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Kamenev-Type Oscillation Criteria for Second Order Generalized Delay Difference Equations

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Abstract : Some oscillation criteria are established by Raccati transformation techniques for the second-order nonlinear generalized neutral difference equation.
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1. INTRODUCTION

In recent years, the oscillation or asymptotic behavior of second-order difference equations was the subject of investigation by many authors (see for example [1, 2, 4-11]).

In this article, we are concerned with a class of second-order nonlinear delay difference equations of the form

$$\Delta_\ell(p(k)(\Delta_\ell(u(k)+c(k)u(k-\tau\ell)))^\gamma)+q(k)u^\beta(k-\sigma\ell) = 0, k\in[0,\infty), \ell\in(0,\infty) \quad (1)$$

where Δ_ℓ denotes the generalized forward difference operator $\Delta_\ell u(k) = u(k+\ell)-u(k)$ for any real valued function $u(k), \gamma>0$ and $\beta>0$ are quotients of odd positive integers, τ, σ are fixed nonnegative integer, $0\le c(k)<1$, $p(k)$, and $q(k)$ are any two real valued functions such that $p(k)>0, q(k)\ge 0,$ and $q(k)$ has a positive real valued function, and for some $k_0>0$,

$$\sum_{r=0}^{\infty}\left(\frac{1}{p(k_0+r\ell)}\right)^{\frac{1}{\gamma}} = \infty, \quad (2)$$

or

$$\sum_{r=0}^{\infty}\left(\frac{1}{p(k_0+r\ell)}\right)^{\frac{1}{\gamma}} = \infty. \quad (3)$$

When $c(k)=0,$ equation (1) reduces to the following equation

$$\Delta_\ell(p(k)(\Delta_\ell u(k))^\gamma)+q(k)u^\beta(k-\sigma\ell) = 0,\ k\in[\sigma\ell,\infty). \quad (4)$$

We say that equation (1) or equation (4) is strictly superlinear if $\beta>1$; strictly sublinear if $0<\beta<1$; and linear if $\beta=1$.

In the superlinear case, when $\gamma=\beta>1,$ and $\ell=1$ the oscillation of the solution of equation (1) was discussed in[11] under the condition (2). But when $\gamma\ge\beta>1$, the oscillation is not known.

In the sublinear case, when $0<\beta<1, \gamma>1,$ and $\ell=1$ some authors studied the equation (4) under the condition (2) or (3), see the article[9]. But it is necessary to point out that their proof under the condition (3) is wrong, so the corresponding theorem does not hold.

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The objectives of this article are:

1. In the superlinear case, when $\gamma \geq \beta > 1$, we establish the oscillation criteria for equation (1) under the condition (2), which improve and include several oscillation criteria in[11].
2. In the sublinear case, when $0 < \beta < 1, \gamma > 1$, we correct and generalized the Theorem 2.3 and its proof in[9], precisely, we obtain a new oscillation or asymptotic criteria for equation (4) under the condition (3).

2. MAIN RESULTS

First, we consider the case, where (2) holds and in the superlinear case where $\gamma \geq \beta > 1$.

Theorem 2.1. Assume that (2) holds. Furthermore, assume that there exists a positive real valued functions $\rho(k), k \in \mathbb{N}(0)$ such that for some positive number M,

$$\limsup_{k\to\infty} \sum_{r=0}^{k} \left[\rho(k_0+r\ell)\,\mathrm{Q}(k_0+r\ell) - \frac{p^{\beta/\gamma}(k_0+(r-\sigma)\ell)(\Delta_\ell\,\rho(k_0+r\ell))^2}{4\rho(k_0+r\ell)\,\mathrm{M}^{(\gamma-\beta)/\gamma}} \right] = \infty, \tag{5}$$

where $\mathrm{Q}(k) = q(k)(1-c(k-\sigma\ell))^{\gamma}$. Then, every solution of equation (1) oscillates.

Proof. Suppose that, on the contrary, $u(k)$ is an eventually nonoscillatory solution of equation (1). Without loss of generality, we may assume that $u(k)$ is an eventually positive solution of equation (1) such that $u(k-\sigma\ell) > 0$ for all $k > k_0$.

Set
$$w(k) = u(k) + c(k)u(k-\tau\ell). \tag{6}$$

By assumption, we have $w(k) > 0$ for $k \geq k_0$ and from (1) it follows that

$$\Delta_\ell\,(p(k)(\Delta_\ell\, w(k))^{\gamma}) = -q(k)u^{\beta}(k-\sigma\ell) \leq 0, \tag{7}$$

for $k \geq k_0$, and so $p(k)(\Delta_\ell w(k))^{\gamma}$ is an eventually nonincreasing real valued function. We show that $p(k)(\Delta_\ell w(k))^{\gamma}$ is eventually positive. Indeed $p(k)(\Delta_\ell w(k))^{\gamma}$ is either eventually positive or negative. We first show that $p(k)(\Delta_\ell w(k))^{\gamma} > 0$ for $k \geq k_0$. In fact, if there exists a real $k_1 \geq k_0$ such that $p(k_1)(\Delta_\ell(w(k_1)))^{\gamma} = c < 0$, then $p(k)(\Delta_\ell\, w(k))^{\gamma} \leq p(k_1)(\Delta_\ell\, w(k_1))^{\gamma} = c$ for $k \geq k_1$, that is, $(\Delta_\ell w(k))^{\gamma} \leq \dfrac{c}{p(k)}$, and hence $w(k) \leq w(k_1) + c^{\frac{1}{\gamma}} \displaystyle\sum_{r=0}^{n^*-1} \left(\frac{1}{p(k_1+r\ell)} \right)^{\frac{1}{\gamma}} \to -\infty$, which contradicts the fact that $w(k) > 0$ for $k \geq k_0$. Hence $p(k)(\Delta_\ell w(k))^{\gamma}$ is eventually positive. Therefore, we have

$$w(k) > 0, \Delta_\ell w(k) \geq 0,\ \Delta_\ell\,(p(k)(\Delta_\ell w(k))^{\gamma}) \leq 0, k \geq k_0. \tag{8}$$

Then, from (8) and (6) we have $u(k) \geq (1-c(k))w(k)$ and this implies that for $k \geq k_1 = k_0 + \sigma\ell$, $u(k-\sigma\ell) \geq (1-c(k-\sigma\ell))w(k-\sigma\ell)$, and by (7),

$$\Delta_\ell(p(k)(\Delta_\ell w(k))^{\gamma} + Q(k)w^{\beta}(k-\sigma\ell) \leq 0, k \geq k_1. \tag{9}$$

Define the sequence $z(k)$ by
$$z(k) = \rho(k)\frac{p(k)(\Delta_\ell w(k))^{\gamma}}{w^{\beta}(k-\sigma\ell)}. \tag{10}$$

Then, $z(k) > 0$ and
$$\Delta_\ell z(k) = \Delta_\ell\rho(k)\frac{z(k+\ell)}{\rho(k+\ell)} + \rho(k)\Delta_\ell\left(\frac{p(k)(\Delta_\ell w(k))^{\gamma}}{w^{\beta}(k-\sigma\ell)} \right)$$

$$= \Delta_\ell\rho(k)\frac{z(k+\ell)}{\rho(k+\ell)} + \frac{\rho(k)\Delta_\ell(p(k)(\Delta_\ell w(k))^{\gamma})w^{\beta}(k-(\sigma-1)\ell) - p(k+\ell)(\Delta_\ell w(k+\ell))^{\gamma}\Delta_\ell w^{\beta}(k-\sigma\ell)}{w^{\beta}(k-(\sigma-1)\ell)w^{\beta}(k-\sigma\ell)}$$

$$\leq \Delta_\ell\rho(k)\frac{z(k+\ell)}{\rho(k+\ell)} - \rho(k)Q(k) - \frac{\rho(k)p(k+\ell)(\Delta_\ell w(k+\ell))^{\gamma}\Delta_\ell(w^{\beta}(k-\sigma\ell))}{w^{\beta}(k-(\sigma-1)\ell)w^{\beta}(k-\sigma\ell)}. \tag{11}$$

Now, by the inequality (see[3])

$$u^{\beta} - v^{\beta} \geq (u-v)^{\beta}, \tag{12}$$

for all $u \geq v$ and $\beta \geq 1$. we have

$$\begin{aligned}\Delta_{\ell} w^{\beta}(k-\sigma\ell) &= w^{\beta}(k-(\sigma-1)\ell) - w^{\beta}(k-\sigma\ell) \geq (w(k-(\sigma-1)\ell) - w(k-\sigma\ell))^{\beta} \\ &= (\Delta_{\ell} w(k-\sigma\ell))^{\beta}. \end{aligned}\tag{13}$$

Then $$\Delta_{\ell} z(k) \leq -\rho(k)Q(k) + \frac{\Delta_{\ell}\rho(k)}{\rho(k+\ell)} z(k+\ell) - \frac{\rho(k)p(k+\ell)(\Delta_{\ell} w(k+\ell))^{\gamma}(\Delta_{\ell} w(k-\sigma\ell))^{\beta}}{(w^{\beta}(k-(\sigma-1)\ell))^{2}}. \tag{14}$$

By (8), we have

$$\begin{aligned} p(k-\sigma\ell)(\Delta_{\ell} w(k-\sigma\ell))^{\gamma} &\geq p(k+\ell)(\Delta_{\ell} w(k+\ell))^{\gamma}, \\ \Delta_{\ell} w(k-\sigma\ell) &\geq \left(\frac{p(k+\ell)}{p(k-\sigma\ell)}\right)^{1/\gamma} \Delta_{\ell} w(k+\ell), \\ (\Delta_{\ell} w(k-\sigma\ell))^{\beta} &\geq \left(\frac{p(k+\ell)}{p(k-\sigma\ell)}\right)^{\beta/\gamma} (\Delta_{\ell} w(k+\ell))^{\beta}, \end{aligned}\tag{15}$$

So that

$$\begin{aligned}\Delta_{\ell} z(k) &\leq -\rho(k)Q(k) + \frac{\Delta_{\ell}\rho(k)}{\rho(k+\ell)} z(k+\ell) \\ &\quad - \frac{\rho(k)p(k+\ell)(\Delta_{\ell} w(k+\ell))^{\gamma}\left(\frac{p(k+\ell)}{p(k-\sigma\ell)}\right)^{\beta/\gamma}(\Delta_{\ell} w(k+\ell))^{\beta}}{(u^{\beta}(k-(\sigma-1)\ell))^{2}} \\ &= -\rho(k)Q(k) + \frac{\Delta_{\ell}\rho(k)}{\rho(k+\ell)} z(k+\ell) \\ &\quad - \frac{\rho(k)p(k+\ell)(\Delta_{\ell} w(k+\ell))^{\gamma}\left(\frac{p(k+\ell)}{p(k-\sigma\ell)}\right)^{\beta/\gamma}(\Delta_{\ell} w(k+\ell))^{\gamma}}{u^{\beta}(k-(\sigma-1)\ell)^{2}(\Delta_{\ell} w(k+\ell))^{\gamma-\beta}} \\ &= -\rho(k)Q(k) + \frac{\Delta_{\ell}\rho(k)}{\rho(k+\ell)} z(k+\ell) \\ &\quad - \frac{\rho(k)}{p^{\beta/\gamma}(k-\sigma\ell)(p(k+\ell))^{(1-\beta)/\gamma}}\left(\frac{z(k+\ell)}{\rho(k+\ell)}\right)^{2}\frac{1}{(\Delta_{\ell} w(k+\ell))^{\gamma-\beta}}. \end{aligned}\tag{16}$$

Now, from the fact that $p(k)(\Delta_{\ell} w(k))^{\gamma}$ is a positive and nonincreasing real valued function, there exists a $k_2 > k_1$ sufficiently large such that $p(k)(\Delta_{\ell} w(k))^{\gamma} \leq 1/M$, holds for some positive constant M and $k > k_2$. And hence by (8) we have $p(k+\ell)(\Delta_{\ell} w(k+\ell))^{\gamma} \leq 1/\mathrm{M}$, so that

$$\frac{1}{(\Delta_{\ell} w(k+\ell))^{\gamma-\beta}} \geq (\mathrm{M}\, p(k+\ell))^{\frac{\gamma-\beta}{\gamma}}. \tag{17}$$

Therefore $$\Delta_{\ell} z(k) \leq -\rho(k)\mathrm{Q}(k) + \frac{\Delta_{\ell}\rho(k)}{\rho(k+\ell)} z(k+\ell) - \mathrm{R}(k) z^{2}(k+\ell), \tag{18}$$

where $$\mathrm{R}(k) = \frac{\rho(k)\mathrm{M}^{\frac{\gamma-\beta}{\gamma}}}{\rho^{2}(k+\ell)p^{\frac{\beta}{\gamma}}(k-\sigma\ell)}$$

This implies that

$$\Delta_\ell z(k) \leq -\rho(k)Q(k) + \frac{1}{4\mathrm{R}(k)}\frac{(\Delta_\ell \rho(k))^2}{\rho^2(k+\ell)} - \left[\sqrt{\mathrm{R}(k)}z(k+\ell) - \frac{1}{2\sqrt{\mathrm{R}(k)}}\frac{\Delta_\ell \rho(k)}{\rho(k+\ell)}\right]^2$$

$$\leq -\left[\rho(k)\mathrm{Q}(k) - \frac{p^{\beta/\gamma}(k-\sigma\ell)(\Delta_\ell \rho(k))^2}{4\rho(k)M^{(\gamma-\beta)/\gamma}}\right]. \tag{19}$$

Summing (19) from k_2 to k, we obtain

$$-z(k_2) < z(k+\ell) - z(k_2) \leq -\sum_{r=0}^{k}\left[\rho(k_2+r\ell)Q(k_2+r\ell) - \frac{p^{\beta/\gamma}(k_2+(r-\sigma)\ell)(\Delta_\ell \rho(k_2+r\ell))^2}{4\rho(k_2+r\ell)M^{(\gamma-\beta)/\gamma}}\right],$$

which yields

$$\sum_{r=0}^{k}\left[\rho(k_2+r\ell)Q(k_2+r\ell) - \frac{p^{\beta/\gamma}(k_2+(r-\sigma)\ell)(\Delta_\ell \rho(k_2+r\ell))^2}{4\rho(k_2+r\ell)M^{(\gamma-\beta)/\gamma}}\right] = c_1,$$

for all large k, and this is contrary to (5). The proof is completed.

In the following theorem, we provide another sufficient condition for oscillation of equation (1). This result is discrete analogy of Philos-type condition for oscillation of second-order differential equations.

Theorem 2.2 Assume that (2) holds. Let $\rho(k), k \in \mathbb{N}(0)$ be a positive real valued function. Further, we assume that there exists a double function $\{\mathrm{H}(m,k) : m \geq k \geq 0\}$ such that (i) $\{\mathrm{H}(m,k) : m \geq k \geq 0\}$ for $m \geq 0$, (ii) $\mathrm{H}(m,k) > 0$, $m > k \geq 0$, and (iii) $\Delta_{2(\ell)}\mathrm{H}(m,k) = \mathrm{H}(m, k+\ell - \mathrm{H}(m,k)) \leq 0$ for $m \geq k \geq 0$. If

$$\limsup_{m\to\infty}\frac{1}{\mathrm{H}(m,0)}\sum_{r=0}^{m-1}\Big[\mathrm{H}(m,k_0+r\ell)\rho(k_0+r\ell)\mathrm{Q}(k_0+r\ell)$$

$$-\frac{(\rho(k_0+(r+1)\ell))^2}{4\overline{\rho(k_0+r\ell)}}\left(h(m,k_0+r\ell) - \frac{\Delta_\ell \rho(k_0+r\ell)}{\rho(k_0+(r+1)\ell)}\sqrt{\mathrm{H}(m,k_0+r\ell)}\right)^2\Bigg] = \infty, \tag{20}$$

where

$$h(m,k) = -\frac{\Delta_{2(\ell)}\mathrm{H}(m,k)}{\sqrt{\mathrm{H}(m,k)}}, m \geq k \geq 0,$$

$$\overline{\rho(k)} = \frac{\rho(k)\mathrm{M}^{(\gamma-\beta)/\gamma}}{p^{\beta/\gamma}(k-\sigma\ell)},$$

then every solution of equation (1) oscillates.

Proof. We proceed as in Theorem 2.1. Assume that equation (1) has a nonoscillatory solution, say $u(k-\sigma\ell) > 0$ for all $k \geq k_0$. From the proof of Theorem 2.1, we obtain (18) for all $k \geq k_2$. From (18) we have for all $k \geq k_2$,

$$\Delta_\ell z(k) \leq -\rho(k)q(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)}w(k+\ell) - \frac{\overline{\rho(k)}}{(\rho(k+\ell))^2}w^2(k+\ell),$$

where $\overline{\rho(k)} = \dfrac{\rho(k)\mathrm{M}^{\frac{\gamma-\beta}{\gamma}}}{p^{\beta/\gamma}(k-\sigma\ell)}$. That is,

$$\rho(k)q(k) \leq -\Delta_\ell z(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)}z(k+\ell) - \frac{\overline{\rho(k)}}{(\rho(k+\ell))^2}w^2(k+\ell).$$

Therefore, we have

$$\sum_{r=0}^{m-1}\mathrm{H}(m,k+r\ell)\rho(k+r\ell)q(k+r\ell) \;\le\; -\sum_{r=0}^{m-1}\mathrm{H}(m,k+r\ell)\Delta_\ell z(k+r\ell)$$

$$+\sum_{r=0}^{m-1}\mathrm{H}(m,k+r\ell)\frac{\Delta_\ell\rho(k+r\ell)}{\rho(k+(r+1)\ell)}z(k+(r+1)\ell)-\sum_{r=0}^{m-1}\mathrm{H}(m,k+r\ell)\frac{\overline{\rho(k+r\ell)}}{(\rho(k+(r+1)\ell))^2}z^2(k+(r+1)\ell).$$

The remainder procedure of the proof is fairly routine and is similar to that of the Theorem 2.2 in[9] when $\ell=1$, so we omit it.

Remark 2.3. By choosing the sequence H(m, k) in appropriate manners, we can derive several oscillation criteria for equation (1). For example, set

$$\mathrm{H}(m,k) \;=\; (m-k)_\ell^{(\lambda)},\ \lambda\in\mathbb{N}(1),\ m\ge k\ge 0,$$

we have the following result.

Theorem 2.4. Assume that (2) holds. Furthermore, assume that there exists a positive real valued function $\rho(k)$ for $k\in\mathbb{N}(0)$ and $\lambda\in\mathbb{N}(1)$ such that

$$\limsup_{m\to\infty}\frac{1}{m_\ell^\lambda}\sum_{r=0}^{m-1}(m-(k_0+r\ell))_\ell^\lambda\left[\rho(k_0+r\ell)\mathrm{Q}(k_0+r\ell)-\psi(k_0+r\ell)\frac{p^{\beta/\gamma}(k_0+(r-\sigma)\ell)(\rho(k_0+(r+1)\ell))^2}{4\rho(k_0+r\ell)M^{(\gamma-\beta)/\gamma}}\right]=\infty, \tag{21}$$

for some $k_0\ge 0$, where

$$\psi(k) \;=\; \left[\frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}-\frac{\lambda(m-k-\ell)_\ell^{(\lambda-1)}}{(m-k)_\ell^{(\lambda)}}\right]^2.$$

Then, every solution of equation (1) oscillates.

Remark 2.5. When $\gamma=\beta\ge 1$, equation (1) reduces to the difference equation

$$\Delta_\ell(p(k)(\Delta_\ell(u(k)+c(k)u(k-\tau\ell)))^\gamma)+q(k)u^\gamma(k-\sigma\ell) \;=\; 0, k\in[\sigma\ell,\infty),$$

and the condition (21) in Theorem 2.4 reduces to

$$\limsup_{m\to\infty}\frac{1}{m^\lambda}\sum_{r=0}^{m-1}(m-(k_0+r\ell))^\lambda\left[\rho(k_0+r\ell)Q(k_0+r\ell)-\frac{p(k_0+(r-\sigma)\ell)(\rho(k_0+(r+1)\ell))^2}{4\rho(k_0+r\ell)}\psi(k_0+r\ell)\right]=\infty, \tag{22}$$

which is the same as Theorem 2.1 in[11] when $\ell=1$. So our Theorem 2.4 extend and include several oscillation criteria in[11] when $\ell=1$.

Next, we consider the case where (3) holds and the sublinear case where $0<\beta<1,\ \gamma>1$.

Theorem 2.6. Assume that (3) holds and $\Delta_\ell p(k)\ge 0$. Furthermore, assume that there exists a positive real valued function $\rho(k),\ k\in\mathbb{N}(0)$ such that for every $\gamma\ge 1$ and positive number M,

$$\limsup_{n\to\infty}\sum_{r=0}^{k}\left[\rho(k_0+r\ell)q(k_0+r\ell)-\frac{(\rho(k_0+(r-\sigma)\ell))^{1/\gamma}\alpha^{1-\beta}(k_0+(r-\sigma)\ell)^{1-\beta}(\Delta_\ell\rho(k_0+r\ell))^2}{4\beta(\mathrm{M})^{\frac{\gamma-1}{\gamma}}\rho(k_0+r\ell)}\right]=\infty \tag{23}$$

and $$\sum_{s=0}^{\infty}\left(\frac{1}{p(k_0+r\ell)}\sum_{r=0}^{s-1}q(k_0+r\ell)\right)^\gamma \;=\; \infty,$$

for some $k_0 > 0$, then every solution of equation (4) oscillates or $\lim_{k\to\infty} u(k) = 0.$

Proof. **S**uppose that, on the contrary, $u(k)$ is an eventually positive solution of (4) such that $u(k-\sigma\ell) > 0$ for all $k \geq k_0$. We shall consider only this case because the substitution $v(k) = -u(k)$, transforms equation (4) into an equation of the same form. From equation (4) we have

$$\Delta_\ell(p(k)(\Delta_\ell u(k))^\gamma) = -q(k)u^\beta(k-\sigma\ell) \leq 0, k \geq k_0,$$

and so $p(k)(\Delta_\ell u(k))^\gamma$ is an eventually nonincreasing function, and then there exist two possible cases of $p(k)(\Delta_\ell u(k))^\gamma$, that is $p(k)(\Delta_\ell u(k))^\gamma$ is eventually nonnegative or eventually negative, from this there exist two possible cases of $\Delta_\ell u(k)$.

In the case, where $\Delta_\ell u(k)$ is eventually nonnegative, we may follow the proof of Theorem 2.1 in[9] and obtain a contradiction.

If $\Delta_\ell u(k)$ is eventually negative, then $\lim_{k\to\infty} u(k) = b \geq 0.$ We assert that $b = 0$. If not then $u^\beta(k-\sigma\ell) \to b^\beta > 0$ as $k \to \infty$, and hence there exists $k_1 \geq k_0$ such that $u^\beta(k-\sigma\ell) \geq b^\beta$. Therefore, we have

$$\Delta_\ell(p(k)(\Delta_\ell u(k))^\gamma) \leq -q(k)b^\beta. \tag{25}$$

Summing the last inequality from k_1 to $k-1$, we have

$$p(k)(\Delta_\ell u(k))^\gamma < p(k)(\Delta_\ell u(k))^\gamma - p(k_1)(\Delta_\ell u(k_1))^\gamma \leq -b^\beta \sum_{r=0}^{k-1} q(k_1 + r\ell), \tag{26}$$

and then
$$\Delta_\ell u(k) \leq -b^{\frac{\beta}{\gamma}} \left(\frac{1}{p(k)} \sum_{r=0}^{k-1} q(k_1 + r\ell) \right)^{\frac{1}{\gamma}}, k \geq k_1.$$

Summing the above inequality from k_1 to $k-1$, we obtain

$$u(k+\ell) \leq u(k_1) - b^{\beta/\gamma} \sum_{s=0}^{k} \left(\frac{1}{p(k_1 + s\ell)} \sum_{r=0}^{s-1} q(k_1 + r\ell) \right)^{\frac{1}{\gamma}}$$

Condition (24) implies that $u(k)$ is eventually negative, which is a contradiction. The proof is completed.

Remark 2.7. From Theorem 2.6, we obtain

$$x(k) \leq x(k_2) - b^\beta \sum_{r=0}^{k-1} \beta(k_2 + (r+1)\ell)q(k_2 + r\ell) + \sum_{s=k_2}^{k-1} (p(k_2 + r\ell)\Delta_\ell \beta(k_2 + r\ell))(\Delta_\ell u(k_2 + r\ell))^\gamma \tag{28}$$

cannot conclude that

$$x(k) \leq x(k_2) - b^\beta \sum_{s=k_2}^{k-1} \beta(k_2 + r\ell + 1)q(k_2 + r\ell)$$

$$+(p(k_2 + r\ell)\Delta_\ell \beta(k_2 + r\ell))(\Delta_\ell \beta(k_2 + r\ell))(\Delta_\ell u(k_2 + r\ell))^\gamma \Big|_{r=0}^{k} - \sum_{r=0}^{k-1} \Delta_\ell (p(k_2 + r\ell)\Delta_\ell \beta(k_2 + r\ell))(\Delta_\ell u(k_2 + r\ell + 1))^\gamma$$

by parts the last term in the right-hand side of (28). Hence, when $\ell = 1$ Theorem 2.6 actually corrects the correspondence Theorem 2.3 and its proof in[9], and it is essentially new.

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