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A STUDY OF MODELING AND SIMULATION OF GRAPH PEBBLING WITH IMPETUS OF INTERDISCIPLINARY EXPLORATION

Jitendra Binwal* and Aakanksha Baber**

Abstract: Consider a connected graph G with finite number of vertices and edges whose vertices are assigned with whole numbers represents the number of pebbles on each vertex. Pebbling step and pebbling number of a connected graph G are playing a vital role. In this paper, we discuss and give an overview of mathematical modeling and simulation of graph pebbling on undirected graphs as well as directed graphs with available literature of what arrangement of directed graphs allow for pebbling to be meaningful. We also discuss the pebbling numbers of various orientations of directed graphs such as directed wheel graphs, directed complete graphs , directed demonic graphs etc. In this continuation, we discuss the importance of demonic directed graphs with that the sharp upper bound and lower bound of the pebbling numbers of the directed graphs is the same as that of the undirected graphs. These are the impetus of interdisciplinary exploration in context of mathematical modeling and simulation for certification of graph isomorphism and decyclization of graphs. We also propose the condition for two strongly connected digraphs are having equal pebbling numbers [3, 4].

Keywords: Pebbles, Pebbling Number, Isomorphism, Decyclization.

AMSC 2010: 05C07, 05C12, 05C20, 97K30, 68R10, 81Q30.

1. INTRODUCTION

A graph $G = (V, E)$ is an well-ordered pair of sets, where V is non-empty set of vertices and E is a set of pairs of elements of V. The cardinality of vertex set of a graph is the order [8]. The graph is said to be not directed graph if the elements of E are unordered pairs and not an undirected graph if they are ordered pairs. Consider a graph G, Assign whole numbers to the vertices of G. If the vertex v is related with the integer label m, we say that m pebbles are placed on v. If the sum of all the integers' labels on G is n, we say n pebbles are distributed on G.

1.1. Pebbling Step

A pebbling step is defined as an activity of subtracting two pebbles from the label of a vertex and adding one pebble to the label of an adjacent vertex.

1.2 Pebbling Number ($f(G)$)

The pebbling number of a graph G is the maximum value of the pebbling numbers of all the vertices in G. For an example, $(f(G)) = 4$ in figure 1.1.

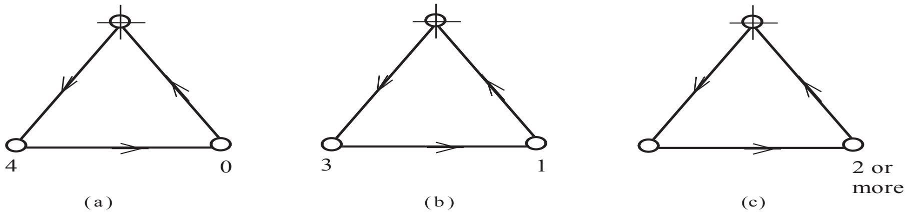


Figure 1.1: A graph with distribution of four pebbles

1.3 Strong Graph

A directed graph is strong or strongly connected if every two vertices are mutually reachable [3, 6, 8]. Pebbling on a directed graph is possible only if every vertex can be reached from any other vertex. Therefore, a directed graph must be strong in order to do any pebbling. A vertex that is only adjacent to other vertices is a Source. A vertex that is only adjacent from other vertices is Sink.

In this paper, we propose the condition for two strongly connected digraphs are having equal pebbling numbers. These are the motivation of multiple explorations in view of mathematical modeling and simulation for invariance of graph isomorphism and decyclization of graphs.

2. PEBBLING ON GRAPHS

2.1 Pebbling on Undirected Graphs

There is a expanding literature on pebbling on undirected graphs [9, 10]. Let G be any undirected graph and u, v, and w be vertices of G. A demonic graph or class zero is a graph, whose pebbling number is equal to its number of vertices. If u is a distance d from v, and $(2^d - 1)$ pebbles are placed on u, and these are all the pebbles on the graph, then no pebble can be moved to v. We know that $f(G) \geq \max\{|V(G)|\ ,\ 2^d\}$, where $|V(G)|$ is the number of the vertices of G, and d is the diameter of the graph G and $f(G) \leq 2^{|V(G)|\text{ -1}}$. We can say that there is a range of values that the pebbling number of not directed graph G, with order p, takes on: $p \leq f(G) \leq 2^{p-1}$.

2.2 Pebbling on Directed Graphs

Consider a finite connected graph G whose vertices contains whole numbers represents the number of pebbles on each vertex. Consider an digraph G_D, with pebbles placed on few of its vertices. Suppose that, for any arc (u,v) of G, we are allowed to change the distribution of pebbles by removing two pebbles from u and adding one pebble on v. Then for a vertex u of G_D, if n exists such that, however n pebbles are placed on G_D, one pebble can always be sent to u, we let $f(u,G_D)$ be the smallest such n. For not an undirected graph, $p \le f(G_D) \le 2^{p-1}$, where p is the number of vertices of G_D [9,12,18].

2.2.1 Cycle Graphs, C_n

The only two strongly connected orientations on a cycle graph C_n are the following in figure 1.2:

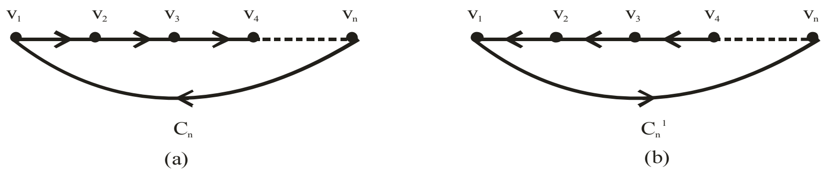


Figure 1.2: Cycle Graphs with different orientations

Theorem 1 The pebbling number of undirected cycles, $f(C_{2k}) = 2^k$ and $f(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$. [11]

Theorem 2 Let C_n be a cycle graph of order n, with a strong orientation.

Then $f(C_n) = 2^{n-1}$. [11]

2.2.2 Alternating Wheel Graphs, W_n

The figure 1.3 is an example of alternating wheel graph.

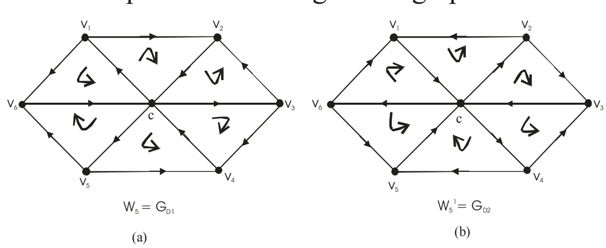


Figure 1.3: Alternating wheel graphs with different orientations

In figure 1.3(a)	In figure 1.3(b)
$d(v_1,v_2)=1$	$d(v_6,v_1)=1$
$d(v_1,v_3)=3$	$d(v_6,v_2)=3$
$d(v_1,v_4)=4$	$d(v_6,v_3)=4$
$d(v_1,v_5)=3$	$d(v_6,v_4)=3$
$d(v_1,v_6)=1$	$d(v_6,v_5)=1$

Table 1.1

It is clear from the table 1.1; the diameter of an alternating wheel is four.

Theorem 3 If W_n is an alternating wheel graph. Then for $n \geq 6$, $f(W_n) = 10 + n$.

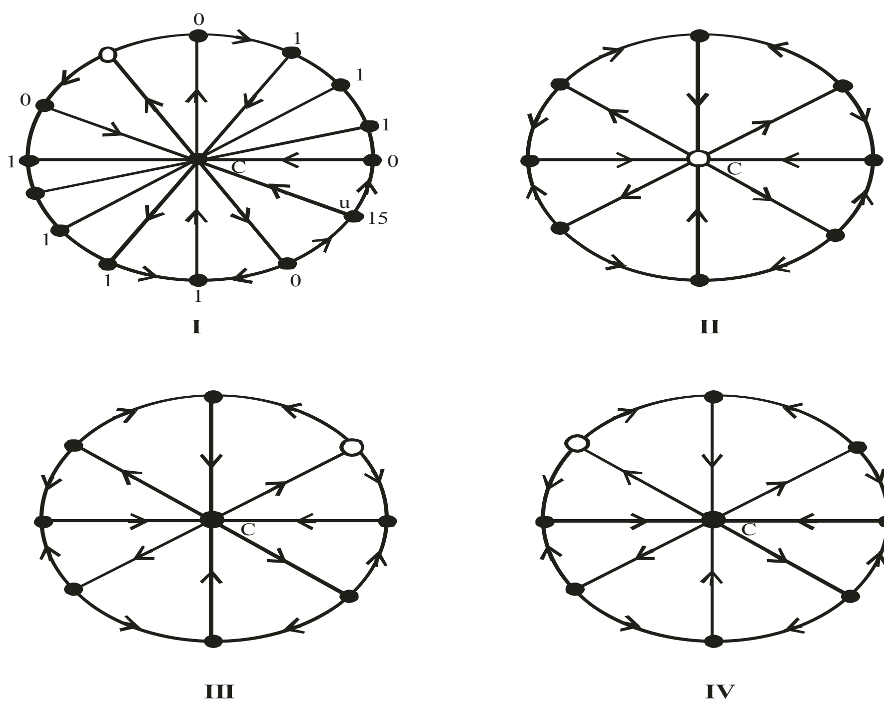
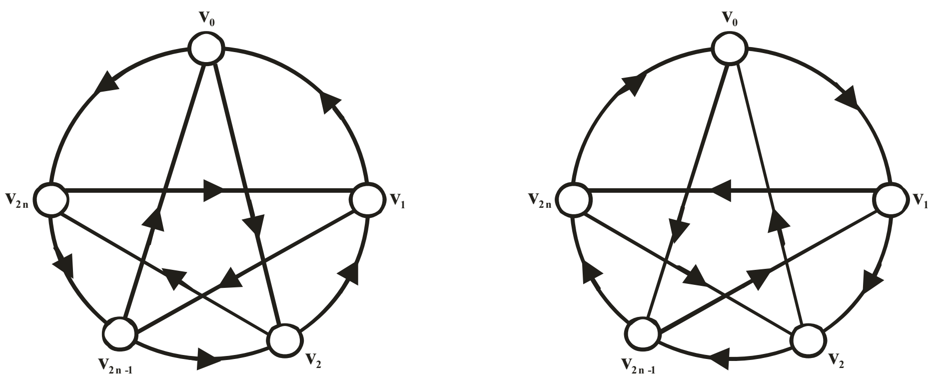


Figure 1.4: Demonstration of Pebbles on Alternating Wheel Graph

2.2.3 Alternating Complete Graphs, K_{2n+1}

An alternating complete graph, K_{2n+1}, is a directed graph with an odd number of vertices, $\{v_0, v_1, ..., v_{2n}\}$ where v_i is adjacent to v_j if and only if $(i-j) \bmod 2n+1$ is odd.

Theorem 4 For an alternating complete graphs, for $n \geq 2$, $f(K_{2n+1}) = 2n+1$.



(a) **(b)**

Figure 1.5: Alternating complete graphs with different orientations

3. EQUALITY OF PEBBLING NUMBERS ON DIRECTED GRAPHS

3.1 Main Theorem

Statement: The pebbling numbers of two strong directed graphs G_{D_1} and G_{D_2} with the same number of vertices and edges with different directions are equal if the following conditions are satisfied

1. Their corresponding adjacency matrices are symmetrical to each other.
2. Their corresponding matrices K_1 and K_2 defined by
3. $K_1 = X^1 + X^2 + ... + X^n$, $K_2 = Y^1 + Y^2 + ... + Y^n$, have no zero entry. X and Y are the adjacency matrices corresponding to G_{D_1} and G_{D_2} respectively.

Proof: We know that if X is the adjacency matrix of a digraph G, then X^T is the transpose matrix which is the adjacency matrix of a digraph G^R acquired by reversing the direction of every edge in G. By using this property, the first condition is obvious.

For proving the second condition, we will use Principal of Mathematical Induction.

- Basis of Induction: The condition is trivially true for $n = 1, 2, 3,..$

- Induction Hypothesis: We assume that the second condition holds for $r = n-1$.
- Induction Step: We will prove the second condition for $r = n$.
  For proving induction step, we use method of contradiction.
  If we assume that
  $K_1 = X^1 + X^2 + \ldots + X^n$, has at least one zero entry then digraph will be disconnected.
  But digraph is connected, and it will give contradiction that digraph is connected.

  Then (i, j) entry in $K_1 = \sum_{r=1}^{n-1} X^r \quad + X^n$ (3.1.1)

  The sum in equation (3.1.1) contains all non-zero entries.

3.2 Illustration By Examples:

Cycle Graphs with order 5 and different orientations (In Figure 1.2)

It is clear that, $Y^T = Y_1 = \begin{array}{c c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$

It is clear that, $f(C_5) = f(C_5^1) = 16$.

And, using MATLAB, we have

$$Y^1 + Y^2 + Y^3 + Y^4 + Y^5 = \begin{array}{c c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

4. CONCLUDING REMARKS

In this paper, we find , if G_{D_1} and G_{D_2} are strong directed graphs with the same number of vertices and edges with different orientations, then $f(G_{D_1}) = f(G_{D_2})$ by considering the pebbling numbers of various orientations of directed wheel graphs, directed complete graphs and cycle graphs. In this direction, we have proposed that, "The pebbling numbers of two strongly connected directed graphs G_{D_1} and G_{D_2} with the same set of vertices and edges with different orientations are equal if their corresponding adjacency matrices are symmetrical to each other and their corresponding matrices K_1 and K_2 defined by $K_1 = X^1 + X^2 + ... + X^n$, $K_2 = Y^1 + Y^2 + ... + Y^n$, have no zero entry. X and Y are the adjacency matrices corresponding to G_{D_1} and G_{D_2} respectively." The motivation of equality of pebbling number on directed graph arise from transporting devices from starting positions to final positions that allow them to record the entire graph. One should not neglect the chance of graph pebbling will have powerful impact. For example, we can think of the loss of a pebble during a pebbling steps ads as a loss of information, fuel or electrical charge.

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Jitendra Binwal
Associate Professor and Head,
jitendrabinwal.cash@modyuniversity.ac.in
Aakanksha Baber
Department of Mathematics,
School of Sciences
Mody University of Science and Technology, India
baberaakanksha8@gmail.com