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OPERATOR FUNCTIONS GENERATED BY FUNCTIONALS AND THEIR APPLICATION TO PROBABILITY THEORY

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Abstract. We consider special operator functions generated by analytic functionals. For these functions, the variational derivatives and ordinary derivatives are calculated. A formula is obtained for solving the initial problem for a linear inhomogeneous system of differential equations with ordinary and variational derivatives. A general explicit formula is obtained for the mathematical expectation of the solution of a linear inhomogeneous system of ordinary differential equations, the coefficients of which are random processes. As an example, a calculation is made for a system with Gaussian and uniformly distributed random processes.

1. Operator functions

Let $\mathbb{R}$ be a set of real numbers, $t \in \mathbb{R}, X$ be a finite-dimensional linear space with the norm $\|\cdot\|$ and $L(X,X)$ space of linear operators acting in the space X with the norm $\|A\|_L$.

In mathematics, operator functions generated by analytical functions play an important role [1]. For example, if $A \in L(X,X), E$ is the identical operator, $\exp A = E + A + \frac{A^2}{2!} + ...$ Operator function $\exp(At)$ is a solution of the Cauchy operator problem

$$\frac{dY}{dt} = AY, Y(0) = E,$$

where $Y : \mathbb{R} \to L(X,X)$ the required operator function [1].

Let $L_1(T)$ be the space of summable functions on the segment $T = [t_0, t_1]$ with the norm $\|u\|_1 = \int_T |u(t)|dt$ and $f : L_1(T) \to \mathbb{C}$ have the form

$$f(u) = \int_T ... \int_T b(s_1, s_2, ..., s_k)u_1(s_1)u(s_2)...u(s_k)ds_1 ds_2 ... ds_k,$$

where b is a function that is symmetric for each pair of variables.

Let $\chi(s) = \chi(t_0, t, s)$ be a function of the variable $s \in \mathbb{R}$ defined as follows: $\chi(t_0, t, s) = sign(s - t_0)$ if s belongs to a segment $[min\{t_0, , t\}, max\{t_0, t\}]$ and $\chi(t_0, t, s) = 0$, if s no belongs to a segment. Note that equality is true

$$\chi(t_0, t + \Delta t, s) = \chi(t_0, t, s) + \chi(t, t + \Delta t, s). \quad (1)$$

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We will use the concept of variational derivative. Recall its definition [2, p. 14]. Let $Y : L_1(T) \to L(X,X), u \in L_1(T), h \in L_1(T)$. If

$$Y(u+h) - Y(u) = \int_T \varphi(t,u)h(t)dt + o(h),$$

where integral (Lebesgue) is a linear bounded on $L_1(T)$ operator and $o(h)$ is infinitesimal of higher order relative to h, then $\varphi : T \times L_1(T) \to L(X,X)$ is called the variational (functional) derivative of mapping Y at the point u and is denoted by $\frac{\delta Y(u)}{\delta u(t)}$.

Consider the special operator function

$$\begin{gathered} Y(t,u) = f(uE + a\chi(t_0,t)A) = \\ = \int_T \dots \int_T b(s_1, s_2, \dots s_k)(u(s_1)E \\ +a(s_1)\chi(t_0,t,s_1)A)(u(s_2)E + a(s_2)\chi(t_0,t,s_2)A)\dots(u(s_k)E \\ +a(s_k)\chi(t_0,t,s_k)A)ds_1 ds_2 \dots ds_k. \end{gathered}$$

We have not seen such functions in the literature.

Further $\frac{\delta_p Y(t,u)}{\delta u(t)}$ denotes the partial variational derivative with respect to the variable u.

Theorem 1.1. *If a is a continuous function on T and b is a continuous function that is symmetric for each pair of variables, then there exists the derivative $\frac{\delta_p Y(t,u)}{\delta u(t)}$ and*

$$\begin{gathered} \frac{\delta_p Y(t,u)}{\delta u(t)} = k \int_T \dots \int_T b(s_1, s_2, \dots s_{k-1}, t)(u(s_1)E + a(s_1)\chi(t_0,t,s_1)A)(u(s_2)E \\ +a(s_2)\chi(t_0,t,s_2)A)\dots(u(s_{k-1})E + a(s_{k-1})\chi(t_0,t,s_{k-1})A)ds_1 ds_2 \dots ds_{k-1}. \end{gathered} \tag{2}$$

Proof. Let $h \in L_1(T)$ be an increment of the variable u. Since b is a function that is symmetric for each pair of variables, then

$$\begin{gathered} Y(t,u+h) - Y(t,u) = \\ = \int_T \dots \int_T b(s_1, s_2, \dots, s_k)[(u(s_1)E + a(s_1)\chi(t_0,t,s_1)A \\ +h(s_1)E))(u(s_2)E + a(s_2)\chi(t_0,t,s_2)A \\ +h(s_2)E)\dots(u(s_k)E + a(s_k)\chi(t_0,t,s_k)A + h(s_k)E)ds_1 ds_2 \dots ds_k - \\ (u(s_1)E + a(s_1)\chi(t_0,t,s_1)A)(u(s_2)E + a(s_2)\chi(t_0,t,s_2)A)\dots(u(s_k)E \\ +a(s_k)\chi(t_0,t,s_k)A)]ds_1 ds_2 \dots ds_k = \\ = k \int_T \dots \int_T b(s_1, s_2, \dots, s_{(k-1)}, t)[(u(s_1)E + a(s_1)\chi(t_0,t,s_1)A)\dots(u(s_{k-1})E \\ +a(s_{k-1})\chi(t_0,t,s_{k-1})A)h(s_k)E ds_1 ds_2 \dots ds_{k-1} + \\ + \int_T \dots \int_T \sum_{m=2}^{k} C_k^m (u(s_1)E + a(s_1)\chi(t_0,t,s_1)A)\dots(u(s_{k-m})E \\ +a(s_{k-m})\chi(t_0,t,s_{k-m})A)h(s_{k-m+1})\dots h(s_k)E ds_1 ds_2 \dots ds_k. \end{gathered}$$

Here C_k^m is the number of combinations of k by m. Let's estimate the norm of the last term

$$\frac{1}{\|h\|_1}\|\int_T ... \int_T \sum_{m=2}^{k} C_k^m (u(s_1)E + a(s_1)\chi(t_0,t,s_1)A)...(u(s_{k-m})E$$
$$+a(s_{k-m})\chi(t_0,t,s_{k-m})A)h(s_{k-m+1})...h(s_k)Eds_1ds_2...ds_k\|_L \leq$$
$$\leq \frac{1}{\|h\|_1} C_k^m B(\||u(s_1)| + max_{s_1\in T}|a(s_1)|\|A\|_L\|_{L_1})^{k-m}\|h\|_1^m \to 0$$

for $\|h\|_1 \to 0, m \geq 2$. Here $B = max_{s_i\in T}|b(s_1,s_2,...,s_k)|$. According to the definition of the variational derivative, $\frac{\delta_p Y(t,u)}{\delta u(t)}$ exists and the equality (2) is valid. The theorem is proved.

Theorem 1.2. *Under the conditions of theorem 1, there exists the partial derivative* $\frac{\partial Y(t,u)}{\partial t}$*, while*

$$\frac{\partial Y(t,u)}{\partial t} = ka(t)A\int_T ... \int_T b(s_1,s_2,...s_{k-1},t)(u(s_1)E + a(s_1)\chi(t_0,t,s_1)A)(u(s_2)E$$
$$+a(s_2)\chi(t_0,t,s_2)A)...(u(s_{k-1})E + a(s_{k-1})\chi(t_0,t,s_{k-1})A)ds_1ds_2...ds_{k-1}. \quad (3)$$

Proof. Let Δt be the increment of the variable t. Then

$$\frac{1}{\Delta t}(Y(t+\Delta t,u)) - Y(t,u)) =$$
$$= \frac{1}{\Delta t}\int_T ... \int_T b(s_1,s_2,...,s_k)[(u(s_1)E$$
$$+a(s_1)\chi(t_0,t,s_1)A + a(s_1)\chi(t,t+\Delta t,s_1)A))(u(s_2)E$$
$$+a(s_2)\chi(t_0,t,s_2)A + a(s_2)\chi(t,t+\Delta t,s_2)A)...(u(s_k)E$$
$$+a(s_k)\chi(t_0,t,s_k)A + a(s_k)\chi(t,t+\Delta t,s_k)A)ds_1ds_2...ds_k-$$
$$(u(s_1)E + a(s_1)\chi(t_0,t,s_1)A)(u(s_2)E + a(s_2)\chi(t_0,t,s_2)A)...(u(s_k)E$$
$$+a(s_k)\chi(t_0,t,s_k)A)]ds_1ds_2...ds_k =$$
$$= \frac{k}{\Delta t}\int_T ... \int_T b(s_1,s_2,...,s_k)(u(s_1)E + a(s_1)\chi(t_0,t,s_1)A)...(u(s_{k-1})E$$
$$+a(s_{k-1})\chi(t_0,t,s_{k-1})A)Aa(s_k)\chi(t,t+\Delta t,s_k)ds_1ds_2...ds_k+$$
$$+\frac{1}{\Delta t}\int_T ... \int_T \sum_{m=2}^{k} C_k^m (u(s_1)E + a(s_1)\chi(t_0,t,s_1)A)...(u(s_{k-m})E$$
$$+a(s_{k-m})\chi(t_0,t,s_{k-m})A)a(s_{k-m+1}\chi(t,t+\Delta t,s_{k-m+1})A...\cdot$$
$$\cdot a(s_k)\chi(t,t+\Delta t,s_k)Ads_1ds_2...ds_k. \quad (4)$$

Further

$$\frac{1}{|\Delta t|}\|\int_T ... \int_T \sum_{m=2}^{k} C_k^m (u(s_1)E + a(s_1)\chi(t_0,t,s_1)A)...(u(s_{k-m})E$$
$$+a(s_{k-m})\chi(t_0,t,s_{k-m})A)a(s_{k-m+1})\cdot$$
$$\cdot\chi(t,t+\Delta t,s_{k-m+1})A...a(s_k)\chi(t,t+\Delta t,s_k)Ads_1ds_2...ds_k\|_L \leq$$
$$\leq \frac{1}{|\Delta t|} C_k^m (max_{s\in T}|a(s)|)^m \|A\|_L^m B[\int_T (|u(s_1)|$$

$$+max_{s_1\in T}|a(s_1)|\|A\|_L)ds_1]^{k-m}|\Delta t|^m =$$

$$= C_k^m(max_{s\in T}|a(s)|)^m\|A\|_L^m B\||u(s_1)| + max_{s_1\in T|}|a(s_1)|\|A\|_L\|_1^{k-m}|\Delta t|^{m-1} \to 0$$

for $\Delta t \to 0, m \geq 2$. Using the definition of function χ and the mean value theorem, we go to the limit in (4) for $\Delta t \to 0$, and get (3). The theorem is proved.

Corollary. If the conditions of theorem 1 are satisfied, $Y(t,u)$ is a solution of the operator equation

$$\frac{\partial Y}{\partial t} = a(t)\frac{\delta Y}{\delta u(t)} \tag{5}$$

with the initial condition

$$Y(t_0,u) = f(u)E. \tag{6}$$

Indeed, substituting equality (3), (4) into equation (5), we get the equality. Next

$$Y(t_0,u) = \int_T \ldots \int_T b(s_1,s_2,...,s_k)u(s_1)u(s_2)...u(s_k)E^k ds_1 ds_2...ds_k = f(u)E.$$

2. More general initial conditions

Let $f : L_1(T) \to \mathbb{C}$ be decomposed into a power series that converges for all $u \in L_1(T)$

$$f(u) = \sum_{k=0}^{\infty}\int_T\int_T \ldots \int_T b_k(s_1,s_2,...,s_k)u(s_1)u(s_2)...u(s_k)ds_1ds_2...ds_k, \tag{7}$$

where b_k is symmetric for each pair of variables of the function.

Theorem 2.1. *If $a : T \to \mathbb{C}$ be a continuous function, b_k be continuous symmetric functions for each pair of variables, then the operator function*

$$Y(t,u) = f(uE + a\chi(t_0,t)A) =$$

$$= \sum_{k=0}^{\infty}\int_T\int_T \ldots \int_T b_k(s_1,s_2,...,s_k)(u(s_1)E + a(s_1)\chi(t_0,t,s_1)A)(u(s_2)E$$

$$+a(s_2)\chi(t_0,t,s_2)A)...(u(s_k)E + a(s_k)\chi(t_0,t,s_k)A)ds_1ds_2...ds_k \tag{8}$$

is a solution of operator equation (5) with the initial condition $Y(t_0,u) = f(u)E$.

Proof. According to the previous corollary, each term of series (7) satisfies equation (5), then (formally)

$$\frac{\partial Y}{\partial t} = a(t)A\frac{\delta_p Y}{\delta u(t)}.$$

In this equation, power series are on the left and on the right, so $Y(t,u)$ satisfies equation (5). In this case,

$$Y(t_0,u) = \sum_{k=0}^{\infty}\int_T\int_T \ldots \int_T b_k(s_1,s_2,...,s_k)u(s_1)u(s_2)...u(s_k)Eds_1ds_2...ds_k$$

$$= f(u)E,$$

in other words, the initial condition is also satisfied. The theorem is proved.

Comment. In the particular case $f(u) = \sum_{k=0}^{\infty}(\int_T b_k(s)u(s)ds)^k$ we get

$$Y(t,u) = f(uE + a\chi(t_0,t)A)) = \sum_{k=0}^{\infty}(\int_T b_k(s)(u(s)E + a(s)\chi(t_0,t,s)A)ds)^k.$$

3. Vector problem

Consider the vector problem

$$\frac{\partial y}{\partial t} = a(t)A\frac{\delta_p y}{\delta u(t)}. \tag{9}$$

$$y(t_0) = f(u)\xi. \tag{10}$$

Here $y : T \times L_1(T) \to X, \xi \in X$.

Theorem 3.1. *If $a : T \to \mathbb{C}$ is a continuous function and f expands into power series (7), $\xi \in X$, then*

$$y(t,u) = f(uE + a\chi(t_0,t)A)\xi$$

is a solution of problem (9), (10).

Proof. We have $\frac{\partial y(t,u)}{\partial t} = a(t)\frac{\delta f(uE+a\chi(t_0,t)A)}{\delta u(t)} = a(t)\frac{\delta y(t,u)}{\delta u(t)}$. Therefore y is a solution to equation (9). Further $y(t_0,u) = f(uE)\xi = f(u)E\xi = f(u)\xi$. The theorem is proved.

Let the initial condition have more general form

$$y(t_0,u) = F(u) = \sum_{k=1}^{n} f_K(u)\xi_k. \tag{11}$$

Here $\xi_1, \xi_2, ..., \xi_n$ is a basis in X. Then

$$y(t,u) = \sum_{k=1}^{n} f_K(uE + a\chi(t_0,t)A)\xi_k$$

is a solution of problem (9), (11). This formula can be formally written in mind $y(t,u) = F(uE + a\chi(t_0,t)A))$

4. Linear inhomogeneous problem

Consider the linear inhomogeneous problem

$$\frac{\partial y}{\partial t} = a(t)A\frac{\delta_p y}{\delta u(t)} + b(t,u). \tag{12}$$

$$y(t_0) = F(u). \tag{13}$$

Here $y : T \times L_1(T) \to X, F : L_1(T) \to X, b : T \times L_1(T) \to X$.

Theorem 4.1. *If $a : T \to \mathbb{C}$ is a continuous function, $b = \sum_{j=1}^{n} b_j\xi_j$ and b_j expands into power series*

$$b_j(t,u) = \sum_{k=0}^{\infty}\int_T ... \int_T b_{jk}(t,s_1,s_2,...,s_k)u(s_1)u(s_2)...u(s_k)ds_1ds_2...ds_k,$$

$$F(u) = \sum_{k=1}^{n} f_k(u)\xi_k,$$

then

$$y(t,u) = F(uE + a\chi(t_0,t)A) + \int_{t_0}^{t} b(s, uE + a\chi(s,t)A)ds \tag{14}$$

is a solution of problem (12), (13).

Proof. Since

$$\frac{\partial F(uE + a\chi(t_0,t)A)}{\partial t} = a(t)A\frac{\delta F(uE + a\chi(t_0,t)A)}{\delta u(t)},$$

we have

$$\frac{\partial y}{\partial t} = a(t)A\frac{\delta F(uE + a\chi(t_0,t)A)}{\delta u(t)} + b(t,uE) + \int_{t_0}^{t} aA\frac{\delta b(S, uE + a\chi(t_0,t)A)}{\delta u(t)}ds,$$

$$\frac{\delta_p y}{\delta u(t)} = \frac{\delta F(uE + a\chi(t_0,t)A)}{\delta u(t)} + \int_{t_0}^{t} \frac{\delta b(s, uE + a\chi(t_0,t)A)}{\delta u(t)}ds$$

Obviously y satisfies equation (12) and

$$y(t_0,u) = F(uE) = \sum_{k=1}^{n} f_k(uE)\xi_k = \sum_{k=1}^{n} f_k(u)\xi_k = F(u),$$

The theorem is proved.

5. Application to probability theory

Consider the system of differential equations with random coefficients

$$\frac{dx}{dt} = \varepsilon_1(t)Ax + \varepsilon_2(t)\xi \tag{15}$$

with the initial condition

$$x(t_0) = x_0. \tag{16}$$

Here $x \in X, A \in L(X,X), g : T \to \mathbb{R}, \xi$ is a random vector, $\varepsilon_1(t), \varepsilon_2(t), g$ are random processes, x_0 is a random vector. Let

$$w = exp(i\int_T (\varepsilon_1(s)u(s) + \varepsilon_2(s)v(s))ds), u, v \in L_1(T)$$

and let the characteristic functional be known [1, p. 30] $\psi(u,v) = E[w], u, v \in L_1(T), E$ means the mathematical expectation of the distribution functions ε, g. We need to find the mathematical expectation of the solution of problem (15), (16).

We multiply (15), (16) by w and take the mathematical expectations of the obtained equalities

$$E[\frac{dx}{dt}w] = E[\varepsilon_1(t)Axw] + E[\varepsilon_2(t)\xi w]$$

$$E[x(t_0)w] = E[x_0 w]$$

We introduce the mapping $y(t,u,v) = E[x(t)w]$. Note that $y(t,0) = E[x(t)]$. The last equations can be written as

$$\frac{\partial y}{\partial t} = -iA\frac{\delta_p y}{\delta u(t)} - i\frac{\delta_p \psi}{\delta v(t)}\xi, \quad (17)$$

$$y(t_0,u,v) = E[x_0]\psi(u,v). \quad (18)$$

Here we assume that x_0 does not depend on $\varepsilon_1, \varepsilon_2$.

Problem (17), (18) has the form of problem (12), (13). Using formula (14) we find

$$y(t,u,v) = \psi(u - i\chi(t_0,t)A, v)E[x_0] - i\int_{t_0}^{t} \frac{\delta_p \psi(u - ichi(s,t)A, v)}{\delta v(s)}\xi ds.$$

Assuming $u = 0, v = 0$, we easily find

$$E[x(t)] = \psi(-i\chi(t_0,t)A, 0)E[x_0] - i\int_{t_0}^{t} \frac{\delta_p \psi(-ichi(s,t)A, 0)}{\delta v(s)}\xi ds.$$

Let $\varepsilon_1, \varepsilon_2$ be given by the characteristic functional

$$\psi(u,v) = exp(i\int_T E[\varepsilon_1(s)u(s)ds$$

$$-\frac{1}{2}\int_T\int_T b(s_1,s_2)]u(s_1)u(s_2)ds_1ds_2)\frac{sin\int_T B(s)u(s)ds}{\int_T B(s)u(s)ds}exp(i\int_T E[\varepsilon_2(s)v(s)ds])$$

This means that ε_1 is a Gaussian random process, and ε_2 is a uniformly distributed random process, $b(s_1,s_2) = E[\varepsilon_1(s_1)\varepsilon(s_2)] - E[\varepsilon_1(s_1)]E[\varepsilon(s_2)], B(s) \geq 0, u \in L_1(T), v \in L_1(T)$. Using the formula, we find

$$E[x(t)] = exp(i\int_{t_0}^{t} AE[\varepsilon_1(s)ds$$

$$+\frac{A^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} b(s_1,s_2)]ds_1ds_2)\frac{sin\int_{t_0}^{t} B(s)ds}{\int_{t_0}^{t} B(s)ds}exp(i\int_{t_0}^{t} E[\varepsilon_2(s)ds])$$

$$\int_{t_0}^{t} exp(i\int_{s}^{t} AE[\varepsilon_1(\tau)d\tau$$

$$+\frac{A^2}{2}\int_{s}^{t}\int_{s}^{t} b(s_1,s_2)]ds_1ds_2)\frac{sin\int_{t_0}^{t} B(s)ds}{\int_{t_0}^{t} B(s)ds}exp(i\int_{t_0}^{t} E[\varepsilon_2(s)ds])E[\varepsilon_2(s)]ds\xi.$$

Similarly we can find the other moment functions for the solution of problem (15),(16).

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