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FLUCTUATION PROPERTIES OF COMPOUND POISSON–ERLANG LÉVY PROCESSES

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ABSTRACT. We derive an expression in terms of the Wright function for the density of the first-passage times (or FPT's) for the Poisson-Erlang Lévy processes. For Poisson-exponential processes, we establish an analogue of Zolotarev space-time duality between the original process and its FPT process "truncated" at zero. We show that an asymptotic duality holds in the sense of weak convergence, thereby providing an interpretation of the Letac-Mora reciprocity. The corresponding limits in the sense of convergence in mean and in mean square have an additional multiplier, which is also present in the asymptotic relationship between the marginals of a Poisson-Erlang process and its "truncated" FPT process. We prove that for Poisson-exponential processes, the FPT and the overshoot are independent.

1. Introduction

The main focus of this work is to derive new fluctuation properties of the members of a proper subclass of the family of the Poisson-gamma Lévy processes and to stress connections between various distributions pertaining to this important class of stochastic processes and the Theory of Special Functions. Our principal results are presented in Section 3 as Theorems 3.1, 3.7, 3.9, 3.12, and 3.14. Theorem 3.1 provides a closed-form expression in terms of the *Wright special function* (2.1) for the probability density function (or the *p.d.f.*) of the law of the first-passage time (or *FPT*) for a generic compound Poisson–Erlang Lévy process. See also Remark 3.2.i.

Several of our results provide a probabilistic explanation of an important analytical property which is hereinafter referred to as the *Letac–Mora reciprocity* (cf., for example, [19, p. 25]). Theorem 3.12, which stipulates that for a certain family of the *incremental processes* constructed starting from the FPT process, and for which the limits in the sense of convergence in mean and in mean square differ from that in the sense of weak convergence, is quite surprising (see also Remark 3.13). Letac–Mora reciprocity is specified by Lemma 2.4. (We refer to [28] for a more comprehensive consideration of this property in a related context.) In addition, we anticipate that our paper will motivate further studies of more general FPT stochastic processes *"truncated" at zero*, which are considered here for the first time (see below).

Each representative of the class of spectrally positive Poisson-gamma Lévy processes is constructed starting from the corresponding member of the three-parameter family of

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compound Poisson-gamma distributions, which is described by Definition 2.3 and formulas (2.13)–(2.18). The class of the Poisson-gamma distributions was introduced in [6, p. 223] for modelling "*the total rainfall for a given period*". More recently, these univariate probability laws were widely used for fitting diverse clustered data. Thus, they often provide the mathematical foundation for various stochastic models of Property and Casualty Insurance. We refer to [9, Introduction], [28, Introduction], [18] and [8] for a review of numerous applications and properties of the Poisson–gamma laws and various related stochastic processes.

The class of Poisson-gamma Lévy processes is defined by (3.1). Its proper subclass, which comprises the totality of the compound Poisson–Erlang Lévy processes, is described just below that formula. For all the members of this subclass, it is relatively easy to employ a simplified version of the *Pecherskii–Rogozin identity* in order to write down both a representation for the upper tail of the law of the FPT in terms of the cumulative distribution function of a specific marginal of the corresponding compound Poisson process and the double Laplace transform of the p.d.f. of the law of the FPT (see [3, p. 97, formula (1) and p. 94, formula (5)], respectively). However, "*an attractive general formula is unlikely to lead to explicit answers*" (compare [3, p. 94, below formula (5)]). Therefore, we regard the new closed-form representation (3.7) for the p.d.f. of the law of the FPT for a generic compound Poisson–Erlang Lévy process to be a significant result. Moreover, an attempt to extend our solution of this problem to the entire class of the Poisson–gamma Lévy processes would require a prior development of new analytical methods (see Remark 3.2.ii).

It is important that the class of Poisson-gamma distributions belongs to a wider *power-variance family* (or the *PVF*) of the univariate probability laws. This family is indexed by the *power parameter* p (whose domain is $\mathbf{R}^1 \setminus (0,1)$) as well as the location and scaling parameters, which are hereinafter denoted by μ and λ, respectively. The members of the class of compound Poisson-gamma probability laws correspond to $p \in (1,2)$. Also, the PVF includes the classes (2.11) and (2.12) of the scaled Poisson and gamma distributions, as well as exponentially tilted positive stable laws with index $\alpha \in (0,1)$ and exponentially tilted spectrally negative stable laws with index $\alpha \in (1,2]$ and skewness $\beta = -1$. (The case $\alpha = 2$ corresponds to the normal class.) More details on the PVF are given in [12, Chapter 4], [24], [27], [28].

It is important that when considering the Letac–Mora reciprocity for the PVF, for which $p \in \mathbf{R}^1 \setminus (0,1)$, one should exclude the values of $p \in (2,3)$ (see a comment below formula (2.20) for more detail). For $p \in (-\infty, 0] \cup [3, +\infty)$, the well-known *probabilistic* interpretation of the Letac–Mora reciprocity is often called *Zolotarev space–time duality*.

It is remarkable that for these values of p, Zolotarev duality can be expressed as an elegant relationship between the p.d.f. of a marginal of a related *spectrally negative* Lévy process and that of its FPT process (compare [23, Theorem 4.1 and Corollary 4.1]). For $p = 0$, Zolotarev space–time duality is equivalent to the celebrated result on the law of the FPT of scaled Brownian motion with a drift, which is employed for pricing *American* stock options (compare [4, Corollary 7.2.6] or [23, Remark 4.1]). Other applications in finance are discussed in [14]. In addition, some subtle results in this direction, which pertain to certain natural exponential families and Lévy processes, are given in [15, Theorem 1 and comment below that theorem], and [16, Theorem 2 and Corollary 3].

It was demonstrated in [28] that Zolotarev–type duality per se does not hold in the Poisson-gamma case for which $p \in (1, 2)$ (with the exception of the *self-reciprocal* case of $p = 3/2$ in Definition 2.3). In this case, Theorem 3.7 provides a probabilistic interpretation of the self–reciprocity of the class of Poisson–exponential distributions. Specifically, we demonstrate that in this case, an analogue of Zolotarev space–time duality still holds, but only between the "*density components"* of the *spectrally positive* compound Poisson–exponential Lévy process which is constructed starting from its own Poisson–exponential distribution, and the corresponding FPT process, which should be *"truncated" at zero* for the result to remain valid. In addition, Theorem 3.9 extends the exact result of Theorem 3.7 to the case of an arbitrary Poisson–Erlang Lévy process stipulating the asymptotic pseudo space-time duality. Its assertion also involves a truncation of the FPT process at zero. Remark 3.8 addresses related aspects of Zolotarev space–time duality for the spectrally negative members of the PVF.

Theorem 3.14 provides a simple probabilistic illustration of the analytical results [17, Theorems 1–2]. See also Remark 3.15.

In Section 2, we introduce and derive some properties of the special functions which are required in Section 3, assemble relevant facts on the class of Poisson-gamma distributions, and address some issues related to Letac–Mora reciprocity.

This paper is not self-contained. Therefore, we refer to [12] or [24] for the PVF, to [1] or [21] for the properties of Lévy processes, to [17] for the *Pecherskii–Rogozin identity* and *Wiener–Hopf factorization* for general univariate Lévy processes, and to [20] for properties of the Wright and the generalized Mittag-Leffler functions.

2. Auxiliary Definitions and Results

First, we summarize some relevant notation and terminology. We will follow the custom of formulating various statements of distribution theory in terms of the properties of random variables (or *r.v.'s*), even when such results pertain only to their *distributions*. Hereinafter, $\mathbf{R}^1_+$ stands for the set of all positive reals. In what follows, the sign "$\stackrel{\mathrm{d}}{=}$" will denote the fact that the distributions of (univariate) r.v.'s coincide. In the sequel, the symbol "$\stackrel{d}{\to}$" will be understood as *weak convergence*. We denote the càdlàg space equipped with the Skorohod metric by $\mathbf{D}[0,\infty)$, and the sign "$\stackrel{\mathbf{D}[0,\infty)}{=\!=\!=}$" is understood as the fact that the laws of two stochastic processes coincide in this space. An empty sum is interpreted as zero, and the expressions $a \vee b$ and $a \wedge b$ are understood as $\max(a, b)$ and $\min(a, b)$, respectively. In the sequel, we will denote a sequence of i.i.d.r.v.'s which possess the same distribution as r.v. $\mathcal{Y}$ by $\{\mathcal{Y}^{(n)},\ n \geq 1\}$.

Throughout this work, we will use the family of *Wright special functions* which is introduced by the following definition (compare [20, Section 2.3]):

Definition 2.1. The complex-valued *Wright function* $\phi(\rho, \delta, z)$ of argument $z \in \mathbb{C}$, which is indexed by the real-valued parameter $\rho \in (-1, 0) \cup (0, \infty)$ and the complex-valued parameter $\delta \in \mathbb{C}$ is hereinafter defined by the following convergent series:

$$\phi(\rho, \delta, z) := \sum_{k=0}^{\infty} \frac{z^k}{k! \cdot \Gamma(\rho k + \delta)} \, . \tag{2.1}$$

For $\delta = 0$, (2.1) is simplified as follows:

$$\phi(\rho, 0, z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k! \cdot \Gamma(\rho k)} \,. \tag{2.2}$$

Next, given integer $n \geq 1$, set $\mathcal{Z} := (n+1) \cdot (n^{-n} z)^{1/(n+1)}$.

Lemma 2.2. **(i)** *For each fixed integer* $k \geq 1$*,*

$$\phi'(n, k, z) \equiv \phi(n, n+k, z) = \mathcal{O}(\mathcal{Z}^{-n-k+1/2} \cdot e^{\mathcal{Z}}) \quad \text{as} \quad z \to +\infty. \tag{2.3}$$

(ii) *For* $z \in \mathbb{C}$*,*

$$\phi(n, 0, z) \equiv nz \cdot \phi'(n, 1, z). \tag{2.4}$$

Proof. **(i)** From [20, Section 2.3] we have

$$\phi(n, \delta, z) = \mathcal{O}(\mathcal{Z}^{1/2-\delta} \cdot e^{\mathcal{Z}}) \quad \text{as} \quad z \to +\infty. \tag{2.5}$$

See also [28, Appendix A.1] for more detail on the terminology used below. We note that, although additional exponential terms are present in the expansion of $\phi(n, \delta, z)$ when $n > 1$, these additional terms are *subdominant* compared to $e^{\mathcal{Z}}$ on the positive real axis so that the dominant behavior is given by (2.5) for $n \geq 1$. Then $\phi'(n, k, z) \equiv \phi(n, n + k, z) = \mathcal{O}(\mathcal{Z}^{-n-k+1/2} \cdot e^{\mathcal{Z}})$ as $z \to +\infty$, which proves (2.3).
(ii) The validity of (2.4) follows by differentiation of (2.2) and comparing the result of differentiation with (2.1). □

In the proof of Lemma 3.11 and Theorem 3.12.i, we will employ the family $\mathcal{E}_{\rho,\delta}(z)$ of *generalized Mittag-Leffler special functions*. By analogy to [20, p. 190], the complex-valued generalized Mittag-Leffler function $\mathcal{E}_{\rho,\delta}(z)$ of argument $z \in \mathbb{C}$, which is indexed by the parameters ρ and δ such that $\rho \in \mathbf{R}^1_+$, $\delta \in \mathbf{R}^1_+$, and $\rho j + \delta \notin \{0, -1, -2, \ldots\}$ $(j \in \{0, 1, 2, \ldots\})$ is hereinafter defined by the following convergent series:

$$\mathcal{E}_{\rho,\delta}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\rho j + \delta)} \,. \tag{2.6}$$

Next, we proceed with the analytical description of the Poisson-gamma class, which is characterized by the triplet of parameters $\{p \in (1, 2), \mu \in \mathbf{R}^1_+, \lambda \in \mathbf{R}^1_+\}$. These parameters are commonly referred to as the *power*, *location* (or *mean*), and *scaling* parameters, respectively; see [24] for more details. Given $p \in (1, 2)$, $\mu \in \mathbf{R}^1_+$, and $\lambda \in \mathbf{R}^1_+$, we introduce the *exponential tilting* parameter

$$\theta_p \ (= \theta(p, \mu, \lambda)) \ := \lambda \mu^{1-p}/(p-1). \tag{2.7}$$

Subsequently, for the same range of the values of the triplet of parameters (p, μ, λ), set

$$\mathcal{A}_p \ (= \mathcal{A}(p, \mu, \lambda)) \ := \lambda \mu^{2-p}/(2-p); \tag{2.8}$$

$$\rho_p := (2-p)/(p-1) \in \mathbf{R}^1_+. \tag{2.9}$$

The probabilistic meaning of the parameters $\mathcal{A}_p$ and ρ_p introduced by (2.8) and (2.9) is clarified by (2.13), (2.17) and by (2.16), respectively. In what follows, log stands for the *natural* logarithm of a positive real argument, whereas $\Gamma(\delta, x) := \int_x^{\infty} u^{\delta-1} \cdot e^{-u} \cdot du$ (with $\delta \in \mathbf{R}^1$ and $x \in \mathbf{R}^1_+$) denotes the *complement of the incomplete gamma function*.

The following definition is consistent with [24, Proposition 1.1, Theorem 1.1, errata].

Definition 2.3. Given $p \in (1,2)$, $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, we define both the corresponding *Poisson-gamma* r.v. $Tw_p(\mu,\lambda)$ and its probability law by means of the following Lévy representation for the cumulant-generating function $\zeta_{p,\mu,\lambda}(\cdot)$:

$$\begin{aligned}\zeta_{p,\mu,\lambda}(s) &= \log \mathbf{E} e^{sTw_p(\mu,\lambda)} \\ &= \mathcal{A}_p\big\{\big(1-\frac{s}{\theta_p}\,\big)^{-\rho_p}-1\big\} = \int_{0+}^{\infty}(e^{sx}-1)\omega_{p,\mu,\lambda}(dx).\end{aligned} \tag{2.10}$$

Here, the values of θ_p, $\mathcal{A}_p$ and ρ_p are given by (2.7)–(2.9), $s < \theta_p$, whereas the spectrally positive Lévy measure $\omega_{p,\mu,\lambda}(\cdot)$ of the infinitely divisible law of the r.v. $Tw_p(\mu,\lambda)$ is such that $\forall y \in \mathbf{R}^1_+$, $\omega_{p,\mu,\lambda}(\{(y,+\infty)\}) = \mathcal{A}_p \cdot \Gamma(\rho_p, \theta_p y)/\Gamma(\rho_p)$.

Throughout this paper, we employ the *mean-value* parameterization for the families of Poisson-gamma, scaled Poisson and gamma distributions, because it is more convenient for describing our results and is consistent with those employed in [12] and [24]. However, since this parameterization of the class of gamma distributions differs from the standard parameterization of this family, we now employ the former one to describe the families of the scaled Poisson and gamma probability laws.

By analogy to [24, formula (2.9)], given $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, we define the corresponding (discrete) scaled Poisson r.v. $Tw_1(\mu,\lambda)$ as follows:

$$Tw_1(\mu,\lambda) \stackrel{\mathrm{d}}{=} \lambda^{-1}\cdot Tw_1(\mu\lambda,1) \stackrel{\mathrm{d}}{=} \lambda^{-1}\cdot \mathcal{P}oiss(\mu\lambda). \tag{2.11}$$

In particular, $Tw_1(\mu,1) \stackrel{\mathrm{d}}{=} \mathcal{P}oiss(\mu)$ (compare to (2.17)).

Subsequently, given arbitrary fixed values of the mean parameter $\mu \in \mathbf{R}^1_+$ and the *shape* (or scaling) parameter $\lambda \in \mathbf{R}^1_+$, the r.v. $Tw_2(\mu,\lambda)$ is characterized by its p.d.f.

$$f_{2,\mu,\lambda}(x) := (\lambda/\mu)^{\lambda}\cdot x^{\lambda-1}\cdot \exp\{-(\lambda/\mu)\cdot x\}/\Gamma(\lambda), \quad \text{where} \quad x \in \mathbf{R}^1_+. \tag{2.12}$$

The r.v. $Tw_2(\mu,\lambda)$ is said to have the gamma law with parameters μ and λ. Also, we extend the leftmost equation in (2.10) and set $\zeta_{p,\mu,\lambda}(s) := \log\,\mathbf{E}\exp\{s\cdot Tw_p(\mu,\lambda)\}$ for arbitrary fixed values of $p \in [1,2]$, $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$.

Next, we review relevant properties of the class of compound Poisson-gamma distributions $\{Tw_p(\mu,\lambda),\ p \in (1,2),\ \mu \in \mathbf{R}^1_+,\ \lambda \in \mathbf{R}^1_+\}$. This can be done via a characterization of the probabilistic structure of the law of each member of this class, which was described analytically in Definition 2.3. First, given $p \in (1,2)$, $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, it follows from (2.10) that

$$\mathbf{P}\{Tw_p(\mu,\lambda)=0\} = \exp\{\lim_{s\to-\infty}\zeta_{p,\mu,\lambda}(s)\} = \exp\{-\mathcal{A}_p\} \tag{2.13}$$

(compare [10, p. 504]; [12, formula (4.29)]). By (2.13), $Tw_p(\mu,\lambda)$ has a positive mass at zero. In addition, this non-negative r.v. has an absolutely continuous component on $\mathbf{R}^1_+$, whose density is denoted by $f_{p,\mu,\lambda}(u)$. Its infinite-series representation can be found in [10, p. 504] or [12, Subsection 4.2.4]. It is important that $f_{p,\mu,\lambda}(u)$ can also be expressed in terms of a member of a proper subclass of the family (2.1), which is characterized by the values of $\rho \in \mathbf{R}^1_+$ and $\delta = 0$ (compare to (2.2)). Thus, given $p \in (1,2)$, $\mu \in \mathbf{R}^1_+$, and $\lambda \in \mathbf{R}^1_+$, [27, formula (3.25)] yields that for $u \in \mathbf{R}^1_+$, $f_{p,\mu,\lambda}(u)$ can be expressed in terms of the parameter $\rho = \rho_p$ as follows:

$$f_{p,\mu,\lambda}(u) = u^{-1} \cdot \exp\Big\{ - \ \frac{\rho+1}{\rho} \ \cdot \lambda \cdot \mu^{\rho/(\rho+1)} - (\rho+1) \cdot \lambda \mu^{-1/(\rho+1)} u \Big\} \\ \times \phi(\rho, 0, (\rho+1)^{\rho+1} \lambda^{\rho+1} u^{\rho}/\rho). \tag{2.14}$$

Next, we proceed with the *compound Poisson representation* for a generic member of the class $\{Tw_p(\mu, \lambda),\ p \in (1,2),\ \mu \in \mathbf{R}^1_+,\ \lambda \in \mathbf{R}^1_+\}$. To this end, fix the arbitrary $p \in (1,2)$, $\mu \in \mathbf{R}^1_+$, $\lambda \in \mathbf{R}^1_+$, and consider independent r.v.'s $\{\mathcal{H}_n,\ n \geq 1\}$ which have a specific common gamma distribution such that for each integer $n \geq 1$, the r.v. $\mathcal{H}_n\ (=\ \mathcal{H}_n(p, \mu, \lambda)) \stackrel{\mathrm{d}}{=} Tw_2(\mu_{\mathcal{H}}, \lambda_{\mathcal{H}})$. The law of $Tw_2(\mu_{\mathcal{H}}, \lambda_{\mathcal{H}})$ is characterized by

$$\mu_{\mathcal{H}} := \rho_p/\theta_p = (2-p) \cdot \mu^{p-1}/\lambda; \tag{2.15}$$

$$\lambda_{\mathcal{H}} := \rho_p. \tag{2.16}$$

Also, consider a Poisson-distributed counting r.v. $\mathcal{T}_p$ with mean $\mathcal{A}_p$, which is assumed to be independent of the i.i.d.r.v.'s $\{\mathcal{H}_n,\ n \geq 1\}$:

$$\mathcal{T}_p \stackrel{\mathrm{d}}{=} Tw_1(\mathcal{A}_p, 1) \stackrel{\mathrm{d}}{=} \mathcal{P}oiss(\mathcal{A}_p). \tag{2.17}$$

In view of [10, p. 505],

$$Tw_p(\mu, \lambda) \stackrel{\mathrm{d}}{=} \sum_{n=1}^{\mathcal{T}_p} \mathcal{H}_n. \tag{2.18}$$

It follows from (2.9) that $\rho_{3/2} = 1$. A subsequent application of (2.16) stipulates that in the case where $p = 3/2$, $\lambda_{\mathcal{H}} = 1$. By (2.18), the r.v. $Tw_{3/2}(\mu, \lambda)$ can be characterized as the Poisson sum of the independent r.v.'s $\{Tw_2^{(n)}(\theta_{3/2}^{-1}, 1), n \geq 1\}$ with common exponential distribution $Tw_2(\theta_{3/2}^{-1}, 1)$ (compare [25, formula (2.4)]). In view of the above, the counting r.v. $Tw_1(\mathcal{A}_{3/2}, 1)$ does not depend on the i.i.d.r.v.'s $\{Tw_2^{(n)}(\theta_{3/2}^{-1}, 1), n \geq 1\}$. Also, it follows from [28, formulas (4.17)–(4.18)] that in the case where $p = 3/2$, (2.14) can be rewritten as follows:

$$f_{3/2,\mu,\lambda}(y) = \frac{1}{y} \cdot e^{-\theta_{3/2} y - \mathcal{A}_{3/2}} \phi(1, 0, 4\lambda^2 y) = \frac{2\lambda}{\sqrt{y}} \cdot e^{-\theta_{3/2}(y+\mu)} I_1(4\lambda\sqrt{y}), \tag{2.19}$$

where $y \in \mathbf{R}^1_+$. Hereinafter, $I_\nu(\cdot)$ denotes the *modified Bessel function of the first kind* of order ν (cf., for example, [7, formula (8.445)]). In this paper, we will consider the case where $\nu \in \{0; 1\}$ (compare to (3.10)). See [9, Sections 1–2]; [25, Section 2] for a review of the class of Poisson-exponential distributions.

The following *reciprocity transformation* of the PVF was described in [19, p. 25]:

$$p \to p' := 3 - p;\ \mu \to \mu' := 1/\mu;\ \lambda' := \lambda. \tag{2.20}$$

Recall that one should exclude values of $p \in (2,3)$. Otherwise, one obtains that $p' \in (0,1)$, which contradicts [12, Proposition 4.2]. It is also relevant that although positive stable distributions with index $\alpha \in (0, 1/2)$ do not possess dual probability distributions per se, but [30, end of Section 2.10] suggests the consideration of the so-called "*trans-stable*" signed measures in lieu of those distributions.

For arbitrary values of $p \in (1,2)$, $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, the mapping (2.20) interchanges the quantities θ_p and $\mathcal{A}_p$ (which are defined by (2.7) and (2.8), respectively). Namely,

$$\mathcal{A}(p',\mu',\lambda') = \theta(p,\mu,\lambda);\ \theta(p',\mu',\lambda') = \mathcal{A}(p,\mu,\lambda). \tag{2.21}$$

Lemma 2.4. *For arbitrary values of* $p \in [1,2]$, $\mu \in \mathbf{R}^1_+$ *and* $\lambda \in \mathbf{R}^1_+$, *the collection* $\{\zeta_{p,\mu,\lambda}(s),\ \zeta_{3-p,1/\mu,\lambda}(-s)\}$ *constitutes a pair of inverse functions. In particular, the sets* $\{\zeta_{3/2,\mu,\lambda}(s), -\zeta_{3/2,1/\mu,\lambda}(-s)\}$ *and* $\{\zeta_{1,\mu,\lambda}(s), -\zeta_{2,1/\mu,\lambda}(-s)\}$ *are pairs of inverse functions.*

Proof. See [24, Proposition 1.1] and [28, Lemma 1]. □

In the case where the reciprocal pair $\{p,\ p'\}$ coincides with the two–point set $\{1;2\}$, and for an arbitrary fixed $\mu \in \mathbf{R}^1_+$, a probabilistic interpretation of the reciprocity of the pair $\{Tw_1(\mu,1), Tw_2(1/\mu,1)\}$ is given by the well-known result on the *Poisson flow of arrivals* when the inter-arrival times are independent and exponential.

3. Main Results

Fix arbitrary values of $p \in (1,2)$, $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, and consider the Poisson-gamma Lévy process $\{X_{p,\mu,\lambda}(t), t \geq 0\}$ (See [18] or [26, Definition 3.1] for the description of a more general class of *Hougaard processes*, for which $p \in \mathbf{R}^1 \setminus (0,1)$.)

By [1, p. 11], it suffices to define a Lévy process by virtue of its increment in unit time. Therefore, set

$$X_{p,\mu,\lambda}(1) \stackrel{\mathrm{d}}{=} Tw_p(\mu,\lambda). \tag{3.1}$$

In view of (2.10) and (3.1), each such Lévy process is a *subordinator*.

The proper subclass of the family of the Poisson-gamma Lévy processes, whose members are called the *compound Poisson–Erlang Lévy processes*, corresponds to the values of $p = 1 + 1/(n+1)$ for some integer $n \geq 1$, whereas μ and λ take on arbitrary fixed values in $\mathbf{R}^1_+$. Evidently, (2.9) yields that the condition $p = 1 + 1/(n+1)$ is equivalent to assuming that $\rho_p = n$ for some integer $n \geq 1$. (A subsequent combination of (2.16) and (2.18) stipulates that the shape (or scaling) parameter $\lambda_{\mathcal{H}}$ of the gamma-distributed r.v.'s $\{\mathcal{H}_n,\ n \geq 1\}$ is a positive integer and hence, the i.i.d.r.v.'s which emerge in the compound Poisson representation (2.18) of the increment of this process in unit time are *Erlang*-distributed.)

By [26, Proposition 3.2.i], this process has the following marginals:

$$X_{p,\mu,\lambda}(t) \stackrel{\mathrm{d}}{=} Tw_p(\mu \cdot t, \lambda \cdot t^{p-1}), \quad \text{where} \quad t \in \mathbf{R}^1_+ \quad \text{is arbitrary and fixed.} \tag{3.2}$$

A combination of (2.14), (3.1), (3.2) yields that for $p \in (1,2)$ and $y > 0$, the "density component" $f_{X_{p,\mu,\lambda}(t)}(y)$ of the Poisson-gamma r.v. $X_{p,\mu,\lambda}(t)$ is as follows:

$$\begin{aligned} f_{X_{p,\mu,\lambda}(t)}(y) &= y^{-1} \cdot \phi(\rho, 0, (\rho+1)^{\rho+1}(\lambda t^{1/(\rho+1)})^{\rho+1} y^{\rho}/\rho) \\ &\times \exp\Big\{ -\frac{\rho+1}{\rho} \cdot \lambda t^{1/(\rho+1)}(\mu t)^{\rho/(\rho+1)} - (\rho+1)\lambda t^{1/(\rho+1)}(\mu t)^{-1/(\rho+1)} y \Big\} \\ &= \frac{1}{y} \cdot e^{-\mathcal{A}_p t - \theta_p y} \phi(\rho, 0, (\rho+1)^{\rho+1}\lambda^{\rho+1} t y^{\rho}/\rho) \\ &= \frac{1}{y} \cdot e^{-\mathcal{A}_p t - \theta_p y} \cdot \phi(\rho, 0, \mathcal{A}_p \theta_p^{\rho} t y^{\rho}). \end{aligned} \tag{3.3}$$

Also, it follows from (2.13) and (3.2) that for each $t > 0$,

$$\mathbf{P}\{X_{p,\mu,\lambda}(t) = 0\} = \exp\{-\mathcal{A}_p \cdot t\}. \tag{3.4}$$

Given real $u \geq 0$, we introduce the *FPT* and the *overshoot* (over the level u) as follows:

$$\tau_u := \inf\{t > 0 : X_{p,\mu,\lambda}(t) > u\}; \tag{3.5}$$

$$\gamma_u := X_{p,\mu,\lambda}(\tau_u+) - u. \tag{3.6}$$

A subsequent combination of the contradiction argument with the strong law of large numbers for Lévy processes (cf., for example, [1, p. 92]) implies that $\forall u \in \mathbf{R}^1_+$, the r.v. τ_u is finite a.s. Hence, in this case one does not have to impose an additional assumption that $\tau_u < \infty$ in the definition of the overshoot by (3.6) for the specific process $\{X_{p,\mu,\lambda}(t), t \geq 0\}$ (compare [22, p. 203]). Also, it follows from a combination of (2.10) with [1, p. 77, Theorem 4] that for $u \in \mathbf{R}^1_+$, $\tau_u = \inf\{t > 0 : X_{p,\mu,\lambda}(t) > u\} = \inf\{t > 0 : X_{p,\mu,\lambda}(t) \geq u\}$ (compare [22, p. 203], where the versions of (3.5)–(3.6) were employed).

Set $\mathcal{X} := \theta_p u$. The following assertion generalizes [3, Section 8.4, formulas (11)–(12)] to the case of a compound Poisson–Erlang Lévy process with $\rho_p \in \mathbf{N}$.

Theorem 3.1. *Consider a compound Poisson–Erlang Lévy process $\{X_{p,\mu,\lambda}(t),\ t \geq 0\}$ which is introduced below (3.1) with $p = 1+1/(n+1)$ for some integer $n \geq 1$, and where the parameters μ and λ take on arbitrary fixed values in $\mathbf{R}^1_+$. Then given level $u \in \mathbf{R}^1_+$, the law of a non-negative r.v. τ_u defined by (3.5) is absolutely continuous and possesses the following p.d.f.:*

$$q_{n,u}(y) = \mathcal{A}_p \cdot \exp\{-\mathcal{X} - \mathcal{A}_p y\} \cdot \sum_{k=1}^{n} \mathcal{X}^{k-1} \cdot \phi(n, k, \mathcal{A}_p \mathcal{X}^n y), \quad \text{where} \quad y \in \mathbf{R}^1_+. \tag{3.7}$$

Proof. Recall that the condition $p = 1+1/(n+1)$ is equivalent to assuming that $\rho_p = n$ for some integer $n \geq 1$. Similar to [3, p. 97], define the Laplace transforms

$$q^*_n(u; w) := \int_0^\infty e^{-wy} f(y)\, dy \quad \text{and} \quad {}^*q^*_n(s; w) := \int_0^\infty e^{-su} q^*_n(u; w)\, du.$$

Here, the argument $w > -\mathcal{A}_p$. Set $a\ (= a(n, w)) := (\mathcal{A}_p/(\mathcal{A}_p + w))^{1/n}$. It then follows from [3, p. 94, formula (5)] that

$${}^*q^*_n(s; w) = \frac{\mathcal{A}_p\{(\theta_p + s)^n - \theta_p^n\}}{s\{(\mathcal{A}_p + w)(\theta_p + s)^n - \mathcal{A}_p \theta_p^n\}} = \frac{a^n\{(\theta_p + s)^n - \theta_p^n\}}{s\{(\theta_p + s)^n - (a \cdot \theta_p)^n\}}\ .$$

Since ${}^*q^*_n(s; w) \to 0$ uniformly as $|s| \to \infty$, the inverse Laplace transform $q^*_n(u; w)$ is given by

$$q^*_n(u; w) = \frac{a^n}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{e^{s \cdot u}\{(\theta_p + s)^n - \theta_p^n\}}{\{(\theta_p + s)^n - (a \cdot \theta_p)^n\}}\, \frac{ds}{s}\ , \tag{3.8}$$

where $c > 0$ is chosen so that the integration path lies to the right of all the poles of the integrand. The denominator in braces can be written as

$$(\theta_p + s - ae_1)(\theta_p + s - ae_2) \dots (\theta_p + s - ae_n), \qquad \text{where } e_r := \exp(2\pi i r/n),$$

so that the integrand possesses simple poles at $s_r = (a \cdot e_r - 1)\theta_p$, where $1 \leq r \leq n$. The residue of $\{(\theta_p + s)^n - (a \cdot \theta_p)^n\}^{-1}$ is readily shown to be $e_r/(n(a \cdot \theta_p)^{n-1})$. Hence, by the residue theorem

$$q_n^*(u;w) = \frac{e^{-\mathcal{X}}}{n}\sum_{r=1}^{n}\exp\{ae_r\mathcal{X}\}\frac{ae_r(1-a^n)}{1-ae_r} = \frac{e^{-\mathcal{X}}}{n}\sum_{r=1}^{n}\exp\{ae_r\mathcal{X}\}\sum_{k=1}^{n}(ae_r)^k,$$

since the sum of the finite geometric series

$$\sum_{k=1}^{n}(ae_r)^k = \frac{ae_r(1-(ae_r)^n)}{1-ae_r} = \frac{ae_r(1-a^n)}{1-ae_r}\,.$$

Thus,

$$q_n^*(u;w) = \frac{e^{-\mathcal{X}}}{n}\sum_{m=0}^{\infty}\frac{\mathcal{X}^m}{m!}\sum_{k=1}^{n}\sum_{r=1}^{n}(ae_r)^{m+k} = \frac{e^{-\mathcal{X}}}{n}\sum_{m=0}^{\infty}\frac{\mathcal{X}^m}{m!}\sum_{k=1}^{n}a^{m+k}\xi_k,$$

where

$$\xi_k = \sum_{r=1}^{n}e_r^{m+k} = \frac{1-e^{2\pi i(m+k)}}{1-e^{2\pi i(m+k)/n}} = \begin{cases} n & \text{if } m = n(j+1)-k,\ j=0,1,2,\dots \\ 0 & \text{otherwise.}\end{cases}$$

This yields that

$$\begin{aligned} q_n^*(u;w) &= e^{-\mathcal{X}}\sum_{k=1}^{n}\sum_{j=0}^{\infty}\frac{\mathcal{X}^{n(j+1)-k}}{(n(j+1)-k)!}\,a^{n(j+1)} \\ &= e^{-\mathcal{X}}\sum_{k=1}^{n}\sum_{j=0}^{\infty}\frac{\mathcal{X}^{nj+k-1}}{\Gamma(nj+k)}\left(\frac{\mathcal{A}_p}{\mathcal{A}_p+w}\right)^{j+1} \end{aligned} \tag{3.9}$$

upon making the change of summation index $k \mapsto n-k+1$.

The inverse Laplace transform of $(\mathcal{A}_p/(\mathcal{A}_p+w))^{j+1}$ is $\mathcal{A}_p^{j+1}e^{-\mathcal{A}_p y}y^j/j!$. Hence, upon taking the inverse Laplace transform of the last expression with respect to w, we finally obtain that

$$q_n(u,y) = \mathcal{A}_p\exp\{-\mathcal{X}-\mathcal{A}_p y\}\sum_{k=1}^{n}\mathcal{X}^{k-1}\cdot\sum_{j=0}^{\infty}\frac{(\mathcal{A}_p\mathcal{X}^n y)^j}{j!\,\Gamma(nj+k)}\,.$$

The infinite sum which emerges on the right-hand side of the above equation is recognized as $\phi(n,k,\mathcal{A}_p\mathcal{X}^n y)$, thus proving the theorem. □

Remark 3.2. **(i)** In the case where $p = 3/2$ and for each fixed $u \in \mathbf{R}_+^1$, the distribution of τ_u was derived in [3, Section 8.4, formulas (11)–(12)]. Namely, it follows from [3] that given $u \in \mathbf{R}_+^1$, the law of the non-negative r.v. τ_u is absolutely continuous and possesses the following p.d.f.:

$$\mathcal{A}_{3/2}\cdot e^{-(\theta_{3/2}\cdot u+\mathcal{A}_{3/2}\cdot y)}\cdot I_0(4\lambda\cdot\sqrt{uy}),\quad \text{where}\quad y \geq 0. \tag{3.10}$$

We point out that $I_0(2\cdot z^{1/2}) \equiv \phi(1,1,z)$ (compare to (2.4), (2.19) and [28, formula (4.16)]). Hence, (3.10) is consistent with (3.7) when $n = 1$. Moreover, [3, p. 98] anticipated a possibility of the derivation of (3.7) for $n \geq 2$.

(ii) When ρ is non-integer, the integrand in (3.8) (with the index n replaced by ρ) has, in addition to the simple poles s_r, a branch point at $s = -\theta_p$. It then becomes necessary to introduce a branch cut in the complex s-plane along $(-\infty, -\theta_p]$, which results in an additional contribution to (3.7) when inverting the Laplace transform in the form of a complicated infinite integral.

Next, a combination of (2.18), (3.1)–(3.5) implies that the r.v.

$$\tau_0 \overset{\mathrm{d}}{=} Tw_2(\mathcal{A}_p^{-1}, 1). \tag{3.11}$$

Similar to the derivation of (3.11), a combination of (2.18) with [22, p. 204, line 11] implies that the overshoot over the level 0, i.e., γ_0, is distributed as follows:

$$\gamma_0 \overset{\mathrm{d}}{=} Tw_2(\rho_p/\theta_p, \rho_p). \tag{3.12}$$

In what follows, we will denote the p.d.f. of γ_0 by $\mathbf{f}(y)$. By (2.12),

$$\mathbf{f}(y) = \theta_p^{\rho_p} \cdot y^{\rho_p - 1} \cdot \exp\{-\theta_p y\}/\Gamma(\rho_p), \text{ where } y \in \mathbf{R}_+^1. \tag{3.13}$$

For $p = 3/2$, (3.9) implies that for a given level $u \in \mathbf{R}_+^1$, the Laplace transform

$$q_1^*(u; w) = \frac{\mathcal{A}_{3/2}}{\mathcal{A}_{3/2} + w} \cdot \exp\Big\{ - \frac{\theta_{3/2} w}{\mathcal{A}_{3/2} + w} \cdot u \Big\}, \text{ where } w > -\mathcal{A}_{3/2}. \tag{3.14}$$

Observe that a combination of (3.14) with (2.7)–(2.10) yields the following new important decomposition of the r.v. τ_u into the sum of two *independent* components:

$$\tau_u \overset{\mathrm{d}}{=} Tw_2(\mathcal{A}_{3/2}^{-1}, 1) + Tw_{3/2}(u/\mu, \lambda\sqrt{u}), \quad \text{where } \ u \in \mathbf{R}_+^1 \ \text{ is fixed.} \tag{3.15}$$

Also, it easily follows from (3.2) and (3.11) that in this case, the laws of the independent r.v.'s which appear on the right-hand side of (3.15) coincide with those of the r.v.'s τ_0 and $X_{3/2,1/\mu,\lambda}(u)$, respectively. We will pursue this observation further in Theorem 3.7.

It is common to study the FPT stochastic process $\{\tau_u, u \geq 0\}$, where the level $u \geq 0$ is regarded as the *time argument*. In this paper, we also pursue the investigation of the FPT stochastic process "*truncated*" at zero:

$$\mathcal{R}_x := \tau_x - \tau_0, \quad \text{where} \quad x \geq 0. \tag{3.16}$$

Evidently, $\mathcal{R}_0 = 0$. In addition, it follows from [22, p. 204] that $\forall x \in \mathbf{R}_+^1$,

$$\mathcal{R}_x = \begin{cases} \widehat{\tau}_{x-\gamma_0} & \text{if } \gamma_0 < x, \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

Hereinafter, $\{\widehat{\tau}_y,\ y \geq 0\}$ is the FPT process constructed according to (3.5) starting from the process $\widehat{X}_{p,\mu,\lambda}(t) := X_{p,\mu,\lambda}(t + \tau_0) - X_{p,\mu,\lambda}(\tau_0)$. Here, the time argument $t \geq 0$. It follows from [22, pp. 197 and 204] that since $\widehat{X}_{p,\mu,\lambda}(t) \overset{\mathbf{D}[0,\infty)}{=\!=} X_{p,\mu,\lambda}(t)$, the stochastic process $\widehat{\tau}_y \overset{\mathbf{D}[0,\infty)}{=\!=} \tau_y,\ y \geq 0$ (Recall that here, the level $y \geq 0$ is regarded as the time variable.)

Lemma 3.3. *For each fixed* $u \geq 0$*, the r.v.'s* $\mathcal{R}_u$ *and* τ_0 *are independent.*

Proof. By [22, p. 197, Theorem 18 and p. 204, lines 20–21], the r.v.'s $\mathcal{R}_u$ and γ_0 do not depend on the r.v. τ_0. The remainder of the proof follows from (3.17) and the fact that a constant does not depend on any r.v. □

Lemma 3.4. *Consider the stochastic process $\{\mathcal{R}_u,\ u \geq 0\}$ defined by (3.16). Then*
(i) For arbitrary fixed $\rho_p \in \mathbf{R}^1_+$ and for an arbitrary fixed $u \in \mathbf{R}^1_+$, the r.v. $\mathcal{R}_u$ has the following point mass at zero:

$$\mathbf{P}\{\mathcal{R}_u = 0\} = \Gamma(\rho_p, \mathcal{X})/\Gamma(\rho_p). \tag{3.18}$$

(ii) Suppose that $\rho\ (=\ \ \rho_p) = n \in \mathbf{N}$. Then for arbitrary $u \in \mathbf{R}^1_+$, the Lebesgue decomposition of the distribution of the non-negative r.v. $\mathcal{R}_u$ does not have a continuous-singular component. The density $g_{\mathcal{R}_u}(y)$ of the absolutely continuous component of the law of the r.v. $\mathcal{R}_u$ admits the following closed-form representation over $y \in \mathbf{R}^1_+$:

$$g_{\mathcal{R}_u}(y) = \exp\{-\mathcal{X} - \mathcal{A}_p y\} \cdot \mathcal{A}_p \cdot \mathcal{X}^n \cdot \sum_{k=1}^{n} \mathcal{X}^{k-1} \cdot \phi(n, n+k, \mathcal{A}_p \mathcal{X}^n y). \tag{3.19}$$

(iii) For $p = 3/2$, the corresponding marginals of the stochastic process $\{\mathcal{R}_u, u \geq 0\}$ coincide in law with those of the Lévy process $\{X_{3/2,1/\mu,\lambda}(u), u \geq 0\}$.

Proof. **(i)** Given $u \in \mathbf{R}^1_+$, a combination of (3.16)–(3.17) yields that the following events are identical: $\{\mathcal{R}_u = 0\} = \{\tau_u = \tau_0\} = \{\gamma_0 > u\}$. The validity of (3.18) is then obtained by combining the previous identity between the three events with (2.12) and (3.12).
(ii) First, we combine the formula of total probability with (3.13), (3.17) and Lemma 3.3. Hence, one ascertains that for fixed $x \in \mathbf{R}^1_+$ and $u \in \mathbf{R}^1_+$,

$$\begin{aligned}\mathbf{P}\{\mathcal{R}_x \leq u\} &= \int_0^\infty \mathbf{P}\{\mathcal{R}_x \leq u | \gamma_0 = v\}\mathbf{f}(v) \cdot dv = \int_0^x \mathbf{P}\{\widehat{\tau}_{x-v} \leq u\}\mathbf{f}(v) dv \\ &+ \int_x^\infty \mathbf{P}\{0 \leq u | \gamma_0 = v\}\mathbf{f}(v) dv = \mathbf{P}\{Tw_2(n/\theta_p, n) > x\} \\ &\quad + \int_0^x \mathbf{P}\{\tau_{x-v} \leq u\} \cdot \Gamma(n)^{-1} \cdot \theta_p^n \cdot v^{n-1} \cdot e^{-\theta_p v} \cdot dv.\end{aligned} \tag{3.20}$$

We differentiate the rightmost expression in (3.20) with respect to u and recall (3.7) to obtain that $\forall x \in \mathbf{R}^1_+$ and $\forall u \in \mathbf{R}^1_+$,

$$\begin{aligned}\frac{d}{du}\, \mathbf{P}\{\mathcal{R}_x \leq u\} &= \Gamma(n)^{-1}\theta_p^n \cdot \int_0^x q_{n,x-v}(u) \cdot v^{n-1} \cdot \exp\{-\theta_p v\} \cdot dv \\ &= \Gamma(n)^{-1}\theta_p^n \mathcal{A}_p \cdot \exp\{-\theta_p x - \mathcal{A}_p u\} \cdot \sum_{k=1}^{n} \theta_p{}^{k-1} \\ &\quad \times \int_0^x (x-v)^{k-1} v^{n-1} \cdot \phi(n, k, \mathcal{A}_p \theta_p^n (x-v)^n \cdot u) \cdot dv.\end{aligned} \tag{3.21}$$

Next, we make the change of variable $w = \theta_p \cdot v$. Hence, the expression which emerges on the right-hand side of (3.21) can be rewritten as follows:

$$\begin{aligned}&\frac{\mathcal{A}_p}{\Gamma(n)} e^{-\mathcal{X} - \mathcal{A}_p u} \sum_{k=1}^{n} \int_0^{\mathcal{X}} (\mathcal{X} - w)^{k-1} w^{n-1} \phi(n, k, \mathcal{A}_p (\mathcal{X} - w)^n u)\, dw \\ &= \frac{\mathcal{A}_p}{\Gamma(n)} e^{-\mathcal{X} - \mathcal{A}_p u} \sum_{k=1}^{n} \sum_{r=0}^{\infty} \frac{(\mathcal{A}_p u)^r}{r!\, \Gamma(nr+k)} \int_0^{\mathcal{X}} (\mathcal{X} - w)^{nr+k-1} w^{n-1} dw\end{aligned}$$

$$= \frac{\mathcal{A}_p}{\Gamma(n)} e^{-\mathcal{X}-\mathcal{A}_p u} \sum_{k=1}^{n} \mathcal{X}^{n+k-1} \sum_{r=0}^{\infty} \frac{(\mathcal{A}_p u \mathcal{X}^n)^r}{r!\,\Gamma(nr+k)} \int_0^1 (1-y)^{nr+k-1} y^{n-1} dy$$

$$= \mathcal{A}_p \mathcal{X}^n e^{-\mathcal{X}-\mathcal{A}_p u} \sum_{k=1}^{n} \mathcal{X}^{k-1} \sum_{r=0}^{\infty} \frac{(\mathcal{A}_p u \mathcal{X}^n)^r}{r!\,\Gamma(nr+n+k)}$$

upon evaluation of the integral as a beta function. Identification of the inner sum in terms of the Wright function (2.1) then shows that this equals the expression on the right-hand side of (3.19).

To complete the proof of part (ii), we point out that it easily follows from (3.16) that the r.v. $\mathcal{R}_u$ is non-negative, since $\tau_u \geq \tau_0$. Also, (2.1) yields that the "density component" $g_{\mathcal{R}_u}(y)$ is strictly positive on $\mathbf{R}^1_+$. Hence, it remains to demonstrate that the law of the r.v. $\mathcal{R}_u$ is comprised of the point mass at zero given by (3.18) and the absolutely continuous component whose "density" is specified by (3.19). To this end, fix an integer $n \in \mathbf{N}$. Recall that $\mathcal{X} = \theta_p \cdot u$, and consider the following function:

$$\mathcal{F}_n(\mathcal{X}) := \mathbf{P}\{\mathcal{R}_u = 0\} + \int_0^{\infty} g_{\mathcal{R}_u}(y)\, dy$$

$$= \frac{\Gamma(n,\mathcal{X})}{\Gamma(n)} + \int_0^{\infty} e^{-\mathcal{X}-\mathcal{A}_p y} \mathcal{A}_p \mathcal{X}^n \sum_{k=1}^{n} \mathcal{X}^{k-1} \phi(n, n+k, \mathcal{A}_p \mathcal{X}^n y)\, dy$$

by (3.18)–(3.19). Next, the integral term can be evaluated as follows:

$$\mathcal{A}_p \mathcal{X}^n e^{-\mathcal{X}} \sum_{k=1}^{n} \mathcal{X}^{k-1} \int_0^{\infty} e^{-\mathcal{A}_p y} \phi(n, n+k, \mathcal{A}_p \mathcal{X}^n y)\, dy$$

$$= \mathcal{A}_p \mathcal{X}^n e^{-\mathcal{X}} \sum_{k=1}^{n} \mathcal{X}^{k-1} \cdot \sum_{r=0}^{\infty} \frac{(\mathcal{A}_p \mathcal{X}^n)^r}{r!\Gamma(nr+n+k)} \int_0^{\infty} e^{-\mathcal{A}_p y} y^r dy$$

$$= \mathcal{X}^n e^{-\mathcal{X}} \sum_{k=1}^{n} \mathcal{X}^{k-1} \sum_{r=0}^{\infty} \frac{\mathcal{X}^{nr}}{\Gamma(nr+n+k)} = \mathcal{X}^n e^{-\mathcal{X}} \sum_{r=0}^{\infty} \mathcal{X}^{nr} \sum_{k=1}^{n} \frac{\mathcal{X}^{k-1}}{\Gamma(nr+n+k)}.$$

At the same time, the ratio

$$\frac{\Gamma(n,\mathcal{X})}{\Gamma(n)} = 1 - \mathcal{X}^n e^{-\mathcal{X}} \sum_{r=0}^{\infty} \frac{\mathcal{X}^r}{\Gamma(r+n+1)}.$$

In this last sum we divide the index of summation r into $(0, n, 2n, \ldots)$, $(1, n+1, 2n+1, \ldots)$, ... , $(n-1, 2n-1, 3n-1, \ldots)$ to obtain that $\forall n \geq 1$, the subtrahend

$$\mathcal{X}^n e^{-\mathcal{X}} \sum_{r=0}^{\infty} \frac{\mathcal{X}^r}{\Gamma(r+n+1)} = \mathcal{X}^n e^{-\mathcal{X}}$$

$$\times \left\{ \sum_{r=0}^{\infty} \frac{\mathcal{X}^{nr}}{\Gamma(nr+n+1)} + \sum_{r=0}^{\infty} \frac{\mathcal{X}^{nr+1}}{\Gamma(nr+n+2)} + \cdots + \sum_{r=0}^{\infty} \frac{\mathcal{X}^{nr+n-1}}{\Gamma(nr+2n)} \right\}$$

$$= \mathcal{X}^n e^{-\mathcal{X}} \sum_{r=0}^{\infty} \mathcal{X}^{nr} \sum_{k=1}^{n} \frac{\mathcal{X}^{k-1}}{\Gamma(nr+n+k)}.$$

Hence, $\mathcal{F}_n(\mathcal{X}) \equiv 1$, which excludes the existence of a continuous-singular component.
(iii) For $u = 0$, this is trivial. Hence, suppose that $u \in \mathbf{R}^1_+$. A combination of (2.21), (3.4) and (3.18) implies that in the case where $p = 3/2$, $\mathbf{P}\{\mathcal{R}_u = 0\} = \Gamma(1, \theta_{3/2}u)/\Gamma(1) = e^{-\theta_{3/2} \cdot u} = \mathbf{P}\{X_{3/2,1/\mu,\lambda}(u) = 0\}$. Therefore, the values of the point mass at zero for the r.v.'s $\mathcal{R}_u$ and $X_{3/2,1/\mu,\lambda}(u)$ coincide.

Next, the closed-form expression for the density component of the r.v. $X_{3/2,1/\mu,\lambda}(u)$ is obtained by combining (2.20)–(2.21) with (3.3). The verification of the fact that it is identical to that of the r.v. $\mathcal{R}_u$ in the case where $n = 1$ (which emerges on the right-hand side of (3.19)) is straightforward and involves an application of (2.3)–(2.4). □

In turn, Lemma 3.4.iii yields the following corollary, which pertains to the "density components" $g_{\mathcal{R}_u}(\cdot)$ and $f_{X_{3/2,\mu,\lambda}(t)}(\cdot)$ of the r.v.'s $\mathcal{R}_u$ and $X_{3/2,\mu,\lambda}(t)$, respectively.

Corollary 3.5. *Suppose that $\rho_p = n = 1$, and fix arbitrary $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$. Then given $t > 0$ and $u > 0$,*

$$\begin{aligned} t \cdot g_{\mathcal{R}_u}(t) &= t \cdot f_{X_{3/2,1/\mu,\lambda}(u)}(t) = t \cdot f_{3/2,u/\mu,\lambda\sqrt{u}}(t) \\ &= u \cdot f_{3/2,\mu t,\lambda\sqrt{t}}(u) = u \cdot f_{X_{3/2,\mu,\lambda}(t)}(u). \end{aligned} \tag{3.22}$$

Proof. It follows from a combination of Lemma 3.4.iii, (2.19), (3.2) and some algebra. □

Remark 3.6. It relatively easily follows from [22, p. 204] that for each $p \in (1, 2)$, the "truncated" FPT stochastic process $\{\mathcal{R}_u, u \geq 0\}$ is a process with independent increments which starts from zero such that for each value of the "time variable" $u \in \mathbf{R}^1_+$, the r.v. $\mathcal{R}_u$ has a positive mass at zero. However, $\forall p \in (1, 2) \setminus \{3/2\}$, it is not a compound Poisson *Lévy* process. This is obtained by combining (3.18) with the well-known fact that the expression on the right-hand side of (3.18) is equivalent to $(\theta_p u)^{\rho_p - 1} \cdot e^{-\theta_p u}/\Gamma(\rho_p)$ as $u \to \infty$. It remains to employ a contradiction argument, since in view of this asymptotics, the logarithm of the expression (3.18) is a linear function of the argument u if and only if $p = 3/2$. Clearly, this contradicts (3.4).

The following assertion partly supports the statement made in [19, p. 24, lines 19–20] in the self-reciprocal case of $p = 3/2$.

Theorem 3.7. *Assume that $\rho_p = n = 1$, and fix arbitrary $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$. Suppose that the "time" variable $u \in [0, \infty)$. Then $\mathcal{R}_u \overset{\mathbf{D}[0,\infty)}{=\!=\!=} X_{3/2,1/\mu,\lambda}(u)$.*

Proof. First, fix an arbitrary $u > 0$. By Lemma 3.4.iii, the r.v. $\mathcal{R}_u \overset{\mathrm{d}}{=} X_{3/2,1/\mu,\lambda}(u)$. Next, since by Remark 3.6, the "truncated" FPT process $\{\mathcal{R}_u, u \geq 0\}$ (which starts from the origin) has independent increments, it follows with some effort from a combination of (3.2) with [24, Theorem 2.1] and [26, Proposition 3.2] that for $p = 3/2$ (or $\rho_p = 1$), this process also has *stationary* increments. Hence, it is a compound Poisson-exponential Lévy process such that $\mathcal{R}_1 \overset{\mathrm{d}}{=} X_{3/2,1/\mu,\lambda}(1) \overset{\mathrm{d}}{=} Tw_{3/2}(1/\mu, \lambda)$. The rest is straightforward, since by [1, p. 11], a Lévy process is completely characterized by its increment in unit time. □

Remark 3.8. Corollary 3.5 and Theorem 3.7 provide the probabilistic interpretation of the self-reciprocity of the Poisson-exponential class of distributions in terms of the fluctuation properties of the related Poisson-exponential Lévy processes. In contrast to [23, Theorem 4.1], which holds for Hougaard processes constructed starting from the spectrally negative stable laws with index $\alpha \in (1,2]$ and skewness $\beta = -1$, here one should subtract τ_0 in Lemma 3.4.iii, Corollary 3.5, and Theorem 3.7 in order to ascertain that the marginals of such modified FPT processes remain within the PVF. Nevertheless, Theorem 3.7 is similar to the fluctuation property of the class of the exponentially tilted spectrally negative stable processes with $1 < \alpha \leq 2$. In the latter case, the analogue of (3.22) goes back to [29, p. 203, line 1]. See also [21, formula (46.12)].

The following result is an asymptotic version of (3.22), which is valid for any $\rho_p \geq 1$.

Theorem 3.9. *Suppose that $p = 1+1/(n+1)$, where $n \geq 1$ is a fixed integer, real $u > 0$, and the time variable $t \to +\infty$. Then the "density components" $f_{X_{p,\mu,\lambda}(t)}(\cdot)$ and $g_{\mathcal{R}_u}(\cdot)$ of the r.v.'s $X_{p,\mu,\lambda}(t)$ and $\mathcal{R}_u$, respectively, are related as follows:*

$$t \cdot g_{\mathcal{R}_u}(t) \sim n^{-1} u f_{X_{p,\mu,\lambda}(t)}(u). \tag{3.23}$$

Proof. Set $z := t\mathcal{A}_p\mathcal{X}^n = t\mathcal{A}_p\theta_p^n u^n$. A combination of Lemma 3.4 with (2.3) yields that

$$\begin{aligned} t \cdot g_{\mathcal{R}_u}(t) &= e^{-\mathcal{A}_p t - \theta_p u} \cdot t \cdot \frac{d}{dt}\, \phi(n,k,\mathcal{A}_p \cdot \mathcal{X}^n \cdot t) \\ \times \sum_{k=1}^{n} \mathcal{X}^{k-1} \frac{\phi(n,n+k,\mathcal{A}_p\mathcal{X}^n t)}{\phi(n,n+1,\mathcal{A}_p\mathcal{X}^n t)} &\sim e^{-\mathcal{A}_p t - \theta_p u} \cdot t\mathcal{A}_p\mathcal{X}^n \cdot \phi'(n,1,z)|_{z=t\mathcal{A}_p\mathcal{X}^n} \end{aligned} \tag{3.24}$$

as $t \to +\infty$, since by (2.3), the finite sum which emerges in (3.24) is either 1 (if $n = 1$) or

$$1 + \sum_{k=2}^{n} \mathcal{O}((\mathcal{X}/t)^{(k-1)/(n+1)})$$

in this limit (if $n \geq 2$). A subsequent combination of (2.4) with (3.24) stipulates that

$$t \cdot g_{\mathcal{R}_u}(t) \sim \frac{1}{n} \cdot e^{-\mathcal{A}_p t - \theta_p u} \cdot \phi(n,0,\mathcal{A}_p\theta_p^n t \cdot u^n) \quad \text{as} \quad t \to +\infty. \tag{3.25}$$

To complete the proof, it remains to combine (3.3) and (3.25), since $\rho = n$. ☐

Remark 3.10. In the case where $p = 3/2$, (3.23) is consistent with (3.22).

Next, for arbitrary real $u \geq 0$ and $\mathcal{L} \geq 0$, we introduce the family of the *incremental* stochastic processes which is indexed by the level u:

$$\mathcal{I}_u(\mathcal{L}) := \mathcal{R}_{u+\mathcal{L}} - \mathcal{R}_u = \tau_{u+\mathcal{L}} - \tau_u. \tag{3.26}$$

Hereinafter, $\mathcal{L}$ is recognized as the "time" argument.

Lemma 3.11. *Fix $\rho_p = n \geq 1$, and suppose that the level $u \to +\infty$. Then*

$$\mathbf{E}\mathcal{R}_u = \frac{\mathcal{X}}{n\mathcal{A}_p} - \frac{n-1}{2n\mathcal{A}_p} + \mathcal{O}\Big(\frac{1}{\mathcal{X}}\Big) \sim \frac{\rho_p u}{\mu} = \rho_p \cdot \mathbf{E}X_{p',\mu',\lambda'}(u); \tag{3.27}$$

$$\begin{aligned} \mathbf{Var}\mathcal{R}_u &= \frac{(n+1)\mathcal{X}}{(n\mathcal{A}_p)^2} - \frac{5n^2-6n+1}{12(n\mathcal{A}_p)^2} + \mathcal{O}\Big(\frac{1}{\mathcal{X}}\Big) \\ &\sim \frac{\rho_p u}{\lambda\mu^{3-p}} = \rho_p \cdot \mathbf{Var}X_{p',\mu',\lambda'}(u). \end{aligned} \tag{3.28}$$

Proof. First, a combination of (3.9), (3.11) and Lemma 3.3 yields that the moment-generating function $Q_n^*(u;v)$ of the r.v. $\mathcal{R}_u$ (with the argument $v = -w < \mathcal{A}_p$) is as follows:

$$Q_n^*(u;v) = e^{-\mathcal{X}} \sum_{k=1}^{n} \sum_{j=0}^{\infty} \frac{\mathcal{X}^{nj+k-1}}{\Gamma(nj+k)} \left(\frac{\mathcal{A}_p}{\mathcal{A}_p - v} \right)^j . \tag{3.29}$$

We differentiate (3.29) with respect to v at $v = 0$ to obtain the mean given by

$$\mathbf{E}\mathcal{R}_u = \frac{e^{-\mathcal{X}}}{\mathcal{A}_p} \sum_{k=1}^{n} \mathcal{X}^{k-1} \sum_{j=0}^{\infty} \frac{j\mathcal{X}^{nj}}{\Gamma(nj+k)} = \frac{e^{-\mathcal{X}}}{\mathcal{A}_p} \sum_{k=1}^{n} \mathcal{X}^{k-1} \Theta \mathcal{E}_{n,k}(\mathcal{X}^n),$$

where the differential operator $\Theta := \mathcal{X}^n \cdot (d/d\mathcal{X})^n$ and $\mathcal{E}_{n,k}(\mathcal{X}^n)$ is the generalized Mittag-Leffler function defined in (2.6).

Simple calculations show that

$$\begin{aligned} \Theta\mathcal{E}_{n,k}(\mathcal{X}^n) = \frac{1}{n} \sum_{j=0}^{\infty} \frac{nj\mathcal{X}^{nj}}{\Gamma(nj+k)} = \frac{1}{n} \Bigg\{ \sum_{j=0}^{\infty} \frac{\mathcal{X}^{nj}}{\Gamma(nj+k-1)} \\ - (k-1) \sum_{j=0}^{\infty} \frac{\mathcal{X}^{nj}}{\Gamma(nj+k)} \Bigg\} = \frac{1}{n} \{ \mathcal{E}_{n,k-1}(\mathcal{X}^n) - (k-1)\mathcal{E}_{n,k}(\mathcal{X}^n) \}. \end{aligned} \tag{3.30}$$

Hence,

$$\mathbf{E}\mathcal{R}_u = \frac{e^{-\mathcal{X}}}{n\mathcal{A}_p} \left\{ \sum_{k=1}^{n} \mathcal{X}^{k-1} \mathcal{E}_{n,k-1}(\mathcal{X}^n) - \sum_{k=1}^{n} (k-1) \mathcal{X}^{k-1} \mathcal{E}_{n,k}(\mathcal{X}^n) \right\}.$$

But

$$\begin{aligned} \sum_{k=1}^{n} \mathcal{X}^{k-1} \mathcal{E}_{n,k-1}(\mathcal{X}^n) &= \sum_{j=0}^{\infty} \frac{\mathcal{X}^{nj}}{\Gamma(nj)} + \sum_{j=0}^{\infty} \frac{\mathcal{X}^{nj+1}}{\Gamma(nj+1)} + \cdots + \sum_{j=0}^{\infty} \frac{\mathcal{X}^{nj+n-1}}{\Gamma(nj+n-1)} \\ &= \sum_{j=0}^{\infty} \frac{\mathcal{X}^j}{\Gamma(j)} = \mathcal{X} e^{\mathcal{X}}, \end{aligned}$$

so that

$$\mathbf{E}\mathcal{R}_u = \frac{\mathcal{X}}{n\mathcal{A}_p} - \frac{e^{-\mathcal{X}}}{n\mathcal{A}_p} \sum_{k=1}^{n} (k-1) \mathcal{X}^{k-1} \mathcal{E}_{n,k}(\mathcal{X}^n).$$

From [5, Section 18.1] we have the dominant behavior

$$\mathcal{E}_{n,k}(\mathcal{X}^n) \sim \frac{1}{n} \mathcal{X}^{1-k} e^{\mathcal{X}} \quad \text{as} \quad \mathcal{X} \to +\infty, \tag{3.31}$$

whence

$$\sum_{k=1}^{n} (k-1) \mathcal{X}^{k-1} \mathcal{E}_{n,k}(\mathcal{X}^n) \sim \frac{e^{\mathcal{X}}}{n} \sum_{k=1}^{n} (k-1) = \frac{1}{2}(n-1) e^{\mathcal{X}}.$$

It then follows that as $\mathcal{X} \to +\infty$,

$$\mathbf{E}\mathcal{R}_u \sim \frac{\mathcal{X}}{n\mathcal{A}_p} - \frac{n-1}{2n\mathcal{A}_p}, \tag{3.32}$$

which is linear in $\mathcal{X}$ (or u).

In order to find the variance, we differentiate (3.29) twice with respect to v and determine the value of the second derivative at $v = 0$. One ascertains that

$$\mathbf{E}(\mathcal{R}_u^2) = (\mathbf{E}\mathcal{R}_u)^2 + \mathbf{Var}\mathcal{R}_u = \frac{e^{-\mathcal{X}}}{\mathcal{A}_p^2} \sum_{k=1}^{n} \mathcal{X}^{k-1} \sum_{j=0}^{\infty} \frac{j(j+1)\mathcal{X}^{nj}}{\Gamma(nj+k)}$$

$$= \frac{e^{-\mathcal{X}}}{\mathcal{A}_p^2} \sum_{k=1}^{n} \mathcal{X}^{k-1} \Theta(\Theta+1) \mathcal{E}_{n,k}(\mathcal{X}^n) = \frac{\mathbf{E}\mathcal{R}_u}{\mathcal{A}_p} + \frac{e^{-\mathcal{X}}}{\mathcal{A}_p^2} \sum_{k=1}^{n} \mathcal{X}^{k-1} \Theta^2 \mathcal{E}_{n,k}(\mathcal{X}^n).$$

From (3.30)

$$\Theta^2 \mathcal{E}_{n,k}(\mathcal{X}^n) = \frac{1}{n^2} \{\mathcal{E}_{n,k-2}(\mathcal{X}^n) - (k-2)\mathcal{E}_{n,k-1}(\mathcal{X}^n)$$

$$- (k-1)(\mathcal{E}_{n,k-1}(\mathcal{X}^n) - (k-1)\mathcal{E}_{n,k}(\mathcal{X}^n))\} = \frac{1}{n^2}\{\mathcal{E}_{n,k-2}(\mathcal{X}^n)$$

$$- (2k-3)\mathcal{E}_{n,k-1}(\mathcal{X}^n) + (k-1)^2 \mathcal{E}_{n,k}(\mathcal{X}^n)\}$$

and

$$\sum_{j=0}^{\infty} \mathcal{X}^{k-1} \mathcal{E}_{n,k-2}(\mathcal{X}^n) = \sum_{j=0}^{\infty} \frac{\mathcal{X}^{nj}}{\Gamma(nj-1)} + \sum_{j=0}^{\infty} \frac{\mathcal{X}^{nj+1}}{\Gamma(nj)} + \cdots + \sum_{j=0}^{\infty} \frac{\mathcal{X}^{nj+n-1}}{\Gamma(nj+n-2)}$$

$$= \sum_{j=0}^{\infty} \frac{\mathcal{X}^j}{\Gamma(j-1)} = \sum_{j=0}^{\infty} \frac{\mathcal{X}^{j+2}}{j!} = \mathcal{X}^2 e^{\mathcal{X}}.$$

Hence,

$$\mathbf{Var}\mathcal{R}_u = \frac{\mathcal{X}^2}{(n\mathcal{A}_p)^2} + \frac{\mathbf{E}\mathcal{R}_u}{\mathcal{A}_p} - (\mathbf{E}\mathcal{R}_u)^2$$

$$+ \frac{e^{-\mathcal{X}}}{(n\mathcal{A}_p)^2} \sum_{k=1}^{n} \mathcal{X}^{k-1} \{(k-1)^2 \mathcal{E}_{n,k}(\mathcal{X}^n) - (2k-3)\mathcal{E}_{n,k-1}(\mathcal{X}^n)\}.$$

From (3.32) we easily obtain the following asymptotic relation as $\mathcal{X} \to +\infty$:

$$(\mathbf{E}\mathcal{R}_u)^2 \sim \frac{\mathcal{X}^2}{(n\mathcal{A}_p)^2} - \frac{(n-1)\mathcal{X}}{(n\mathcal{A}_p)^2} + \frac{(n-1)^2}{(2n\mathcal{A}_p)^2}. \tag{3.33}$$

A subsequent combination of (3.31) and (3.33) yields that as $\mathcal{X} \to +\infty$,

$$\mathbf{Var}\mathcal{R}_u \sim \frac{(2n-1)\mathcal{X}}{(n\mathcal{A}_p)^2} - \frac{(n-1)(3n-1)}{(2n\mathcal{A}_p)^2}$$

$$+ \frac{e^{-\mathcal{X}}}{(n\mathcal{A}_p)^2} \sum_{k=1}^{n} \mathcal{X}^{k-1} \{(k-1)^2 \mathcal{E}_{n,k}(\mathcal{X}^n) - (2k-3)\mathcal{E}_{n,k-1}(\mathcal{X}^n)\}$$

$$\sim \frac{(2n-1)\mathcal{X}}{(n\mathcal{A}_p)^2} - \frac{(n-1)(3n-1)}{(2n\mathcal{A}_p)^2} + \frac{1}{n^3 \mathcal{A}_p^2} \sum_{k=1}^{n} \{(k-1)^2 - (2k-3)\mathcal{X}\}.$$

It remains to combine this result with the evaluations $\sum_{k=1}^{n}(k-1)^2 = n^3/3 - n^2/2 + n/6$ and $\sum_{k=1}^{n}(2k-3) = n(n-2)$ to find that

$$\mathbf{Var}\mathcal{R}_u \sim \frac{(n+1)\mathcal{X}}{(n\mathcal{A}_p)^2} - \frac{5n^2 - 6n + 1}{12(n\mathcal{A}_p)^2} \quad \text{as} \quad \mathcal{X} \to +\infty.$$

□

Theorem 3.12. *Fix $\rho_p = n \geq 1$, and real $\mathcal{L} > 0$. Suppose that the level $u \to +\infty$. Then (i) the family of positive r.v.'s*

$$\mathcal{I}_u(\mathcal{L}) \overset{\mathrm{d}}{\to} X_{p',\mu',\lambda'}(\mathcal{L}) \ (\overset{\mathrm{d}}{=} Tw_{3-p}(\mathcal{L}/\mu, \lambda\mathcal{L}^{2-p})). \tag{3.34}$$

(ii) The first two moments of the incremental process $\mathcal{I}_u(\mathcal{L})$ possess the following asymptotics:

$$\mathbf{E}\mathcal{I}_u(\mathcal{L}) \to \rho_p \cdot \mathbf{E}X_{p',\mu',\lambda'}(\mathcal{L}) = \rho_p\mathcal{L}/\mu; \tag{3.35}$$

$$\mathbf{Var}\mathcal{I}_u(\mathcal{L}) \to \rho_p \cdot \mathbf{Var}X_{p',\mu',\lambda'}(\mathcal{L}) = \rho_p\mathcal{L}\mu^{p-3}/\lambda. \tag{3.36}$$

Proof. (i) It suffices to establish that for all fixed real $v \geq 0$ and $\mathcal{L} > 0$, and as $u \to +\infty$,

$$\begin{aligned}\log Q_n^*(u+\mathcal{L}; v) - \log Q_n^*(u; v) \to \theta_p\mathcal{L} \cdot \Big\{\big(1 - \frac{v}{\mathcal{A}_p}\big)^{-1/n} - 1\Big\} \\ (= \zeta_{3-p,\mathcal{L}/\mu,\mathcal{L}^{2-p}\lambda}(v)).\end{aligned} \tag{3.37}$$

To this end, observe that $a = (\mathcal{A}_p/(\mathcal{A}_p - v))^{1/n}$ with $v < \mathcal{A}_p$ fixed and finite. Then

$$\begin{aligned} Q_n^*(u; v) &= e^{-\mathcal{X}} \sum_{k=1}^{n} \mathcal{X}^{k-1} \sum_{j=0}^{\infty} \frac{\mathcal{X}^{nj}}{\Gamma(nj+k)} \left(\frac{\mathcal{A}_p}{\mathcal{A}_p - v}\right)^j \\ &= e^{-\mathcal{X}} \sum_{k=1}^{n} \mathcal{X}^{k-1} \sum_{j=0}^{\infty} \frac{(a\mathcal{X})^{nj}}{\Gamma(nj+k)} = e^{-\mathcal{X}} \sum_{k=1}^{n} \mathcal{X}^{k-1} \mathcal{E}_{n,k}((a\mathcal{X})^n). \end{aligned}$$

Now, from the asymptotic behavior in (3.31) we see that, as $X \to +\infty$,

$$Q_n^*(u; v) \sim \frac{1}{n} e^{a\mathcal{X} - \mathcal{X}} \sum_{k=1}^{n} a^{1-k} = e^{(a-1)\mathcal{X}} \frac{a^{-n} - 1}{n(a^{-1} - 1)} \tag{3.38}$$

and so, as $\mathcal{X} \to +\infty$,

$$\log\, Q_n^*(u; v) = \mathcal{X}(a-1) + \mathcal{O}(1) = \mathcal{X}\left\{\left(1 - \frac{v}{\mathcal{A}_p}\right)^{-1/n} - 1\right\} + \mathcal{O}(1).$$

Let $\hat{X} := \theta(u + \mathcal{L})$. Then it follows from (3.38) that

$$Q_n^*(u+\mathcal{L}; v) \sim e^{(a-1)\hat{X}}\, \frac{a^{-n} - 1}{n(a^{-1} - 1)}$$

as $u \to +\infty$. Hence,

$$\begin{aligned} \log\, Q_n^*(u+\mathcal{L}; v) - \log\, Q_n^*(u; v) &\sim (a-1)\{\hat{X} - \mathcal{X}\} = \{\theta(u+\mathcal{L}) - \theta u\}\,(a-1) \\ &= \theta_p\mathcal{L} \cdot \Big\{\big(1 - \frac{v}{\mathcal{A}_p}\big)^{-1/n} - 1\Big\} \quad \text{as} \quad u \to +\infty. \end{aligned}$$

(ii) The validity of (3.35) and (3.36) easily follows by combining (3.26), the facts that in view of Remark 3.6 and (3.1), $\{\mathcal{R}_u,\ u \geq 0\}$ and $\{X_{p',\mu',\lambda'}(u),\ u \geq 0\}$ are the processes with *independent* increments which have *additive* mean and variance, with the leftmost representations in (3.27) and (3.28), respectively. □

Remark 3.13. The formula (3.34) implies that the univariate increments of the FPT process "truncated" at zero converge to the corresponding increments of the "reciprocal" compound Poisson-gamma Lévy process $X_{3-p,1/\mu,\lambda}(\cdot)$ as the starting point (i.e., u) of all such increments approaches $+\infty$.

It is well known that the mean-square convergence implies convergence in the mean, and the latter implies weak convergence. There are numerous counter-examples for which weak convergence holds but convergence in the mean does not take place. Theorem 3.12 provides a new natural counter-example, since (3.34) yields weak convergence, whereas (3.35)–(3.36) stipulate that neither the limit in mean nor the mean-square limit coincide with the weak limit. This is due to an additional factor ρ_p in the middle expressions of these formulas, which is generally not 1.

An interesting open problem is to determine the limit of the family of the incremental stochastic processes $\{\mathcal{I}_u(\mathcal{L}),\ \mathcal{L} \geq 0\}$ (which is defined by (3.26)) as $u \to +\infty$ in the sense of J-convergence, i.e., convergence of càdlàg stochastic processes in topology generated by the Skorohod metric (see [11, Chapter VI] for more detail on such convergence). We conjecture that the corresponding limit should be different from the compound Poisson-gamma Lévy process $\{X_{p',\mu',\lambda'}(\mathcal{L}),\ \mathcal{L} \geq 0\}$.

Theorem 3.14. *Suppose that* $n = 1$. *Then*
(i) For each fixed $u \geq 0$, *the overshoot* γ_u *defined by (3.6) is an exponential r.v. with mean* $\theta_{3/2}^{-1}$ *and hence its probability law is invariant with respect to level* u.
(ii) For each fixed $u \geq 0$, *the r.v.'s* τ_u *and* γ_u *are independent.*

Proof. **(i)** In the case where $u = 0$, this assertion coincides with (3.12). For $u \in \mathbf{R}^1_+$, we apply the *Pecherskii–Rogozin identity* (cf., for example, [17, Theorem 2]). In what follows, we slightly modify the notation developed in [17] when necessary. For simplicity, the rest of the proof is carried out in the case where $\mu = \lambda = 1$.

Now, [17, Theorem 1.b] implies that for an arbitrary fixed $w \in \mathbf{R}^1_+$, the Wiener–Hopf factor $\Phi^+_w(iz)$, which corresponds to the Laplace transform of the (exponentially stopped) Poisson-exponential Lévy process $X_{3/2,1,1}(t)$, is as follows:

$$\Phi^+_w(iz) = 1/(w + 2z/(2+z)). \tag{3.39}$$

A subsequent combination of (3.39) with [17, Theorem 2] yields that for arbitrary $w \in \mathbf{R}^1_+$, $\psi \in \mathbf{R}^1_+$ and $z \in \mathbf{R}^1_+$ such that $\psi \neq z$,

$$\begin{aligned}\int_0^\infty e^{-\psi u} \cdot \mathbf{E}e^{-w\tau_u - z\gamma_u} \cdot du &= (1 - \Phi^+_w(i\psi)/\Phi^+_w(iz))/(\psi - z) \\ &= \frac{1}{\psi - z} \cdot \frac{2\psi/(2+\psi) - 2z/(2+z)}{w + 2\psi/(2+\psi)}\ .\end{aligned} \tag{3.40}$$

Next, we may take the limit as $w \downarrow 0$ in (3.40) to obtain that

$$\int_0^\infty e^{-\psi u} \cdot \mathbf{E}e^{-z\gamma_u} du = \frac{1}{\psi - z} \cdot \Big(1 - \frac{z/(2+z)}{\psi/(2+\psi)}\Big) = \frac{1}{\psi(1+z/2)}\ . \tag{3.41}$$

The rest follows from the fact that the rightmost expression in (3.41) is the double Laplace transform of the exponential r.v. $Tw_2(1/2, 1)$ with mean $1/2$.
(ii) For $u = 0$, this is obvious (see also [22, p. 197, Theorem 18]). For $u \in \mathbf{R}^1_+$, (3.14)

yields that the Laplace transform $\mathbf{E}e^{-w\tau_u}$ can be factored as follows:

$$\mathbf{E}e^{-w\tau_u} = q_1^*(u;w) = (2/(2+w))\cdot\exp\{-u\cdot 2w/(2+w)\}, \quad \text{where} \quad w > -2. \tag{3.42}$$

Next, we multiply the expression on the right-hand side of (3.42) by the Laplace transform $1/(1+z/2)$ of the r.v. $\gamma_u \stackrel{\mathrm{d}}{=} Tw_2(1/2,1)$ (see part (i)). Subsequently, consider the Laplace transform (in the new variable ψ) of this product. The verification of the fact that the result of this transformation coincides with the closed-form expression for the double Laplace transform which emerges on the right-hand side of (3.40) is straightforward and left to the reader.

The rest follows by employing a characterization of the independence of a pair of generic positive r.v.'s $(\mathcal{U},\mathcal{Y})$ in terms of existence of the following factorization of their joint Laplace transform into a product of the individual Laplace transforms of these r.v.'s: $\mathbf{E}e^{-v\mathcal{U}-w\mathcal{Y}} = \mathbf{E}e^{-v\mathcal{U}}\cdot\mathbf{E}e^{-w\mathcal{Y}}$ (cf., for example, [2, Exercise 5.21]). Here, v and w take values in $[0,+\infty)$. □

Remark 3.15. Certain results which are similar in spirit to Theorem 3.14 are well known (cf., for example, [13, p. 27]). However, we have not found this assertion in the literature and decided to illustrate herewith [17].

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