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QUASI – STATIC THERMAL STRESSES DUE TO HEAT GENERATION IN A SEMI – INFINITE HOLLOW CYLINDER

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Abstract: The present paper deals with the determination of displacement and thermal stresses in a Semi infinite hollow Cylinder defined by $\alpha \le r \le b$, $o \le z < \infty$ due to internal heat generation within it. Initially the Semi infinite hollow Cylinder is at arbitrary temperature $F(r, z)$. The governing heat conduction equation has been solved by the method of integral transform technique. The results are obtained in a series form in terms of Bessel's functions. The results for displacement and stresses have been computed numerically and illustrated graphically.

Keywords: Thermal stresses, Heat generation.

1. INTRODUCTION

Problems of thermal stress arise in many practical design problems, such as those encountered in the design of stream and gas turbines, diesel engines, jet engines, rocket motors, and nuclear reactors. The high aerodynamic heating rates associated with high speed flight present even more severe thermal-stress problems for the design of spacecraft and missiles.

The steady – state thermal stresses in circular disk subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and the circular edge has been considered by Nowacki [1]. Roy Choudhuri [3] has succeeded in determining the quasi-static thermal stresses in thin circular disk subjected to transient temperature along the circumference of a circle over the upper face with lower face at zero temperature and fixed circular edge thermally insulated. Roy Choudhuri [4] has also determined quasi-static thermal defection of a thin clamped circular disk due to ramp – type heating of a concentric. circular region of the upper face. Wankhade [5] determined the quasi – static thermal stresses in a thin circular disk subjected to arbitrary initial temperature on the upper face with lower face at zero temperature and faxed circular edge thermally insulated. Ahmed [6] studied the thermal stresses problem in non-homogeneous transversely isotropic infinite circular cylinder". El-Naggar [7] has also studied the thermal stresses in an infinite elastic slab.

Gogulawar and Deshmukh [8] determined thermal stresses in a thin circular disk with heat sources. Recently Kulkarni [9] has studied thermal stresses due to heat generation in a Thin hollow circular disk.

However Thermal stress problems in a Semi infinite hollow Cylinder due to heat generation within it has not been analyzed. The results presented here will be useful in engineering problems, particularly in the determination of the state of strain in hollow cylinder constituting foundation of containers for hot gases or liquids, in the foundations for furnaces etc.

2. FORMULATION OF THE PROBLEM

Consider a semi-infinite hollow cylinder defined by $\alpha \le r \le b$, $0 \le z < \infty$. Initially the cylinder is at arbitrary temperature $F(r, z)$. The inner circular boundary ($r = \alpha$) and outer circular boundary ($r = b$) are kept at zero temperature. Also and the boundary surface $z = 0$ is at zero temperature. For time $t > 0$ heat is generated within the Semi-infinite hollow cylinder at a rate of $g(r, z, t)$. Under these conditions, the displacement and thermal stresses in semi-infinite hollow cylinder due to heat generation are required to be determined.

The differential equation governing the displacement potential function $\Psi(r, z, t)$ is given as

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} = (1+v)\,a_t T \quad \text{at} \quad t = 0 \tag{1}$$

$$= 0 \text{ at } r = \alpha \text{ and } r = b \text{ for all time } t \tag{2}$$

The stress function σ_{rr} and $\sigma_{\theta\theta}$ are given as,

$$\sigma_{rr} = -\frac{2\mu}{r}\frac{\partial \Psi}{\partial r} \tag{3}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial \Psi}{\partial r^2} \tag{4}$$

Also, in the plane state of stress within the Hollow cylinder, we have

$$\sigma_{rz} = \sigma_{zz} = \sigma_{\theta z} = 0 \tag{5}$$

Initially $$T = \Psi = \sigma_{rr} = \sigma_{\theta\theta} = F(r, z) \quad \text{at} \quad t = 0 \tag{6}$$

The temperature of the hollow cylinder satisfies the heat conduction equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g(r, z, t)}{K} = \frac{1}{\alpha} \tag{7}$$

with the boundary conditions,

$$T = 0 \quad \text{at} \quad r = \alpha, t > 0 \tag{8}$$

$$T = 0 \quad \text{at} \quad r = b, t > 0 \tag{9}$$

$$T = 0 \quad \text{at} \quad z = 0 \tag{10}$$

and the initial condition

$$T = F(r, z) \quad \text{in} \quad \alpha \le r \le b, 0 \le z < \infty, \quad t = 0 \tag{11}$$

where k and α are thermal conductivity and thermal diffusivity of the material of Hollow cylinder, respectively.

Equations (1) to (11) constitute mathematical formulation of problem.

3. SOLUTION OF THE HEAT CONDUCTION EQUATION

To obtain the expression for temperature $T(r, z, t)$, we introduce the finite Hankel transform over the variable r. This expression and its inverse transform are

$$\overline{T}(\lambda_m, z, t) = \int_b^a r' K_0(\lambda_m, r') T(r', z, t)\, dr' \tag{12}$$

$$T(r, z, t) = \sum_{m=1}^{\infty} K_0(\lambda_m, r) \overline{T}(\lambda_m, z, t) \tag{13}$$

$$K_0(\lambda_m, r) = \frac{\pi}{\sqrt{2}} \frac{\lambda_m J_0(\lambda_m b) Y_0(\lambda m b)}{} \left[\left(\frac{J_0(\lambda_m r)}{J_0(\lambda_m b)} \right) - \left(\frac{Y_0(\lambda_m r)}{Y_0(\lambda m b)} \right) \right] \tag{14}$$

and λ_1, λ_2, ... are the positive roots of the transcendental equation.

$$\frac{J_0(\lambda_m a)}{J_0(\lambda_m b)} - \frac{Y_0(\lambda_n a)}{Y_0(\lambda_n b)} = 0 \tag{15}$$

Where $Jn(z)$ is the Bessel function of first-order n.

Secondly we introduce the Fourier transform F over the variable z and its inverse transform defined in the range $0 \le z < \infty$

$$\bar{\bar{T}}(\lambda_m, \beta_n, t) = \int_0^\infty K(\eta, z')\bar{T}(\lambda_m, z', t)\,dz' \tag{16}$$

$$\bar{T}(\lambda_m, z, t) = \int_0^\infty K(\beta_n, z)\bar{\bar{T}}(\lambda_m, \eta, t)\,d\eta \tag{17}$$

Where
$$K(\eta, z) = \sqrt{\frac{2}{\pi}}\sin \eta z \tag{18}$$

On applying the finite Hankel transform & Fourier transform successively, defined in the equation (12) & (16) and its inverse transform defined in (13) & (17), to the equation (7), one obtains the expression for temperature as

$$T(r,z,t)=\int_0^\infty \sum_{m=1}^\infty K_0(\lambda_m,r)K(\eta,z)e^{-\alpha(\lambda_m^2+\eta^2)t}\left[F(\lambda_m,\eta)+\int_{t'=0}^{t}\frac{\alpha}{k}e^{\alpha(\lambda_m^2+\eta^2)t'}g(\lambda_m,\eta,t')\,dt'\right]d\eta \tag{19}$$

4. DISPLACEMENT POTENTIAL AND THERMAL STRESSES

On inserting the value of temperature form (19) in (1), one obtains.

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} = (1+v)\,a_t \int_0^\infty \sum_{m=1}^\infty K_0(\lambda_m, r)K(\eta, z)e^{-\alpha(\lambda_m^2+\eta^2)t} \left[F(\lambda_m, \eta) + \int_{t'=0}^{t}\frac{\alpha}{k}e^{\alpha(\lambda_m^2+\eta^2)t'}g(\lambda_m, \eta, t')\,dt'\right]d\eta \tag{20}$$

on solving (20), one obtains

$$\Psi = -(1+v)\,a_t \int_0^\infty \sum_{m=1}^\infty \frac{1}{\lambda_m^2} K_0(\lambda_m, r)K(\eta, z)e^{-\alpha(\lambda_m^2+\eta^2)t} \left[F(\lambda_m, \eta) + \int_{t'=0}^{t}\frac{\alpha}{k}e^{\alpha(\lambda_m^2+\eta^2)t'}g(\lambda_m, \eta, t')\,dt'\right]d\eta \tag{21}$$

Substituting the value of displacement potential form equation (21) in equations (3) & (4) one obtains expressions for thermal stresses as

$$\sigma_{rr} = 2(1+v)a_t\mu \int_0^\infty \sum_{m=1}^\infty \frac{1}{\lambda_m r} K_1(\lambda_m, r) K(\eta, z) e^{-\alpha(\lambda_m^2+\eta^2)t}$$
$$\left[F(\lambda_m, \eta) + \int_{t'=0}^{t} \frac{\alpha}{k} e^{\alpha(\lambda_m^2+\eta^2)t'} g(\lambda_m, \eta, t') dt' \right] d\eta \tag{22}$$

$$\sigma_{\theta\theta} = -2(1+v)a_t\mu \int_0^\infty \sum_{m=1}^\infty \frac{1}{\lambda_m} \left(\lambda_m K_0(\lambda_m, r) - \frac{K_1(\lambda_m, r)}{r} \right) K(\eta, z) e^{-\alpha(\lambda_m^2+\eta^2)t}$$
$$\left[F(\lambda_m, \eta) + \int_{t'=0}^{t} \frac{\alpha}{k} e^{\alpha(\lambda_m^2+\eta^2)t'} g(\lambda_m, \eta, t') dt' \right] d\eta \tag{23}$$

Where $$K_1(\lambda_m, r) = \frac{\partial}{\partial r}[K_0(\lambda_m, r)].$$

5. SPECIAL CASE

Set $$F(r, z) = T_0(r^2 - a^2)(r^2 - b^2)e^{-\omega z} \qquad T_0 \succ 0 \tag{24}$$

$$g(r, z, t) = g_i \delta(r - r_1)\delta(z - z_1)\delta(t - \tau) \tag{25}$$

where r is the radius, δ is the Dirac-delta function, $\omega > 0$ the heat source $g(r, z, t)$ is an instantaneous line heat source situated at the center of the cylinder along the radial direction and releases its heat instantaneously at time $t = \tau = 2$ hr.

Applying finite Hankel transform &Fourier transform to the equation (24), one obtains.

$$\int_{r=a}^{a} \int_{\eta=0}^{\infty} r' K_0(\lambda_m r') K(\eta, z')(r'^2 - a^2) e^{-\omega z} dr' d\eta$$
$$= \frac{8[(16 + a^2\lambda_m^2 - 3b^2\lambda_m^2)J_0(\lambda_m a) - (16 + b^2\lambda_m^2 - 3a^2\lambda_m^2)J_0(\lambda_m b)]}{\lambda_m^6 \pi \sqrt{1 - \left(\frac{J_0(\lambda_m b)}{J_0(\lambda_m a)} \right)^2}} \sqrt{\frac{2}{\pi}} \frac{\eta}{\eta^2 + \omega^2} \tag{26}$$

By using equation (25), we get

$$\int_{t'=0}^{t} \frac{\alpha}{k} e^{\alpha(\lambda_m^2+\eta^2)t'} g(\lambda_m, \eta, t') dt' = g_i \sqrt{\frac{2}{\pi}} \sin \eta z_1 k_0(\lambda_m, r) e^{\alpha(\lambda_m^2+\eta^2)\tau} \tag{27}$$

6. NUMERICAL CALCULATIONS

The Numerical Calculations have been carried out for an steel Semi-infinite hollow cylinder. The heat source $g(r, z, t)$ is an instantaneously line heat source of strength g_i =50 Btu/hr.ft, situated at the center of Semi-infinite hollow cylinder coaxially along the radial direction and released its heat instantaneously at the time $t = \tau = 2$ hr.,

$$a = 1 \text{ ft.}, \qquad b = 2 \text{ ft.}, \qquad r_1 = 1.5 \text{ ft. } \& \; z_1 = 2 \text{ ft.}$$

The $\lambda_1 = 3.1230$, $\lambda_2 = 6.2734$, $\lambda_3 = 9.4182$, $\lambda_4 = 12.5614$, $\lambda_5 = 15.7040$ are the positive roots of the transcendental equation.

$$\frac{J_0(\lambda_m a)}{J_0(\lambda_m b)} - \frac{Y_0(\lambda_n a)}{Y_0(\lambda_n b)} = 0 \quad \text{and}$$

We set for convenience, $X = 10^6$, $Y = 10^{10}$, $Z = 10^8$, with the material properties as,

Thermal diffusivity $\alpha = 0.48$ ft^2/hr,

Thermal conductivity $k = 26$ Btu / (hr.ft.^{0}F),

Poisson ratio $\nu = 0.281$,

Coefficient of linear thermal expansion $\alpha_t = 16.4 \times 10^{-6} \, \frac{1}{F}$,

Density p = 490 lb/ft^3,

Specific heat $c_p = 0.211$ Btu/ lb^0F

The numerical calculations have been carried out with the help of computational mathematical software Mathcad.-2000.

7. GRAPHICAL REPRESENTATION

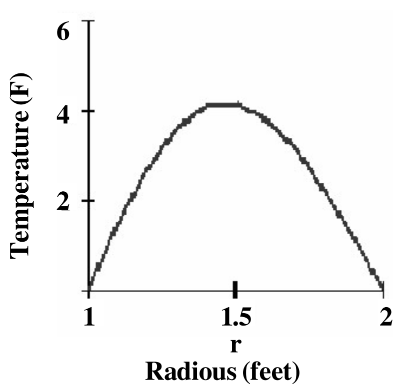

Figure 1: Temperature Distribution $\frac{T}{X}$

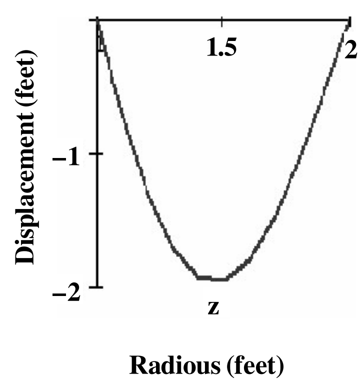

Figure 2: Displacement Function $\frac{\Psi}{Y}$

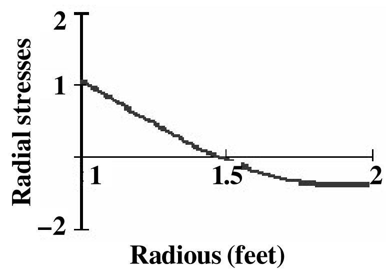

Figure 3: Radial Stress Function $\frac{\sigma_{rr}}{Z}$

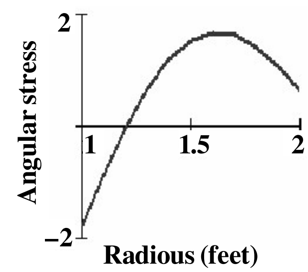

Figure 4: Angular Stress Function $\frac{\sigma_{\theta\theta}}{Z}$

Concluding Remark:

In this paper a Semi infinite hollow cylinder is considered which is subjected to arbitrary known interior temperature and determine the expression for temperature, displacement and stress functions due to heat generation within heat

From Figure1: It is observed that the temperature function T increases within the annular region $1 \leq r \leq 1.5$ and decreases within the annular region $1.5 \leq r \leq 2$ and becomes maximum at the center (r =1.5).

From Figure 2: It is observed that the displacement function ψ decreases within the annular region $1 \leq r \leq 1.5$ and increases within the annular region $1.5 \leq r \leq 2$ and becomes minimum at the center (r = 1.5).

From Figure 3: It is clear that radial stress component σ_{rr} develops tensile stress within the annular region $1 \leq r \leq 1.5$ and Compressive stress within the annular region $1.5 \leq r \leq 2$ and becomes zero at the center (r = 1.5).

From Figure 4: It is observed that the angular stress function $\sigma_{\theta\theta}$ increases from inner circular boundary to outer circular boundary.

Also it can be observed that from the figures of temperature and displacement the direction of heat flow and direction of body displacement are opposite to each other.

The results obtained here are useful in engineering problems particularly in the determination of state of stress in Semi infinite hollow cylinder. Also assigning suitable values to the parameter and function in the expression (19), (22), (23), we can derive any particular case of special interest.

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