0]$ to V with supremum norm. In this paper we are interested in the study of the approximate controllability of the semilinear control system

$$\frac{dx(t)}{dt} = Ax(t) + u(t) + f(t, x_t, u(t)); \quad 0 < t \le T$$

$$x_0(\theta) = \phi(\theta); \quad -h \le \theta \le 0 \tag{1.1}$$

Where the state $x(\cdot)$ takes values in V and control $u \in L_2[0, T; V] = Y$ also takes values in *V*. If $x : [-h, T] \to V$ is a continuous function then x_t is an element in C defined by $x_t(\theta) = x(t+\theta)$; $\theta \in [-h, 0]$, and $\phi \in C$. $f : [0, T] \times C \times V \to V$ is a non linear operator. The mild solution [9] of the above system is given by

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$$x_t(0) = x(t) = S(t)\phi(0) + \int_0^t S(t-s)u(s)ds + \int_0^t S(t-s)f(s, x_s, u(s))ds$$

$$x_0(\theta) = \phi(\theta) \quad -h \le \theta \le 0, 0 < t \le T. \tag{1.2}$$

Where $S(t)$ a C_0 semigroup generated by operator A and $A : D(A) \subset V \to V$ is a closed operator whose domain $D(A)$ is dense in V.

Let $x(T;\phi(0),u)$ be a state value of the system (1.1) at time T corresponding to the control $u \in Y$ and the initial value $\phi(0)$. The system (1.1) is said to be approximate controllable in the time interval $[0,T]$, if for every desired final state x_1 and $\varepsilon > 0$ there exists a control function $u \in Y$, such that the solution $x(T;\phi(0),u)$ of (1.1) satisfies $\| x(T;\phi(0),u) - x_1 \| < \varepsilon$. Now we introduced the reachable set $K_T(f)$ as follows

$$K_T(f) = \{x(T;\phi(0),u) : u \in Y\} \tag{1.3}$$

Definition 1.1. A control system is said to be approximate controllable on $[0,T]$ if $\overline{K_T(f)} = V$, where $\overline{K_T(f)}$ denotes the closure of the set $K_T(f)$.

When $f = 0$, then system (1.1) is called corresponding linear system denoted by $(1.1)^*$ and this system is called approximate controllable if $\overline{K_T(0)} = V$. Kalman (1963) [5] introduced the concept of controllability. The controllability results for abstract linear control systems have been proved by many authors (see [1], [3], [10], [15]). Several authors have extended these concepts to infinite-dimensional nonlinear systems $\frac{dx(t)}{dt} = Ax(t) + Bu(t) + f(t, x(t)) \quad : 0 \le t \le T;$ where $x(0) = x_0$ (see [4], [6], [8], [11], [12], [13], [14]). Dauer and Mahmudov [2] studied the approximate controllability for system (1.1), for any general operator B, by assuming

(a) $S(t)$ is compact for each $t \ge 0$,

(b) f is continuous and uniformly bounded and

(c) Corresponding linear system $(1.1)^*$ is approximate controllable on $[0, T]$.

Remark 1. Since we take B=I, Condition (c) is automatically satisfied for system (1.1) (see [13]).

Mahmudov [7] studied the approximate controllability for system (1.1) with out delay by assuming same conditions as above. In this paper we replace the stronger conditions (a) and (b) with the Lipschitz condition of f and prove the controllability of (1.1).

The next section contains the main results of this paper. In Section 2.1, the approximate controllability of the system (1.1), with out delay, has been proved under the Lipschitz condition on the nonlinear operator f. In Section 2.2 the delay system (1.1) is considered.

2. Controllability Results. First we introduce some notations. For a given operator A, $D(A)$ and $R(A)$ denote the domain and range space of A respectively, $\overline{E}$ denotes the closure of a set E. Let M > 0 be a constant such that $\| S(t) \| \leq M$, for all $t \in [0,T]$.

2.1. CONTROLLABILITY OF SEMILINEAR SYSTEMS WITH OUT DELAY

Consider the linear system given by

$$\frac{dx(t)}{dt} = Ax(t) + u(t)$$

$$x(0) = 0 \tag{2.1}$$

and the following semilinear system

$$\frac{dy(t)}{dt} = Ay(t) + v(t) + f\big(t, y(t), v(t)\big)$$

$$y(0) = 0 \tag{2.2}$$

Assumption [2A]: $f(t, x, u)$ satisfies the Lipschitz condition

$\|f(t,x,u) - f(t,y,v)\|_V \leq l\big(\|x-y\|_V + \|u-v\|_V\big)$ for some constant $l > 0$, for all $x, y, u, v \in V$ and $t \in [0,T]$.

Theorem 1. Under assumption [2A] the semilinear control system (2.2) is approximate controllable if the constant l satisfies the condition $l < 1$.

Proof. The system (2.2) has a unique mild solution for a given control v as f satisfies the Lipschitz condition [9].

Let $x(t)$ be a mild solution of (2.1) corresponding to a control u and consider the following semilinear system

$$\frac{dy(t)}{dt} = Ay(t) + f\big(t, y(t), v(t)\big) + u(t) - f\big(t, x(t), v(t)\big)$$

$$y(0) = 0 \tag{2.3}$$

Hence the control function is $v(t) = u(t) - f\big(t, x(t), v(t)\big)$

The mild solutions of (2.1) and (2.3), respectively, can be written as

$$x(t) = \int_0^t S(t-s)u(s)ds \tag{2.4}$$

$$y(t) = \int_0^t S(t-s) f\big(s, y(s), v(s)\big)ds + \int_0^t S(t-s)u(s)ds - \int_0^t S(t-s) f\big(s, x(s), v(s)\big)ds \tag{2.5}$$

Subtracting (2.5) from (2.4), we get

$$x(t) - y(t) = \int_0^t S(t-s)\big\{f\big(s, x(s), v(s)\big) - f\big(s, y(s), v(s)\big)\big\}ds \tag{2.6}$$

and $$\|x(t) - y(t)\|_V \leq M\int_0^t \big\|f\big(s, x(s), v(s)\big) - f\big(s, y(s), v(s)\big)\big\|_V \, ds$$

Applying Lipschitz condition (2A), we get

$$\|x(t) - y(t)\|_V \leq Ml\int_0^t \|x(s) - y(s)\|_V \, ds$$

Using Gronwall's inequality, we get, $x(t) = y(t)$ for all $t \in [0, T]$. Therefore, every solution of the linear system with control u is also a solution of the semilinear system with control v. Therefore $K_T(f) \supset K_T(0)$, this is dense in V (due to remark 1). Hence the result.

Now it remains to show that there exits a $v(t) \in V$ such that $v(t) = u(t) - f\big(t, x, v(t)\big)$.

Let $v_0 \in V$ and $v_{n+1} = u - f(t, x, v_n)$ $(n = 0,1,2,\ldots)$. So we have

$v_{n+1} - v_n = f(t,x,v_{n-1}) - f(t,x,v_n)$. Therefore by condition (2A)

$$\begin{aligned}\|v_{n+1} - v_n\|_V &\le l\|v_n - v_{n-1}\|_V \\ &\le l^n \|v_1 - v_0\|_V\end{aligned}$$

Since R.H.S. of above inequality tends to zero as $n \to 0$ (because $l < 1$). Hence the sequence $\{v_n\}$ is a Cauchy sequence and since V is complete so $\{v_n\}$ converges to an element v of V. Now

$$\begin{aligned}\|(u - v_{n+1}) - f(t,x,v)\|_V &= \|f(t,x,v_n) - f(t,x,v)\| \\ &\le l\|v_n - v\|_V\end{aligned}$$

Since R.H.S. of above inequality tends to zero as $n \to \infty$. Hence

$$f(t,x,v) = \lim_{n\to\infty}(u - v_n) = u - v \;\Rightarrow v = u - f(t,x,v)$$

It can be proved easily that v is unique.

2.2. CONTROLLABILITY OF SEMILINEAR SYSTEMS WITH DELAY

In this section we proved the controllability result for the system (1.1).

Assumption [2B]: $f(t,x,u)$ satisfies the Lipschitz condition

$\|f(t,x,u) - f(t,y,v)\|_V \le l\left(\|x - y\|_C + \|u - v\|_V\right)$ for some constant $l > 0$, for all $x, y \in C$; $u, v \in V$ and $t \in [0,T]$.

Theorem 2. Under assumption [2B] the semilinear control system (1.1) is approximate controllable if the constant l satisfies the condition $l < 1$.

Let $x(t)$ be a mild solution of $(1.1)^*$ corresponding to a control u and consider the following semilinear system

$$\frac{dy(t)}{dt} = Ay(t) + f\left(t, y_t, v(t)\right) + u(t) - f\left(t, x_t, v(t)\right);\; 0 < t \le T$$

$$y_0(\theta) = \phi(\theta); \quad -h \le \theta \le 0 \tag{2.7}$$

The mild solutions of $(1.1)^*$ and $(2.7)^*$, respectively, can be written as

$$x_t(0) = S(t)\phi(0) + \int_0^t S(t-s)u(s)\,ds$$

$$x_0(\theta) = \phi(\theta) \text{ ; and} \tag{2.8}$$

$$y_t(0) = S(t)\phi(0) + \int_0^t S(t-s) f\left(s, y_s, v(s)\right)ds + \int_0^t S(t-s)u(s)\,ds - \int_0^t S(t-s) f\left(s, x_s, v(s)\right)ds$$

$$y_0(\theta) = \phi(\theta). \tag{2.9}$$

Now from (2.9) and (2.8), for all $0 \le t \le T$ we get

$$x_t(0) - y_t(0) = \int_0^t S(t-s)\left\{f\left(s, x_s, v(s)\right) - f\left(s, y_s, v(s)\right)\right\} ds$$

and $$\| x_t(0) - y_t(0) \|_V \le M \int_0^t \| f\left(s, x_s, v(s)\right) - f\left(s, y_s, v(s)\right) \|_V \, ds .$$

Applying Lipschitz condition (2B), we get

$$\| x_t(0) - y_t(0) \|_V \le M\, l \int_0^t \| x_s - y_s \|_C \, ds . \tag{2.10}$$

Thus, from (2.8), (2.9) and (2.10) we get

$$\|x_t(\theta) - y_t(\theta)\|_V \le M\, l \int_0^t \|x_s - y_s\|_C \, ds ; \ \forall \theta \in [-h, 0], \text{ hence}$$

$$\| x_t - y_t \|_C \le M\, l \int_0^t \| x_s - y_s \|_C \, ds$$

Using Gronwall's inequality, as in theorem 1, we get, $x_t(0) = y_t(0)$ for all $t \in [0, T]$. Further, as in theorem 1, we can complete the proof.

3. Example. Let $V = L_2(0, \pi)$ and $A = \dfrac{d^2}{dx^2}$ with $D(A)$ consisting of all $y \in V$ with $\dfrac{d^2 y}{dx^2} \in V$ and $y(0) = 0 = y(\pi)$. Put $\phi_n(x) = \left(2/\pi\right)^{1/2} \sin(nx), 0 \le x \le \pi, n = 1, 2....,$ then $\{\phi_n : n = 1, 2, ...,\}$ is an orthonormal basis for V and ϕ_n is an eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A, $n = 1,2,...,$. Then the C_0-semigroup $S(t)$ generated by A has $e^{\lambda_n t}$ as the eigenvalues and ϕ_n as their corresponding eigenfunctions.

Consider the control system governed by semilinear heat equation

$$\frac{\partial y(t,x)}{\partial t} = \frac{\partial^2 y(t,x)}{\partial x^2} + u(t,x) + f(t, y(t-h,x), u(t,x)) : 0 < t < T,\ 0 < x < \pi$$

$$\begin{aligned} &y(t,0) = y(t,\pi) = 0;\ t > 0 \\ &y(t,x) = \phi(t,x) \qquad ; -h \le t \le 0 \end{aligned} \tag{3.1}$$

Where $\phi(t,x)$ is continuous. Dauer and Mahmudov [2] proved that a similar system to (3.1) is approximately controllable on $[0,T]$ under the uniform boundedness on the non-linear function *f*. In [4, 8, and 13] the heat equation of the form (3.1) without delay was considered and the approximate controllability was proved under restrictions such as the uniform boundedness on *f* or some inequality constraints. Here the approximate controllability follows from result in section 2.2 for non-uniform bounded function *f* too, satisfying the conditions (2B). For example, consider the function *f* given by

$f(t,x,u) = \alpha\left[\|x\|\phi_3(x) + \|u\|\phi_4(x)\right]$, where α is a positive constant such that $\alpha < 1$.

Here *f* is not uniformly bounded. For this function *f* the approximate controllability of (3.1) follows from the result in section 2.2.

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